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On Jordan's and Kober's inequality

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Abstract. We refine some classical inequalities for trigonometric functions, such as Jordan's inequality, Cusa–Huygens's inequality, and Kober's inequality.

1. Introduction

The study of classical inequalities for trigonometric functions such as the inequalities of Adamović–Mitrinović, Cusa–Huygens, Jordan, Redheffer, Becker–Stark, Wilker, Huygens, and Kober has caught the attention of numerous authors. Since the last ten years, a large number of papers on refinement and generalization of these inequalities has appeared (see, e.g., [1, 2, 5, 6, 8, 9, 10, 14, 16, 15, 17] and the references therein). Motivated by these studies, we refine Jordan's, Kober's and Cusa–Huygens's inequalities.

The well-know Jordan's inequality (see [7]) states that

$$\frac{\pi}{2} \leq \frac{\sin x}{x}, \quad 0 < x \leq \frac{\pi}{2}, \tag{1.1}$$

with equality for $x = \pi/2$.

In 2003, Debnath and Zhao [3] refined the inequality (1.1) as follows:

$$\begin{aligned} d_1(x) &:= \frac{2}{\pi} + \frac{1}{12\pi}(\pi^2 - 4x^2) \leq \frac{\sin x}{x}, \\ d_2(x) &:= \frac{2}{\pi} + \frac{1}{\pi^3}(\pi^2 - 4x^2) \leq \frac{\sin x}{x} \end{aligned} \tag{1.2}$$

for $x \in (0, \pi/2)$, with equality in both inequalities for $x = \pi/2$. Thereafter, another proof of the inequality (1.2) was given by Zhu in [20].

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In 2006, Özban [11] proved the inequality

$$o(x) := \frac{2}{\pi} + \frac{1}{\pi^3}(\pi^2 - 4x^2) + \frac{4(\pi-3)}{\pi^3}\left(x - \frac{\pi}{2}\right)^2 \le \frac{\sin x}{x}, \tag{1.3}$$

for $x \in (0, \pi/2)$, with equality for $x = \pi/2$.

In the same year, the following refinement of (1.1) was proved by Jiang and Yun [4]:

$$j(x) = \frac{2}{\pi} + \frac{\pi^4 - 16x^4}{2\pi^5} < \frac{\sin x}{x}$$

for $x \in (0, \pi/2)$, with equality for $x = \pi/2$.

In [19], Zhang et al. gave the following inequality:

$$zw(x) := \frac{3}{\pi} - \frac{4}{\pi^3}x^2 < \frac{\sin x}{x}, \quad 0 < x < \frac{\pi}{2}.$$

It is easy to see that $d_1(x) < d_2(x)$, $d_2(x) = zw(x)$, and $j(x) < d_2(x) < o(x)$ for $x \in (0, \pi/2)$.

Our first main result refines the inequality (1.3).

Theorem 1. *For $x \in (0, \pi/2)$, we have*

$$o(x) \le 1 + \frac{16(\pi-3)}{\pi^4}x^3 - \frac{4(3\pi-8)}{\pi^3}x^2 \le \frac{\sin x}{x},$$

with equality in both inequalities for $x = \pi/2$.

In literature, the inequalities

$$(\cos x)^{1/3} < \frac{\sin x}{x} < \frac{\cos x + 2}{3}, \quad 0 < |x| < \frac{\pi}{2}, \tag{1.4}$$

are known as Adamović–Mitrinović's inequality (see [7, p. 238]) and Cusa–Huygens's inequality (see [15]), respectively. For a refinement of (1.4), see, e.g., [5, 8, 10, 16, 15, 17] and the bibliography of these papers. Most of the refinements of (1.4) involve very complicated upper and lower bounds of $(\sin x)/x$. In the following theorem, we refine (1.4) by giving the upper and lower bound of $(\sin x)/x$ in terms of much simpler functions, and these functions are also independent of the exponent.

