ACTA ET COMMENTATIONES UNIVERSITATIS TARTUENSIS DE MATHEMATICA
Volume 20, Number 1, June 2016
Available online at http://acutm.math.ut.ee

Wide right Morita contexts in lax-unital bicategories

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Abstract. The classical result in the theory of unital rings that the maps of a Morita context are isomorphisms when they are epimorphisms can be proven in the general setting of wide right Morita contexts in bicategories. There exists a similar result for non-unital rings, but bicategories are not general enough to handle that case. In this paper, we use the more general lax-unital bicategories to prove a version of the result and study some related questions.

1. Introduction

There are many structures whose Morita theory has elements in common with the Morita theory of a class of similar structures, so as to admit a common generalization. For example for unital rings and monoids, there is the Morita theory of enriched monoids [6] and the Morita theory of enriched categories [4]. We will try to unify aspects of the Morita theory of non-unital rings and semigroups. The approach we will use is based on the one taken in [1], where El Kaoutit defined the concept of a wide right Morita context in a bicategory. We will generalize El Kaoutit's result which allows one to deduce the bijectivity of a Morita context's maps from their surjectivity. This result was itself a generalization of a similar result in the classical Morita theory for unital rings, and it is not hard to modify their proof so that it also generalizes a similar result in the Morita theory of monoids or other similar structures. To apply such a result to a particular kind of structure, we need to construct a corresponding bicategory.

We note the following about our notation for bicategories:

- we will write the composition of 1-cells of a (lax-unital) bicategory from left to right and the composition of 2-cells from right to left,

Received June 17, 2015.
2010 *Mathematics Subject Classification.* 18D05.
Key words and phrases. Wide right Morita context, lax-unital bicategory.
http://dx.doi.org/10.12697/ACUTM.2016.20.06

- as is customary, we will usually omit the subscripts from the natural 2-cells $a_{M,N,L}$, l_M and r_M,
- we will often denote the composition of 2-cells f, g in $\mathrm{B}(M, N)$ by $f \circ g$,
- we will write B_0 for the collection of objects of a (lax-unital) bicategory B.

In the case of unital rings, the bicategory we need to construct is well known.

Example 1. There is a bicategory Mod, which consists of the following data:

- the objects of Mod are all unital rings A,
- the 1-cells $M: A \to B$ are the A-B-bimodules and they are composed using the tensor product,
- the 2-cells $f: M \to N$ are bimodule morphisms and are they composed as functions,
- the unit 1-cell of an object A is the bimodule ${}_A A_A$,
- the unitors $l_M: IM \to M$, $r_M: MI \to M$ are the left and right multiplication maps respectively.

In general, for certain monoid-like structures like monoids, semirings, or ordered monoids, the construction goes along the same lines.

There exist similar results about the bijectivity of the maps of a Morita context for certain non-unital rings and modules, and certain semigroups and acts (e.g., in [3], for semigroups with local units and unitary biacts). If we try to apply the bicategorical result to non-unital rings or semigroups, the problem we run into is that these structures and modules between them do not form a bicategory, since we do not have unit 1-cells, because the usual unitors are not invertible. We can not just forget about the units and use semibicategories, because we need unit 1-cells in order to define wide right Morita contexts. This is why we will use the notion of lax units.

2. Lax-unital bicategories

Definition 1 (Lax-unital bicategory)**.** A *lax-unital bicategory* B is given by the same data as a bicategory:

- a collection B_0 of objects,
- for each pair of objects $A, B \in \mathrm{B}_0$, a category $\mathrm{B}(A, B)$, the objects and morphisms of which are called 1-cells and 2-cells, respectively,
- for each triple of objects $A, B, C \in \mathrm{B}_0$, a composition functor

$$\mathrm{B}(A, B) \times \mathrm{B}(B, C) \to \mathrm{B}(A, C),$$

- for each object $A \in \mathrm{B}_0$, a distinguished 1-cell I_A, called a *lax unit*; we will occasionally call these 1-cells *units*,

- for each quadruple of objects $A, B, C, D \in \mathrm{B}_0$ a collection of 2-cells
$$a_{M,N,L}\colon (MN)L \to M(NL)$$
natural in $M \in \mathrm{B}(A,B)$, $N \in \mathrm{B}(B,C)$, $L \in \mathrm{B}(C,D)$, called the *associators*,
- for each pair of objects $A, B \in \mathrm{B}_0$, two collections of 2-cells
$$l_M\colon I_A M \to M, \quad r_M\colon M I_B \to M$$
natural in $M \in \mathrm{B}(A,B)$, called the *left* and *right unitors* respectively.

