New iteration process for a general class of contractive mappings

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In honor of my mentor Professor M. A. Kenku on his 74th birthday

ABSTRACT. Let K be a closed convex subset of X, and let $T : K \to K$ be a self-mapping with the set F_T of fixed points such that $\|Tx - \rho\| \leq \delta\|x - \rho\|$ for all $x \in K$, $\rho \in F_T$ and some $\delta \in (0, 1)$. We introduce a new iteration process called Picard-hybrid iteration and show that this iteration process converges to the unique fixed point of T. It is also shown that our iteration process converges more rapidly than the Picard–Mann and Picard iteration processes. Our result improves a recent result of S. H. Khan and some other results.

1. Introduction

The importance of fixed point theory consists mainly in the fact that a wide class of problems which arise in several branches of mathematical analysis can be studied in the general framework of the nonlinear equations

$$f(x) = 0, \tag{1.1}$$

where f is a linear or non-linear operator. We can equivalently transform them to a fixed point problem

$$x = Tx,$$

where T is a self-mapping of a space X and $x \in X$. The later is solved by some iterative processes of the form

$$x_{n+1} = f(T, x_n), \quad n \geq 1,$$

that converges to a fixed point of T.

As a background to our study, we describe some iteration processes and contractive-type maps.

Let K be a closed convex subset of a Banach space X, and $T : K \to K$ be a self-mapping. Throughout this paper $\mathbb{N}$ denotes the set of non-negative integers. A point $\rho \in K$ satisfying $T\rho = \rho$ is called fixed point of T, and the set of all fixed points of T is denoted by F_T.

One-step iteration method (due to Mann [10]):

$$x_{n+1} = (1 - b_n)x_n + b_n Tx_n, \quad n \in \mathbb{N}, \tag{1.2}$$

where $\{b_n\}_{n=1}^{\infty}$ is a sequence in $[0,1]$. Observe that when $b_n = 1$ $(n \in \mathbb{N})$ in equation (1.2), we have the following iterative process.

Picard iteration process (see [3]):

$$x_{n+1} = Tx_n, \quad n \in \mathbb{N}. \tag{1.3}$$

The iteration process (1.3) is commonly used to approximate fixed point of contraction mappings satisfying

$$\|Tx - \rho\| \le \delta \|x - \rho\|, \quad x \in K, \ \rho \in F_T, \tag{1.4}$$

for some $\delta \in (0,1)$. We observe that the class of mappings satisfying contractive condition (1.4) is fairly large and interesting. In fact, Chidume and Olaleru [4] gave an example to show that the class of contraction mappings with fixed points is a proper subclass of the class of mappings satisfying inequality (1.4).

In recent years, much attention has been given to the study of convergence of various iterative methods for different classes of operators satisfying certain contractive definition (see, for example, [1–9, 11–16]). In several cases, there can be more than one iteration method to approximate fixed points of a particular mapping, e.g., [3, 6, 8]. In such cases, it is reasonable for computation purposes to choose among some given iteration processes by taking into consideration some important criterias such as the efficiency and applicability of such iteration process in the real life sense. In practice, the iteration process with a better speed of convergence to the fixed point saves time. The following definitions about the speed of convergence are given in [3].

Definition 1.1. Let $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ be two sequences of real numbers that converge to α and β, respectively, assume that there exists

$$l = \lim_{n \to \infty} \frac{|\alpha_n - \alpha|}{|\beta_n - \beta|}.$$

(a) If $l = 0$, then we say that $\{\alpha_n\}_{n=1}^{\infty}$ converges faster to α than $\{\beta_n\}_{n=1}^{\infty}$ to β.

(b) If $0 < l < \infty$, then we say that $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ have the same rate of convergence.

It is clear that, when two iteration processes are compared, their convergence rate depends on the choice of their error bounds. Inspired by this, we therefore discuss the following definition.

Definition 1.2. Suppose that $\{u_n\}_{n=1}^{\infty}$ and $\{v_n\}_{n=1}^{\infty}$ are two fixed point iteration processes both converging to the same fixed point ρ of a mapping T with the error estimates

$$
\begin{aligned}
|u_n - \rho| &\leq \alpha_n, \quad n \in \mathbb{N}, \\
|v_n - \rho| &\leq \beta_n, \quad n \in \mathbb{N}.
\end{aligned}
\tag{1.6}.
$$

We say that $\{u_n\}_{n=1}^{\infty}$ converges more rapidly than $\{v_n\}_{n=1}^{\infty}$ and $\{v_n\}_{n=1}^{\infty}$ more slowly than $\{u_n\}_{n=1}^{\infty}$ if

$$
\lim_{n \to \infty} \frac{\alpha_n}{\beta_n} = 0.
$$

Khan [9] introduced the following Picard–Mann hybrid iteration process for single map T.

Picard–Mann hybrid iteration process:

$$
y_n = (1 - b_n)x_n + b_n Tx_n, \; x_{n+1} = Ty_n, \quad n \in \mathbb{N}, \tag{1.7}
$$

where $\{b_n\}_{n=1}^{\infty}$ is a sequence in $(0, 1)$. He showed that iteration process (1.7) converges faster (more rapidly) than the Picard iteration process (1.3) in the sense of Berinde [1] for a different class of contraction maps with fixed point. Khan [9, lines 9–11 of page 2] claimed that "even if $b_n = 1$", then iteration process (1.7) will converge faster (more rapidly) than any case of this type.

A natural question that arises is: *Can this claim be true?* As an answer, we introduce the following new iteration process obtainable when $b_n = 1$ in (1.7).