Theorem 2. *For $x \in (0, \pi)$, we have*

$$\frac{1 + \cos x}{2 - \alpha x^2} < \frac{\sin x}{x} < \frac{1 + \cos x}{2 - \beta x^2},$$

with the best possible constants $\alpha = 1/6 \approx 0.166667$ and $\beta = 2/\pi^2 \approx 0.202642$.

In 1944, Kober [7, 3.4.9] established the inequalities

$$1 - 2\frac{x}{\pi} < \cos x, \quad x \in \left(0, \frac{\pi}{2}\right), \quad \text{and} \quad \cos x < 1 - \frac{x^2}{\pi}, \quad x \in \left(\frac{\pi}{2}, \pi\right).$$

In literature, these inequalities are known as Kober's inequalities.

By studying the function $x \mapsto (1-\cos x)/x$, $x \in (0,\pi/2)$, Sándor [13] refined Kober's inequalities as follows:

$$\cos x < 1 - \frac{2}{\pi}x - \frac{2(\pi-2)}{\pi^2}\left(x - \frac{\pi}{2}\right), \quad 0 < x < \frac{\pi}{2},$$

$$1 - \frac{x^2}{2} < \cos x < 1 - \frac{4x^2}{\pi^2}, \quad 0 < x < \frac{\pi}{2}.$$

In [19], the following refinement appeared:

$$1 - \frac{4-\pi}{\pi}x - \frac{2(\pi-2)}{\pi^2}x^2 < \cos x < 1 - \frac{4}{\pi^2}x^2, \quad 0 < x < \frac{\pi}{2}.$$

By applying Taylor series expansion, one has

$$1 - \frac{x^2}{2} < \cos x < 1 - \frac{x^2}{2} + \frac{x^4}{24}, \quad 0 < x < \frac{\pi}{2}.$$

Using Mathematica Software® [12], we conclude that our next result refines the above Kober's inequalities.

Theorem 3. *For $x \in (0,\pi/2)$, we have*

$$1 - \frac{x^2/2}{1+x^2/12} < \cos x < 1 - \frac{24x^2/(5\pi^2)}{1+4x^2/(5\pi^2)}.$$

Remark 1. For $x \in (0,\pi/2)$, the inequalities

$$\left(\frac{\pi^2-4x^2}{12}\right)^{3/2} < \cos x < \left(1 - \frac{x^2}{3}\right)^{3/2} \tag{1.5}$$

hold. The proof of (1.5) follows from the monotonicity of the function $f_1(x) = (\cos x)^{2/3} + x^2/3$, which is strictly decreasing from $(0,\pi/2)$ onto $(\pi^2/12, 1)$, because by Adamović–Mitrinović's inequality, we have

$$f_1'(x) = \frac{2x}{3} - \frac{2\sin x}{3(\cos x)^{1/3}} = \frac{2x}{3}\left(1 - \frac{(\sin x)/x}{(\cos x)^{1/3}}\right) < 0.$$

Clearly, $\lim_{x\to 0} f_1(x) = 1$ and $\lim_{x\to\pi/2} f_1(x) = \pi^2/12$. Using Mathematica Software®, we can see that the second inequality in (1.5) refines the corresponding inequality in Theorem 3 for $x \in (0,\ 1.1672)$.

2. Proofs of main results

Proof of Theorem 2. For $x \in (0,\pi)$, let

$$g_1(x) = \frac{(1+\cos x)}{x\sin x} - \frac{2}{x^2}.$$

Differentiating g_1 with respect to x, we get

$$\begin{aligned} g_1'(x) &= \frac{4}{x^3} - \frac{1}{x} - \frac{1+\cos x}{x^2 \sin x} - \frac{(1+\cos x)\cos x}{x \sin x} \\ &= \frac{4(1-\cos x)/x^2 - 1 - (\sin x)/x}{x(1-\cos x)}. \end{aligned}$$