The morphisms $a_{M,N,L}$ are required to be invertible, but the morphisms l_M and r_M are not. The natural morphisms $a_{M,N,L}$, l_M and r_M need to be such that the diagrams

$$\begin{tikzcd}
((MN)L)K \arrow[r, "a_{MN,L,K}"] \arrow[dd, "a_{M,N,L}1_K"] & (MN)(LK) \arrow[dr, "a_{M,N,LK}"] & \\
 & & M(N(LK)), \\
(M(NL))K \arrow[r, "a_{M,NL,K}"'] & M((NL)K) \arrow[ur, "1_M a_{N,L,K}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
(MI)N \arrow[d, "a_{M,I,N}"'] \arrow[dr, "r_M 1_N"] & \\
M(IN) \arrow[r, "1_M l_N"'] & MN
\end{tikzcd}$$

commute and the diagrams

$$\begin{tikzcd}
(MN)I \arrow[d, "a_{M,N,I}"'] \arrow[dr, "r_{MN}"] & \\
M(NI) \arrow[r, "1_M r_N"'] & MN,
\end{tikzcd}
\qquad
\begin{tikzcd}
(IM)N \arrow[d, "a_{I,M,N}"'] \arrow[dr, "l_M 1_N"] & \\
I(MN) \arrow[r, "l_{MN}"'] & MN,
\end{tikzcd}
\qquad
\begin{tikzcd}
II \arrow[r, bend left, "l_I"] \arrow[r, bend right, "r_I"'] & I
\end{tikzcd}$$

commute.

Remark 1. The definition of a lax-unital bicategory differs from that of a bicategory only in two ways:

- the natural transformations l and r are not required to be invertible,
- the last three diagrams are required to commute.

The reason why three additional diagrams are included in the definition of a lax-unital bicategory is because we want the analogue of the bicategorical *coherence* theorem to hold. That is to say, we want any two 2-cells that are the results of composing the 2-cells l, r, a and 1 in various ways to coincide whenever their domains are formally the same and codomains are formally the same. The bicategorical version of the result, when originally proven [5], included the additional three diagrams in our definition. It was later shown by Kelly [2] that these diagrams were redundant in the bicategorical case.

One can check that MacLane's proof [5] of the coherence theorem works for lax-unital bicategories. Additionally one can check that Kelly's proof [2] of the redundancy of the last three diagrams holds when the unitors l and r are merely epimorphisms.

We take the definition of a *wide right Morita contexts* from [1] and put it into the context of lax-unital bicategories.

Definition 2 (Wide right Morita context)**.** Let A and B be objects in a lax-unital bicategory B. A *wide right Morita context* from A to B is a quadruple $\Gamma = (P, Q, \theta, \phi)$, with 1-cells

$$P\colon A \to B, \qquad Q\colon B \to A,$$

and 2-cells

$$\theta\colon PQ \to I_A, \qquad \phi\colon QP \to I_B\,,$$

such that the following two diagrams commute:

$$\begin{tikzcd}
Q(PQ) \arrow[r, "1\theta"] & QI \arrow[dr, "r"] & \\
& & Q, \\
(QP)Q \arrow[uu, "a"] \arrow[r, "\phi 1"'] & IQ \arrow[ur, "l"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
P(QP) \arrow[r, "1\phi"] & PI \arrow[dr, "r"] & \\
& & P. \\
(PQ)P \arrow[uu, "a"] \arrow[r, "\theta 1"'] & IP \arrow[ur, "l"'] &
\end{tikzcd}$$

If θ and ϕ are isomorphisms, we call Γ an *adjoint equivalence*, since by inverting θ or ϕ, the axioms of a wide right Morita context are precisely the conditions that make (P, Q) an adjoint pair. We will occasionally use the word *context* to refer to a wide right Morita context.

Lemma 1. *Let (P, Q, θ, ϕ) be a wide right Morita context in a lax-unital bicategory* B*. Then the following diagrams commute:*

$$\begin{tikzcd}
(PQ)(PQ) \arrow[r, "1\theta"] \arrow[d, "\theta 1"'] & (PQ)I \arrow[d, "r"] \\
I(PQ) \arrow[r, "l"'] & PQ,
\end{tikzcd}
\qquad
\begin{tikzcd}
(QP)(QP) \arrow[r, "1\phi"] \arrow[d, "\phi 1"'] & (QP)I \arrow[d, "r"] \\
I(QP) \arrow[r, "l"'] & QP.
\end{tikzcd}$$

Proof. In the diagram

$$
\begin{tikzcd}
(PQ)(PQ) \arrow[rrr, "\theta(1)", "\theta(11)"'] \arrow[dr, "a^{-1}"] \arrow[ddddr, "a"] \arrow[ddddd, "(1)\theta"', "(11)\theta"] & & & I(PQ) \arrow[dl, "a^{-1}"'] \arrow[ddddd, "l"] \\
& ((PQ)P)Q \arrow[r, "(\theta 1)1"] \arrow[d, "a1"'] & (IP)Q \arrow[ddddr, "l1"] & \\
& (P(QP))Q \arrow[r, "(1\phi)1"] \arrow[d, "a"'] & (PI)Q \arrow[d, "a"'] \arrow[dddr, "r1"] & \\
& P((QP)Q) \arrow[r, "1(\phi 1)"] \arrow[d, "1a"'] & P(IQ) \arrow[ddr, "1l"] & \\
& P(Q(PQ)) \arrow[r, "1(1\theta)"] & P(QI) \arrow[dr, "1r"] & \\
(PQ)I \arrow[urr, "a"] \arrow[rrr, "r"'] & & & PQ,
\end{tikzcd}
$$

all the parts commute either because of naturality, coherence, or the axioms of a wide right Morita context.