New iteration (Picard hybrid) method:

$$
y_n = Tx_n, \; x_{n+1} = Ty_n, \quad n \in \mathbb{N}. \tag{1.8}
$$

Our new iteration (1.8) can be seen as a "special case of Picard–Mann hybrid iteration method" or a hybrid of Picard iteration method with itself.

In this paper, we first show that our hybrid iteration process (1.8) can be used to approximate fixed point of a general class of contractive mappings (1.4). Also, we show that our newly introduce iteration process (1.8) converges more rapidly than the Picard–Mann and Picard iteration processes when applied to the class of mappings satisfying (1.4).

2. Main results

We begin with the convergence analysis of our new iteration process.

Proposition 2.1. *Let K be a closed convex subset of a Banach space X, and let $T : K \to K$ be a mapping with $F_T \neq \emptyset$ such that (1.4) holds. Then, the Picard hybrid iteration process (1.8) converges to ρ.*

Proof. The well-known Picard–Banach theorem guarantees the existence and uniqueness of a fixed point ρ. From (1.4) and (1.8) the inequality

$$\|x_{n+1} - \rho\| \leq \delta^{2n}\|x_1 - \rho\|, \quad n \in \mathbb{N}, \tag{2.1}$$

follows. The proof of (2.1) is given by induction method. When $n = 1$, we have

$$\|x_2 - \rho\| = \|Ty_1 - \rho\| \leq \delta\|y_1 - \rho\| = \delta\|Tx_1 - \rho\| \leq \delta^2\|x_1 - \rho\|$$

and so, (2.1) holds for $n = 1$. If we assume that it holds for $n = k$, then

$$\begin{aligned}
\|x_{(k+1)+1} - \rho\| = \|Ty_{k+1} - \rho\| &\leq \delta\|y_{k+1} - \rho\| \\
&= \delta\|Tx_{k+1} - \rho\| \leq \delta^2\|x_{k+1} - \rho\| \\
&\leq \delta^{2(k+1)}\|x_1 - \rho\|
\end{aligned}$$

and so, inequality (2.1) holds for $k + 1$. Thus, by induction, inequality (2.1) holds for all $n \in \mathbb{N}$. Since $\delta \in (0,1)$, it follows that $\{x_n\}_{n=1}^{\infty}$ converges to ρ. $\qquad\square$

Now, we prove that our new iteration process converges more rapidly than the Picard and Picard–Mann iteration processes.

Theorem 2.1. *Let K be a closed convex subset of a Banach space X, and let $T : K \to K$ be a mapping with $F_T \neq \emptyset$ such that (1.4) holds. Assume that $\{u_n\}_{n=1}^{\infty}$ is defined by the Picard iteration process (1.3), $\{x_n\}_{n=1}^{\infty}$ is defined by the iteration process (1.8), and $\{v_n\}_{n=1}^{\infty}$ is defined by the Picard–Mann hybrid iteration process (1.7), where $u_1 = x_1 = v_1$, and $\{b_n\}_{n=1}^{\infty}$ is in $[\lambda, 1 - \lambda]$ for all $n \in \mathbb{N}$ and for some λ in $(0,1)$. Then the sequence $\{x_n\}_{n=1}^{\infty}$ converges more rapidly than $\{v_n\}_{n=1}^{\infty}$ to ρ. That is, our newly introduced iteration process (1.8) converges more rapidly than the Picard–Mann and Picard iteration processes.*

Proof. As proved in Proposition 1 of Khan [9],

$$\|v_{n+1} - \rho\| \leq [\delta(1 - (1 - \delta)\lambda)]^n \|v_1 - \rho\| =: \alpha_n$$

for all $n \geq 1$. Further, by Proposition 2.1, inequality (2.1) is satisfied. Let

$$\beta_n := \delta^{2n}\|x_1 - \rho\|.$$

Then

$$\theta_n := \frac{\beta_n}{\alpha_n} = \frac{\delta^{2n}\|x_1 - \rho\|}{[\delta(1 - (1 - \delta)\lambda)]^n \|v_1 - \rho\|} = \left(\frac{\delta}{1 - (1 - \delta)\lambda}\right)^n.$$

Since

$$\frac{\delta}{1 - (1 - \delta)\lambda} < 1,$$

we have that $\theta_n \to 0$ as $n \to \infty$, i.e., $\{x_n\}_{n=1}^{\infty}$ converges more rapidly than $\{v_n\}_{n=1}^{\infty}$ to ρ and $\{v_n\}_{n=1}^{\infty}$ more slowly than $\{x_n\}_{n=1}^{\infty}$.

In a similar way, from Proposition 1 of Khan [9],

$$\|u_{n+1} - \rho\| \leq \delta^n \|u_1 - \rho\| =: \gamma_n.$$

for all $n \in \mathbb{N}$. Then

$$\theta_n := \frac{\beta_n}{\gamma_n} = \frac{\delta^{2n}\|x_1 - \rho\|}{\delta^n\|u_1 - \rho\|} = \delta^n.$$

Since $\delta \in (0, 1)$, $\theta_n \to 0$ as $n \to \infty$, i.e., $\{x_n\}_{n=1}^{\infty}$ converges more rapidly than $\{u_n\}_{n=1}^{\infty}$ to ρ, and $\{u_n\}_{n=1}^{\infty}$ more slowly than $\{x_n\}_{n=1}^{\infty}$. $\square$

3. Applications

Fixed point theory has been an interesting and fascinating field that takes care of some of challenging nonlinear problems in mathematical analysis (see Berinde's book [3]). Several authors have employed hybrid iteration processes to approximate fixed point of contraction mappings (see [8, 9]) and also applied these iterations to solve problems involving differential equations with retarded argument, constrained minimization problems and split feasibility problems (see [3, 5, 8, 15]). Our newly introduced iteration is obviously well enriched and applicable for these type of problems.

Acknowledgement

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