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Diametral strong diameter two property of Banach spaces is stable under direct sums with 1-norm

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Abstract. We prove that the diametral strong diameter 2 property of a Banach space (meaning that, in convex combinations of relatively weakly open subsets of its unit ball, every point has an "almost diametral" point) is stable under 1-sums, i.e., the direct sum of two spaces with the diametral strong diameter 2 property equipped with the 1-norm has again this property.

All Banach spaces considered in this note are over the real field. The closed unit ball and the unit sphere of a Banach space X will be denoted by B_X and S_X, respectively. Whenever referring to a relative weak topology, we mean such a topology on the closed unit ball of the space under consideration.

Diameter 2 *properties* for a Banach space mean that certain subsets of its unit ball (e.g., slices, nonempty relatively weakly open subsets, or convex combinations of weakly open subsets) have diameter equal to 2. In recent years, these properties have been intensively studied (see, e.g., [1–11] for some typical results and further references).

To clarify the gap between the well-studied Daugavet property [12] and known diameter 2 properties, the *diametral* diameter 2 properties were introduced and studied in the recent preprint [7]. In particular, the stability under p-sums of diametral diameter 2 properties was analyzed. The question whether the 1-sum of two Banach spaces enjoying the diametral *strong* diameter 2 property also has this property, was posed as an open problem in [7]. Below, we shall answer this question in the affirmative.

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Definition (see [7])**.** A Banach space X is said to have the *diametral strong diameter* 2 *property* (DSD2P) if, given $n \in \mathbb{N}$, relatively weakly open subsets $U_1, \dots, U_n$ of B_X, $\lambda_1, \dots, \lambda_n \in [0,1]$ with $\sum_{i=1}^n \lambda_i = 1$, $x \in \sum_{i=1}^n \lambda_i U_i$, and $\varepsilon > 0$, there is a $u \in \sum_{i=1}^n \lambda_i U_i$ satisfying

$$\|x - u\| \geq \|x\| + 1 - \varepsilon.$$

Theorem. *Suppose that Banach spaces X and Y have the DSD2P. Then also the* 1*-sum $X \oplus_1 Y$ has the DSD2P.*

Our proof of Theorem makes use of the following observation:

($\bullet$) in Definition, one may assume that the element x is of the form $x = \sum_{i=1}^n \lambda_i x_i$ where $x_i \in S_X \cap U_i$.

For ($\bullet$), first notice that the space X may be assumed to be infinite dimensional (because clearly no finite dimensional space can have the DSD2P) and the sets $U_1, \dots, U_n$ to be convex (because, since $x = \sum_{i=1}^n \lambda_i u_i$ where $u_i \in U_i$, for every $i \in \{1, \dots, n\}$, it suffices to consider in the role of U_i a convex relatively weakly open neighbourhood V_i of u_i satisfying $V_i \subset U_i$). Now, for ($\bullet$), it suffices to observe that

($\circ$) every $a \in U_i$ can be written in the form $a = (1 - \mu_i) y_i + \mu_i z_i$ where $\mu_i \in [0,1]$ and $y_i, z_i \in S_X \cap U_i$,

because, if ($\circ$) holds, then the element x can be written as

$$x = \sum_{i=1}^n \lambda_i (1 - \mu_i) y_i + \sum_{i=1}^n \lambda_i \mu_i z_i$$

and (by the convexity of $U_1, \dots, U_n$)

$$\sum_{i=1}^n \lambda_i (1 - \mu_i) U_i + \sum_{i=1}^n \lambda_i \mu_i U_i \subset \sum_{i=1}^n \lambda_i U_i.$$

It remains to prove ($\circ$). Let $i \in \{1, \dots, n\}$ and let $a \in U_i$, $\|a\| < 1$. Let $m \in \mathbb{N}$, $x_1^*, \dots, x_m^* \in X^*$, and $\delta > 0$ be such that

$$U_i \supset \bigl\{ b \in B_X \colon |x_j^*(b) - x_j^*(a)| < \delta,\ j = 1, \dots, m \bigr\}.$$

Choose a non-zero $c \in \bigcap_{j=1}^m \ker x_j^*$ (such a c exists when the space X is infinite dimensional), and consider the function $f(t) = \|a + tc\|$, $t \in \mathbb{R}$. Since $f(0) = \|a\| < 1$ and $f(t) \xrightarrow[t \to \pm\infty]{} \infty$, there are $s, t \in (0, \infty)$ such that $f(-s) = f(t) = 1$, but now $y_i := a - sc$, $z_i := a + tc$, and $\mu_i := \dfrac{s}{s+t}$ do the job.

Proof of Theorem. Put $Z := X \oplus_1 Y$, and let $n \in \mathbb{N}$, let $W_1, \dots W_n$ be relatively weakly open subsets of B_Z, let $\lambda_1, \dots, \lambda_n \in [0,1]$ satisfy $\sum_{i=1}^n \lambda_i = 1$, and let $z = \sum_{i=1}^n \lambda_i z_i$ where $z_i = (x_i, y_i) \in S_Z \cap W_i$. We must find a