This proves that the first diagram in the lemma commutes. In a similar way, we can prove that the second diagram commutes. $\square$

Until the end of this paper, $\mathcal{E}$ will denote a class of epimorphic 2-cells of the lax-unital bicategory B under discussion. We will require $\mathcal{E}$ to satisfy:

(1) $\mathcal{E}$ is closed under composition,
(2) all isomorphisms belong to $\mathcal{E}$,
(3) the 1-cell composition functor of B maps members of $\mathcal{E}$ into $\mathcal{E}$,
(4) every monomorphism in $\mathcal{E}$ is an isomoprhism,
(5) if $f \circ g$ and g are in $\mathcal{E}$ then so is f,
(6) the unitors of the unit 1-cells are in $\mathcal{E}$.

The main example for the class $\mathcal{E}$ is the class of all regular epimorphisms in the case of non-unital rings and also in the case of semigroups.

Theorem 1. *Suppose that (P, Q, θ, ϕ) is a wide right Morita context in a lax-unital bicategory* B*, where either all left unitors or all right unitors are epimorphisms. Then, if θ (resp. ϕ) is in $\mathcal{E}$, it is an isomorphism.*

Proof. Suppose that all left unitors are epimorphisms (the proof is similar if all right unitors are epimorphisms). Also suppose that $\theta\colon PQ \to I$ is $\mathcal{E}$. We will show that θ is a monomorphism and therefore an isomorphism by the fourth condition on $\mathcal{E}$. Let $u, v\colon X \to PQ$ be such that $\theta \circ u = \theta \circ v$. If we apply the functor $(PQ) \cdot -$ to this equality, we get

$$(1\theta) \circ (1u) = (1\theta) \circ (1v).$$

We have the diagram

$$\begin{tikzcd}
(PQ)X \arrow[r, shift left, "1u"] \arrow[r, shift right, "1v"'] \arrow[d, "\theta 1"'] & (PQ)(PQ) \arrow[d, "\theta 1"'] \arrow[dr, "1\theta"] & \\
IX \arrow[r, shift left, "1u"] \arrow[r, shift right, "1v"'] \arrow[d, "l"'] & I(PQ) \arrow[d, "l"'] & (PQ)I \arrow[dl, "r"] \\
X \arrow[r, shift left, "u"] \arrow[r, shift right, "v"'] & PQ &
\end{tikzcd}$$

which is commutative with respect to the upper (lower) morphisms of the parallel pairs of 2-cells. The squares with horizontal morphisms $1u$ and horizontal morphisms $1v$ commute because of the functoriality of the multiplication, while the lower squares commute because of naturality. The right part of the diagram commutes by Lemma 1. From this we get

$$u \circ l \circ (\theta 1) = v \circ l \circ (\theta 1),$$

which implies $u = v$, since l and $\theta 1$ are epimorphisms. Therefore θ is a monomorphism and since it is in $\mathcal{E}$, an isomorphism.

Similar arguments work for ϕ. □

Probably the simplest application of the previous theorem is the following.

Corollary 1. *Suppose that* B *is a lax-unital bicategory, where either all left unitors or all right unitors are epimorphisms. Then the unitors of the unit* 1*-cells are isomorphisms.*

Proof. For any object A of B, let $\Gamma = (I_A, I_A, r_I, l_I)$ be the unit wide right Morita context from A to A, as defined in [1]. Since r_I an l_I are in $\mathcal{E}$, we can apply the previous theorem, which means that the unitors of I_A are isomorphisms. □

Among other things, this means that when the conditions of the preceding corollary are satisfied, the relation of adjoint equivalence is reflexive. It is symmetric by definition and by assuming slightly more, it is also transitive.

Corollary 2. *Suppose that* B *is a lax-unital bicategory where either all left unitors or all right unitors are in $\mathcal{E}$. Then the relation of adjoint equivalence is an equivalence relation.*

Proof. As we noted, the relation of adjoint equivalence is reflexive and symmetric in this setting. To show that it is transitive, let A, B and C be objects of B, let $\Gamma_1 = (P_1, Q_1, \theta_1, \phi_1)$ be a wide right Morita context from A to B and let $\Gamma_2 = (P_2, Q_2, \theta_2, \phi_2)$ be a wide right Morita context from B to

C. In [1], it is shown that in the case of bicategories we can compose these contexts to get a composite context

$$\Gamma_1\Gamma_2 = (P_1P_2, Q_2Q_1, \theta_1 * \theta_2, \phi_2 * \phi_1)$$

from A to C. It is easy to check that the construction given there is also valid in the case of lax-unital bicategories. In order to show that the relation of adjoint equivalence is transitive, we need to show that $\theta_1 * \theta_2$ and $\phi_2 * \phi_1$ are isomorphisms. In [1], the 2-cell $\theta_1 * \theta_2$ is given as the composite

$$\begin{tikzcd}
(P_1P_2)(Q_2Q_1) \arrow[r, "a"] \arrow[d, "\theta_1 * \theta_2"'] & P_1(P_2(Q_2Q_1)) \arrow[r, "1a^{-1}"] & P_1((P_2Q_2)Q_1) \arrow[d, "1(\theta_2 1)"] \\
I & P_1Q_1 \arrow[l, "\theta_1"] & P_1(IQ_1). \arrow[l, "1l"]
\end{tikzcd}$$