In order to prove that $g_1'(x) < 0$, we must show that

$$4(1-\cos x)/x^2 < 1 + (\sin x)/x$$

or, equivalently,

$$a(x) = x^2 + x \sin x + 4\cos x - 4 > 0$$

for $x \in (0,\pi)$. This is true, as one has

$$a'(x) = (2+x)\cos x - 3\sin x > 0$$

by Cusa–Huygens's inequality $(\sin x)/x < (2+\cos x)/3$, valid for all $x \in (0,\pi)$ (in fact it holds for all $x \neq 0$, see [7, Problems 5.11 and 5.15], [18, Lemma 2.4]). Thus $a(x) > a(0) = 0$, and it follows that g_1 is strictly decreasing in $x \in (0,\pi)$. By applying l'Hôpital's rule, we get the limiting values. This completes the proof. □

Lemma 1. *The function*

$$f_2(x) = \frac{x - \sin x}{x^3}$$

is strictly decreasing and concave from $(0,\pi)$ *onto* $(1/\pi^2, 1/6)$*. In particular, for* $x \in (0,\pi)$*,*

$$1 - \frac{x^2}{6} < \frac{\sin x}{x} < 1 - \frac{x^2}{\pi^2}.$$

Proof. One has

$$f_2'(x) = \frac{1-\cos x}{x^3} - 3\frac{x-\sin x}{x^4} = \frac{3(\sin x)/x - 2 - \cos x}{x^4},$$

which is negative by Cusa–Huygens's inequality, hence f_2 is strictly decreasing in $x \in (0,\pi)$. Further,

$$\begin{aligned} f_2''(x) &= \frac{2\cos x + x\sin x - 2}{x^5} - \frac{4}{x^5}(3\sin x - x(2+\cos x)) \\ &= \frac{6x(1+\cos x) - (12-x^2)\sin x}{x^5}, \end{aligned}$$

which is negative by Theorem 2. This implies the concavity of the function f_2. □

Proof of Theorem 1. Since by Lemma 1 the function $f_2(x)$ is concave in $(0,\pi/2)$, the tangent line at the point $(\pi/2, f_2(\pi/2))$ is above the graph of $f_2(x)$ on $(0,\pi/2)$. The equation of the tangent line is

$$y = \frac{4(\pi-2)}{\pi^3} + \frac{16(3-\pi)}{\pi^4}(x-\pi/2).$$

After some computations, we get the desired inequality. The first inequality is equivalent to

$$-\frac{4(\pi-3)(\pi-2x)^2 x}{\pi^4} < 0,$$

which is obvious. This completes the proof. □

Proof of Theorem 3. For $x \in (0,\pi/2)$, let

$$f(x) = \frac{x^2(5+\cos x)}{1-\cos x}.$$

We have

$$f'(x) = \frac{2x(5-g(x))}{(\cos x - 1)^2},$$

where

$$g(x) = \cos x(4+\cos x) + 3x\sin x.$$

Further,

$$\begin{aligned} g'(x) &= 3x\cos x - (1+2\cos x)\sin x \\ &= x\cos x\left(3 - \left(2\frac{\sin x}{x} + \frac{\tan x}{x}\right)\right), \end{aligned}$$

which is negative by Huygens's inequality (see [10])

$$2\frac{\sin x}{x} + \frac{\tan x}{x} > 3, \quad 0 < x < \frac{\pi}{2}.$$

Thus, g is decreasing and $\lim_{x\to 0} g(x) = 5$. It follows that $f' > 0$. This implies that f is strictly increasing. Applying l'Hôpital's rule, we get

$$12 = \lim_{x\to 0} f(x) < f(x) < \lim_{x\to 0} f(x) = \frac{5\pi^2}{4} \approx 12.33701,$$

which is equivalent to

$$\frac{6}{1+x^2/12} - 5 < \cos x < \frac{6}{1+4x^2/(5\pi^2)} - 5.$$

This implies the desired inequalities. □

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