$w = (u, v) \in \sum_{i=1}^n \lambda_i W_i$ so that $\|z - w\| \geq \|z\| + 1 - \varepsilon$, i.e., putting $x := \sum_{i=1}^n \lambda_i x_i$ and $y := \sum_{i=1}^n \lambda_i y_i$ (now one has $z = (x, y)$),

$$\|x - u\| + \|y - v\| \geq \|x\| + \|y\| + 1 - \varepsilon.$$

For every $i \in \{1, \dots, n\}$, putting

$$\widehat{x}_i = \begin{cases} \dfrac{x_i}{\|x_i\|}, & \text{if } x_i \neq 0, \\ 0, & \text{if } x_i = 0, \end{cases} \quad \text{and} \quad \widehat{y}_i = \begin{cases} \dfrac{y_i}{\|y_i\|}, & \text{if } y_i \neq 0, \\ 0, & \text{if } y_i = 0, \end{cases}$$

there are relatively weakly open neighbourhoods $U_i \subset B_X$ and $V_i \subset B_Y$ of $\widehat{x}_i$ and $\widehat{y}_i$, respectively, such that $\big(\|x_i\|\, U_i\big) \times \big(\|y_i\|\, V_i\big) \subset W_i$. Indeed, letting $m \in \mathbb{N}$, $z_j^* = (x_j^*, y_j^*) \in S_{Z^*}$, $j = 1, \dots, m$, and $\delta > 0$ be such that

$$W_i \supset \big\{ w \in B_Z \colon\ |z_j^*(w) - z_j^*(z_i)| < \delta,\ j = 1, \dots, m \big\},$$

and defining

$$U_i := \big\{ u \in B_X \colon\ |x_j^*(u) - x_j^*(\widehat{x}_i)| < \delta,\ j = 1, \dots, m \big\},$$
$$V_i := \big\{ v \in B_Y \colon\ |y_j^*(v) - y_j^*(\widehat{y}_i)| < \delta,\ j = 1, \dots, m \big\},$$

one has, whenever $u \in U_i$ and $v \in V_i$, for every $j \in \{1, \dots, m\}$,

$$\begin{aligned}
\Big| z_j^*\big(\|x_i\|u, \|y_i\|v\big) - z_j^*(z_i) \Big| &= \Big| z_j^*\big(\|x_i\|u, \|y_i\|v\big) - z_j^*(x_i, y_i) \Big| \\
&= \Big| x_j^*\big(\|x_i\|u\big) + y_j^*\big(\|y_i\|v\big) - x_j^*(x_i) - y_j^*(y_i) \Big| \\
&= \Big| x_j^*\big(\|x_i\|u\big) + y_j^*\big(\|y_i\|v\big) - x_j^*\big(\|x_i\|\widehat{x}_i\big) - y_j^*\big(\|y_i\|\widehat{y}_i\big) \Big| \\
&= \big| \|x_i\| x_j^*(u - \widehat{x}_i) + \|y_i\| y_j^*(v - \widehat{y}_i) \big| \\
&\leq \|x_i\|\, \big|x_j^*(u - \widehat{x}_i)\big| + \|y_i\|\, \big|y_j^*(v - \widehat{y}_i)\big| \\
&< \big(\|x_i\| + \|y_i\|\big)\delta = \|z_i\|\delta \\
&= \delta.
\end{aligned}$$

Put

$$\alpha := \sum_{i=1}^n \lambda_i \|x_i\| \quad \text{and} \quad \beta := \sum_{i=1}^n \lambda_i \|y_i\|.$$

Notice that

$$\alpha + \beta = \sum_{i=1}^n \lambda_i \big(\|x_i\| + \|y_i\|\big) = \sum_{i=1}^n \lambda_i \|z_i\| = \sum_{i=1}^n \lambda_i = 1.$$

We only consider the case when both $\alpha \neq 0$ and $\beta \neq 0$. (The case when $\alpha = 0$ or $\beta = 0$ can be handled similarly and is, in fact, simpler.)

For every $i \in \{1, \dots, n\}$, letting

$$\alpha_i := \frac{\lambda_i \|x_i\|}{\alpha} \quad \text{and} \quad \beta_i := \frac{\lambda_i \|y_i\|}{\beta},$$

one has $\alpha_i, \beta_i \in [0,1]$, and $\sum_{i=1}^n \alpha_i = \sum_{i=1}^n \beta_i = 1$. Since X and Y have the DSD2P, observing that

$$\frac{x}{\alpha} = \sum_{i=1}^n \frac{\lambda_i \|x_i\|}{\alpha} \widehat{x}_i \in \sum_{i=1}^n \alpha_i U_i \quad \text{and} \quad \frac{y}{\beta} = \sum_{i=1}^n \frac{\lambda_i \|y_i\|}{\beta} \widehat{y}_i \in \sum_{i=1}^n \beta_i V_i,$$

there are $u_0 \in \sum_{i=1}^n \alpha_i U_i$ and $v_0 \in \sum_{i=1}^n \beta_i V_i$ such that

$$\left\| \frac{x}{\alpha} - u_0 \right\| \geq \frac{1}{\alpha}\|x\| + 1 - \varepsilon \quad \text{and} \quad \left\| \frac{y}{\beta} - v_0 \right\| \geq \frac{1}{\beta}\|y\| + 1 - \varepsilon.$$

Finally, putting

$$u := \alpha u_0 \in \sum_{i=1}^n \alpha\alpha_i U_i = \sum_{i=1}^n \lambda_i \|x_i\| U_i,$$
$$v := \beta v_0 \in \sum_{i=1}^n \beta\beta_i V_i = \sum_{i=1}^n \lambda_i \|y_i\| V_i,$$

one has

$$(u,v) \in \sum_{i=1}^n \lambda_i \Big(\big(\|x_i\|\, U_i\big) \times \big(\|y_i\|\, V_i\big) \Big) \subset \sum_{i=1}^n \lambda_i W_i$$

and

$$\|x-u\| + \|y-v\| \geq \|x\| + \|y\| + (\alpha+\beta)(1-\varepsilon) = \|x\| + \|y\| + 1 - \varepsilon,$$

as desired. □

Thus the stability of the diametral strong diameter 2 property under 1- and ∞-sums is similar to that of the Daugavet property. In fact, among all 1-unconditional sums of two Daugavet spaces only the 1- and ∞-sum have the Daugavet property. Whether the diametral strong diameter two property and the Daugavet property coincide remains an open question.

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