If all the left unitors are in $\mathcal{E}$, then $\theta_1 * \theta_2$ is clearly in $\mathcal{E}$, being a composite of 2-cells in $\mathcal{E}$. We could equivalently define $\theta_1 * \theta_2$ in terms of a composite featuring the right unitor. Using coherence it would be easy to check that it would give us the same $\theta_1 * \theta_2$. This means that if all the right unitors are in $\mathcal{E}$, the 2-cell $\theta_1 * \theta_2$ would still be in $\mathcal{E}$. We can therefore use Theorem 1 to deduce that $\theta_1 * \theta_2$ is invertible. Since we can do the same with $\phi_2 * \phi_1$, we have an adjoint equivalence from A to C. $\square$

While we can not apply Theorem 1 to the lax-unital bicategory of non-unital rings or the lax-unital bicategory of semigroups, we can apply the theorem to their certain lax-unital subbicategories. For example, we can apply the theorem to the lax-unital bicategory of semigroups with commuting local units, unitary biacts, and biact morphisms, in which $\mathcal{E}$ is the collection of all regular epimorphisms, which are precisely the surjective epimorphisms. In this case and in several other cases, the theorem gives us an essentially known fact. In the following, we will try to lift wide right Morita contexts in lax-unital bicategories to suitable lax-unital subbicategories, in which the last theorem could potentially be applied.

3. Improving contexts

In this section we will give a few results that allow us to improve wide right Morita contexts so as to satisfy the conditions given in Theorem 1. If B is a lax-unital bicategory, let B_L (B_R) be the lax-unital locally full subbicategory determined by the 1-cells for which the left (right) unitors are in $\mathcal{E}$. The fact that the collection of 1-cells of B_L is closed under composition can be seen from the following diagram, which commutes by coherence:

$$
\begin{tikzcd}
I(MN) \arrow[r, "a^{-1}"] \arrow[dr, "l_{MN}"'] & (IM)N \arrow[d, "l_M 1"] \\
& MN.
\end{tikzcd}
$$

In a similar fashion, one can show that the left unitor of IM is always in $\mathcal{E}$. In fact we have the following proposition.

Proposition 1. *Let* B *be a lax-unital bicategory. Let* $\Gamma = (P, Q, \theta, \phi)$ *be a wide right Morita context from* A *to* B *in* B*, such that* θ *and* ϕ *are in* $\mathcal{E}$*. Then there exists a wide right Morita context* $\Gamma' = (P', Q', \theta', \phi')$ *from* A *to* B *such that* θ'*,* ϕ' *and the unitors of* P' *and* Q' *are in* $\mathcal{E}$*.*

Proof. We define the wide right Morita context $\Gamma' = (P', Q', \theta', \phi')$ by setting $P' = IP$, $Q' = IQ$ and defining θ' and ϕ' using the diagrams

$$
\begin{tikzcd}
(IP)(IQ) \arrow[r, "ll"] \arrow[dr, "\theta'"'] & PQ \arrow[d, "\theta"] \\
& I,
\end{tikzcd}
\qquad
\begin{tikzcd}
(IQ)(IP) \arrow[r, "ll"] \arrow[dr, "\phi'"'] & QP \arrow[d, "\phi"] \\
& I.
\end{tikzcd}
$$

Now we will show that Γ' is indeed a wide right Morita context. This means that the diagrams in the definition of a wide right Morita context commute. We will only check that one of the diagrams commutes, since the other one can be shown to commute in a similar way. We have the following commutative diagram:

$$
\begin{tikzcd}
((IP)(IQ))(IP) \arrow[rrrr, "a"] \arrow[d, "(1l)1"] & & & & (IP)((IQ)(IP)) \arrow[d, "1(l1)"] \\
((IP)Q)(IP) \arrow[rrrr, "a"] \arrow[dd, "(l1)1"'] \arrow[dr, "a \circ (a1)"] & & & & (IP)(Q(IP)) \arrow[dd, "1(1l)"] \\
& I((PQ)(IP)) \arrow[dl, "l"'] \arrow[d, "1(\theta 1)"] \arrow[dr, "1(1l)"] & & I(P(QP)) \arrow[d, "1(1\phi)"] & \\
(PQ)(IP) \arrow[d, "\theta 1"'] & I(I(IP)) \arrow[dl, "l"', "1(1l)"] & I((PQ)P) \arrow[ur, "1a"] \arrow[dll, "1(\theta 1)"] & I(PI) \arrow[dl, "1r"'] & (IP)(QP) \arrow[ul, "a"'] \arrow[d, "1\theta"] \\
I(IP) \arrow[rr, "1l", "l"'] & & IP & & (IP)I. \arrow[ll, "r"] \arrow[ul, "a"']
\end{tikzcd}
$$

All the inner diagrams commute either by naturality, coherence, or because Γ is a wide right Morita context. Therefore, the outer rectangle commutes.

The composition of the left (respectively, right) edge of the outer rectangle is $\theta' 1$ (respectively, $1\phi'$). Therefore, the second diagram in the definition of wide right Morita context commutes.

Now we need to check that the unitors of P' and Q' are in $\mathcal{E}$. For the left unitors this is easily seen from the following diagram, which commutes by coherence:

$$\begin{tikzcd}
I(IP) \arrow[dr, "l_{IP}"] & \\
(II)P \arrow[u, "a"] \arrow[r, "l_I 1_P"'] & IP.
\end{tikzcd}
\qquad
\begin{tikzcd}
I(IQ) \arrow[dr, "l_{IQ}"] & \\
(II)Q \arrow[u, "a"] \arrow[r, "l_I 1_Q"'] & IQ.
\end{tikzcd}$$

We need to check the same thing for the right unitors of P' and Q'. Since it is similar for P' and Q', we will only check it for P'. The diagram

$$\begin{tikzcd}
(IP)(QP) \arrow[rrrr, "(11)\phi"] \arrow[dr, "a"] \arrow[ddd, "a^{-1}"] & & & & (IP)I \arrow[dl, "a"'] \arrow[ddd, "r_{IP}"] \\
& I(P(QP)) \arrow[rr, "1(1\phi)"'] & & I(PI) \arrow[ddr, "1r"] & \\
& I((PQ)P) \arrow[u, "1a"] \arrow[rr, "1(\theta 1)"'] & & I(IP) \arrow[dr, "1l"'] & \\
(I(PQ))P \arrow[ur, "a"'] \arrow[rr, "(1\theta)1"'] & & (II)P \arrow[ur, "a"] \arrow[rr, "r_I 1"'] & & IP
\end{tikzcd}$$

commutes since each smaller part of it commutes either because of coherence, naturality, or because Γ is a wide right Morita context. Since the left composite of the outer diagram is clearly in $\mathcal{E}$, so is the right composite. By using the fifth property of $\mathcal{E}$, we get that the right unitor of IP is in $\mathcal{E}$.

Finally, we need to check that θ' and ϕ' are in $\mathcal{E}$. We will check this only for θ' (the proof for ϕ' is similar). Let Δ be the coherent natural transformation $\Delta_X \colon (II)X \to X$. Clearly, Δ_I is in $\mathcal{E}$. We have the following commutative diagram, where $\sim$ represent the various coherent combinations of associators:

$$\begin{tikzcd}
((PQ)P)((QP)Q) \arrow[rrr, "(\theta 1)(\phi 1)"] \arrow[dr, "(\theta 1)(11)"] \arrow[dddd, "\sim"'] & & & (IP)(IQ) \arrow[dl, "\sim"] \arrow[dddd, "ll"] \\
& (IP)((QP)Q) \arrow[urr, "(11)(\phi 1)"] \arrow[d, "\sim"'] & I((PI)Q) \arrow[d, "1(r1)"] & \\
& I((P(QP))Q) \arrow[ur, "1((1\phi)1)"] \arrow[d, "\sim"'] & I(PQ) \arrow[ddr, "l"] & \\
& I(((PQ)P)Q) \arrow[r, "1((\theta 1)1)"'] \arrow[d, "\sim"'] & I((IP)Q) \arrow[u, "1(l1)"'] \arrow[d, "\sim"'] & \\
((PQ)(PQ))(PQ) \arrow[r, "(\theta 1)1"] \arrow[d, "(\theta\theta)\theta"'] & (I(PQ))(PQ) \arrow[r, "(1\theta)1"] & (II)(PQ) \arrow[r, "\Delta"'] \arrow[dll, "(11)\theta"'] & PQ \arrow[d, "\theta"] \\
(II)I \arrow[rrr, "\Delta"'] & & & I.
\end{tikzcd}$$

The diagram commutes since every small diagram in the interior commutes either because of coherence, naturality, or because Γ is a wide right Morita context. Once again we can use the fifth property of $\mathcal{E}$ to deduce that the right edge of the outer rectangle is in $\mathcal{E}$, but the right edge is just θ', which completes the proof. $\square$

This proposition can help us to use Theorem 1 in situations where it would normally not apply. To do so, we need to restrict the class $\mathcal{E}$ to B_L. Unfortunately, this may cause the fourth condition we required of $\mathcal{E}$ not to hold. This is because B_L might have more monomorphic 2-cells than B, which could potentially not be isomorphisms when in $\mathcal{E}$. To overcome this, we must additionally assume that B_L does not have more monomorphic 2-cells than B. Then we can restrict $\mathcal{E}$ to B_L so that the conditions on $\mathcal{E}$ given in Section 2 are satisfied, and the above proposition gives us a wide right Morita context that lies in B_L and to which Theorem 1 can be applied.

We can use a different method to improve 1-cells and contexts, which can have more pleasing properties. In the paper [3], for example, the method used to make the unitors of acts surjective used the assignment

$${}_SM \mapsto \{sm \mid s \in S,\, m \in M\},$$

which maps an act to the image of the act's unitor. We can generalize this construction to our situation, but we have to make more assumptions. We will need to use orthogonal factorization systems, so we recall the definition now.

Definition 3. Let $\mathcal{C}$ be a category and let $\mathcal{E}$ be a class of epimorphisms and $\mathcal{M}$ a class of monomorphisms belonging to that category. Morphisms $e\colon A \to B$ and $m\colon C \to D$ are said to be *orthogonal*, a situation expressed by writing $e \perp m$, when for each commuting square

$$\begin{tikzcd} A \arrow[r, "e"] \arrow[d] & B \arrow[d] \arrow[dl, dashed, "s"'] \\ C \arrow[r, "m"'] & D \end{tikzcd}$$

there exists a unique diagonal $s\colon B \to C$ making the whole diagram commute. Let $\mathcal{E}$ consist of precisely those morphisms e of $\mathcal{C}$ for which $e \perp m$ for each $m \in \mathcal{M}$, and let $\mathcal{M}$ consist of precisely those morphisms m of $\mathcal{C}$ for which $e \perp m$ for each morphism e in $\mathcal{E}$. If each morphism f of $\mathcal{C}$ factors as $f = me$, we say that $(\mathcal{E}, \mathcal{M})$ is an *orthogonal factorization system* on $\mathcal{C}$.

We will now assume that on each morphism category of our lax-unital bicategory B, the 2-cells in $\mathcal{E}$ and the monomorphic 2-cells constitute an orthogonal factorization system on that category. This means that the collection $\mathcal{E}$ is precisely the collection of all strongly epimorphic 2-cells.

Remark 2. We now have a slightly easier way of checking whether a given class $\mathcal{E}$ satisfies the six conditions we required of it in the beginning. Let us assume that we are given a random lax-unital bicategory B$'$ such that each morphism category has (StrongEpis, Monos) as an orthogonal factorization system. Because of some well knows properties of factorization systems, the class of all strongly epimorphic 2-cells automatically satisfies all but two conditions required of $\mathcal{E}$:

- the 1-cell composition functor must preserve strongly epimorphic 2-cells,
- the unitors of the unit 1-cells must be strongly epimorphic.

The first of these two can often be deduced from other properties of B$'$. For example, if the functor of composing with a 1-cell always has a right adjoint, it automatically preserves strongly epimorphic 2-cells. Such is the case in the lax-unital bicategory of non-unital rings and modules, which means that we can get a suitable lax-unital subbicategory simply by throwing out the rings that do not satisfy the last condition. The same can be done with semigroups and acts and analogous structures.

We can now define an alternate way of improving wide right Morita contexts and the unitors of 1-cells. Since the construction will be lax functorial, let us, for the sake of completeness, list here the definition of a lax functor, which will essentially be the same as the usual definition in the case of bicategories.

Definition 4. Let C and D be lax-unital bicategories. A *lax functor* F from C to D consists of

- for each object A of C, an object $F(A)$ of D,
- for each pair $A, B \in \mathrm{C}_0$, a functor $F_{A,B} \colon \mathrm{C}(A,B) \to \mathrm{D}(F(A), F(B))$, which we will refer to as the local part of the lax functor,
- natural comparison 2-cells $\Phi_{M,N} \colon F(M)F(N) \to F(MN)$,
- comparison 2-cells $\Phi^0_A \colon I_{F(A)} \to F(I_A)$.

The comparison 2-cells need to be such that the following diagrams commute:

$$\begin{tikzcd}[column sep=small]
& (F(M)F(N))F(K) \arrow[r, "a"] \arrow[dl, "\Phi 1"'] & F(M)(F(N)F(K)) \arrow[dr, "1\Phi"] & \\
F(MN)F(K) \arrow[dr, "\Phi"'] & & & F(M)F(NK) \arrow[dl, "\Phi"] \\
& F((MN)K) \arrow[r, "F(a)"'] & F(M(NK)), &
\end{tikzcd}$$

$$\begin{tikzcd}
F(M)I_{F(B)} \arrow[r, "1\Phi^0_B"] \arrow[d, "r_{F(M)}"'] & F(M)F(I_B) \arrow[d, "\Phi_{M,I}"] \\
F(M) & F(MI_B), \arrow[l, "F(r)"]
\end{tikzcd}
\qquad
\begin{tikzcd}
I_{F(A)}F(M) \arrow[r, "\Phi^0_A 1"] \arrow[d, "l_{F(M)}"'] & F(I_A)F(M) \arrow[d, "\Phi_{I,M}"] \\
F(M) & F(I_A M). \arrow[l, "F(l)"]
\end{tikzcd}$$

Let us fix for each 1-cell M in B, an ($\mathcal{E}$,mono)-factorization (e_M, m_M) of l_M. Let A and B be objects of B. We will now define a functor

$$L_{A,B} \colon \mathrm{B}(A,B) \to \mathrm{B}_L(A,B).$$

This functor will depend on the choice of the factorizations (e_M, m_M), but will be unique up to isomorphism. Since $L_{A,B}$ will later turn out to be the local part of a lax functor, we will omit the subscripts A and B when applying the functor. Let $M \colon A \to B$. We define $L(M)$ as the 1-cell through which l_M factors, as seen in the diagram

$$\begin{tikzcd}
IM \arrow[r, "e_M"'] \arrow[rr, bend left, "l_M"] & L(M) \arrow[r, "m_M"'] & M.
\end{tikzcd}$$

We need to check that $L(M)$ is actually in $\mathrm{B}_L(A,B)$. To show this, we will use the following diagram, which commutes because of coherence and the naturality of l:

$$\begin{tikzcd}
 & I(IM) \arrow[r, "1e"] \arrow[dl, "a^{-1}"'] \arrow[d, "l_{IM}"] & IL(M) \arrow[d, "l_{L(M)}"] \\
(II)M \arrow[r, "l_I 1"'] & IM \arrow[r, "e"'] & L(M).
\end{tikzcd}$$

The right composite $l_{L(M)} \circ 1e$ is in $\mathcal{E}$, since the left side composite is the composite of 2-cells in $\mathcal{E}$. This implies that $l_{L(M)}$ is in $\mathcal{E}$, which means that $L(M)$ is a 1-cell in B_L.

Now suppose that $f\colon M \to N$ is a 1-cell in B. Then we can define $L(f)$ to be the unique 2-cell $L(M) \to L(N)$ that makes the following diagram commutative and exists because of the properties of the factorization system:

$$\begin{tikzcd}
IM \arrow[r, "e_M"'] \arrow[d, "1f"'] \arrow[rr, bend left, "l_M"] & L(M) \arrow[r, "m_M"'] \arrow[d, dashed, "L(f)"] & M \arrow[d, "f"] \\
IN \arrow[r, "e_N"] \arrow[rr, bend right, "l_N"'] & L(N) \arrow[r, "m_N"] & N.
\end{tikzcd}$$

It is clear that we can vertically paste the defining diagrams of $L(f)$ and $L(g)$ of a composable pair of 2-cells f and g. It is also clear that the resulting diagram will be the defining diagram of $L(g \circ f)$. Therefore, since $L(g \circ f)$ is the unique 2-cell making the diagram commute, it must equal to $L(g) \circ L(f)$. Since L also clearly takes unit 2-cells to unit 2-cells, we have shown that $L_{A,B}$ is a functor.

Now we will construct an identity-on-objects lax functor $L\colon \mathrm{B} \to \mathrm{B}_L$ with the functors $L_{A,B}$ as the local components. The only data missing is the comparison 2-cells. We define the comparison 2-cell

$$\Phi_{M,N}\colon L(M)L(N) \to L(MN)$$

as the unique 2-cell that makes the following diagram commute:

$$\begin{tikzcd}
(IM)(IN) \arrow[r, "e_M e_N"] \arrow[d, "a \circ (1l)"'] & L(M)L(N) \arrow[d, dashed, "\Phi_{M,N}"'] \arrow[dr, "m_M m_N"] & \\
I(MN) \arrow[r, "e_{MN}"'] & L(MN) \arrow[r, "m_{MN}"'] & MN.
\end{tikzcd}$$

The unique 2-cell exists because of the properties of a factorization system, since ee is in $\mathcal{E}$, m is a monomorphism and the outer composites are equal. To see that the outer composites are equal, we simply need to remember that $m_M \circ e_M = l_M$ and use coherence.

To define Φ^0_A for an object A of B, we need to remember that $l_I \colon II \to I$ is in $\mathcal{E}$. This means that m_I, the monomorphic part of the ($\mathcal{E}$,mono)-factorization of l_I, is an isomorphism. This means that we can define $\Phi^0_A \colon I \to L(I)$ to be $m_I^{-1} \colon I \to L(I)$. Finally we need to check that Φ and Φ^0 satisfy the conditions required of comparison maps and that Φ is natural.

For the first condition, observe that all the small parts of the following diagram commute either because of naturality, functoriality, the definition of Φ, or the definition of L:

$$\begin{tikzcd}
& (L(M)L(N))L(K) \arrow[r, "a"] \arrow[ddl, "\Phi 1"'] \arrow[d, "(mm)1"'] \arrow[dd, bend left, "(mm)m"] & L(M)(L(N)L(K)) \arrow[d, "1(mm)"] \arrow[dd, bend right, "m(mm)"'] \arrow[ddr, "1\Phi"] & \\
& (MN)L(K) \arrow[dd, "1m"', "(11)m"] & L(M)(NK) \arrow[dd, "m(11)"', "m1"] & \\
L(MN)L(K) \arrow[ur, "m1"'] \arrow[dr, "mm"] \arrow[ddr, "\Phi"'] & & & L(M)L(NK) \arrow[ul, "1m"] \arrow[dl, "mm"'] \arrow[ddl, "\Phi"] \\
& (MN)K \arrow[r, "a"'] & M(NK) & \\
& L((MN)K) \arrow[u, "m"'] \arrow[r, "L(a)"'] & L(M(NK)). \arrow[u, "m"] &
\end{tikzcd}$$

One can easily check that the left composite and the right composite of the outer hexagon are equal under m. This means that they are equal, because m is a monomorphism. Therefore, the first condition holds. In a similar fashion, we can see that the second condition holds by looking at the diagram

$$\begin{tikzcd}
L(M)I \arrow[rrr, "1\Phi^0", "1m^{-1}"'] \arrow[drr, "m1"] \arrow[dd, "r_M"'] & & & L(M)L(I) \arrow[dl, "mm"'] \arrow[dd, "\Phi_{M,I}"] \\
& M & MI \arrow[l, "r"] & \\
L(M) \arrow[ur, "m"] & & & L(MI). \arrow[lll, "L(r)"] \arrow[ul, "m"']
\end{tikzcd}$$

One can use a diagram very similar to the previous one to check that the third condition holds. To check the naturality of Φ, let $f \colon M \to U$ and $g \colon N \to V$ be 2-cells between suitably composable 1-cells. Then the commutativity of the following diagram is checked by using the monomorphicity of m, as we did before:

$$\begin{tikzcd}
L(M)L(N) \arrow[rrr, "L(f)L(g)"] \arrow[dd, "\Phi"'] \arrow[dr, "mm"] & & & L(U)L(V) \arrow[dd, "\Phi"] \arrow[dl, "mm"'] \\
& MN \arrow[r, "fg"] & UV & \\
L(MN) \arrow[rrr, "L(fg)"'] \arrow[ur, "m"'] & & & L(UV). \arrow[ul, "m"]
\end{tikzcd}$$

Therefore, L is indeed an identity-on-objects lax functor from B to $\mathrm{B_L}$.

We could go through a similar process for the right unitors, resulting in a lax functor $R\colon \mathrm{B} \to \mathrm{B}_R$ into the locally full lax-unital subbicategory determined by the 1-cells with right unitors in $\mathcal{E}$.

In the paper [1], it is shown that given a wide right Morita context Γ from A to B in a bicategory C, a 2-functor $F\colon \mathrm{C} \to \mathrm{D}$ induces a wide right Morita context from $F(A)$ to $F(B)$ in D. One can follow their proof to easily check that it also works for lax-unital bicategories and that the comparison 2-cells $F(A)F(B) \to F(AB)$ do not have to be invertible for the construction to be valid. This means that the lax functors L and R can be used to improve wide right Morita contexts, just as we did in Proposition 1.

Let us assume that the monomorphic 2-cells in B_L are the ones inherited from B. Then we can restrict $\mathcal{E}$ to B_L so that it satisfies the conditions given in Section 2. The only thing that we need to verify in order for this new construction to have the same power as the one in Proposition 1, is that given a wide right Morita context $\Gamma = (P, Q, \theta\, \phi)$ from A to B in B, with θ and ϕ in $\mathcal{E}$, does the wide right Morita context $\Gamma_L = (P_L, Q_L, \theta_L, \phi_L)$ in B_L, induced by L, have θ_L and ϕ_L in $\mathcal{E}$.

According to [1], a 2-functor F will take a wide right Morita context $\Gamma = (P, Q, \theta\, \phi)$ to the context $(F(P), F(Q), \theta^*, \phi^*)$, where θ^* is the composite

$$F(P)F(Q) \xrightarrow{\Phi} F(PQ) \xrightarrow{F(\theta)} F(I) \xrightarrow{\Phi^{0^{-1}}} I$$

and ϕ^* is the obvious counterpart. If we apply this result to the lax functor L, we see that the 1-cells in Γ_L are given by $P_L = L(P)$, $Q_L = L(Q)$, the 2-cell θ_L is the composite

$$P_L Q_L = L(P)L(Q) \xrightarrow{\Phi} L(PQ) \xrightarrow{L(\theta)} L(I) \xrightarrow{m} I$$

and the 2-cell ϕ_L is a similar composite. To see that θ_L is in $\mathcal{E}$ if θ is in $\mathcal{E}$, observe that the following diagram commutes because of the definitions of

L and Φ:

$$\begin{tikzcd}
(IP)(IQ) \arrow[r, "a \circ (1l)"] \arrow[d, "ee"'] & I(PQ) \arrow[r, "1\theta"] \arrow[d, "e"'] & II \arrow[d, "e"'] \arrow[dr, "l"] & \\
L(P)L(Q) \arrow[r, "\Phi"'] & L(PQ) \arrow[r, "L(\theta)"'] & L(I) \arrow[r, "m"'] & I.
\end{tikzcd}$$

By using naturality and coherence, one can simply verify that the upper composite of the diagram above is just θ' as constructed in Proposition 1. This means that $\theta_L \circ ee = \theta'$ is in $\mathcal{E}$. Therefore, θ_L is in $\mathcal{E}$.

The above can of course also be done for the lax functor R. The construction of Γ_L offers some advantages over the construction of Γ'. First of all, under the assumptions required to construct the lax functor L, it is easy to check that each $\mathrm{B}_L(A, B)$ is a coreflective subcategory of $\mathrm{B}(A, B)$ with $L_{A,B}$ being the right adjoint of the inclusion functor. This allows for an easier transfer of properties between the two categories.

Acknowledgments

The research was partially supported by institutional research funding IUT20-57 of the Estonian Ministry of Education and Research. The author would like to thank the referee and Gabriella Böhm for their helpful comments and suggestions.

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