On translation surfaces in 4-dimensional Euclidean space

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Abstract. We consider translation surfaces in Euclidean spaces. Firstly, we give some results of translation surfaces in the 3-dimensional Euclidean space $\mathbb{E}^3$. Further, we consider translation surfaces in the 4-dimensional Euclidean space $\mathbb{E}^4$. We prove that a translation surface is flat in $\mathbb{E}^4$ if and only if it is either a hyperplane or a hypercylinder. Finally we give necessary and sufficient condition for a quadratic triangular Bézier surface in $\mathbb{E}^4$ to become a translation surface.

1. Introduction

Surfaces of constant mean curvature, H-surfaces and those of constant Gaussian curvature, K-surfaces in the 3-dimensional Euclidean space $\mathbb{E}^3$ have been studied extensively. An interesting class of surfaces in $\mathbb{E}^3$ is that of translation surfaces, which can be parameterized locally as $X(u,v) = (u, v, f(u) + g(v))$, where f and g are smooth functions.

From the definition, it is clear that translation surfaces are double curved surfaces. Therefore, translation surfaces are made up of quadrilateral, that is, four sided, facets. Because of this property, translation surfaces are used in architecture to design and construct free-form glass roofing structures (see [4]). Generally, these glass roofings are made up of triangular glass facets or curved glass panes. But, since quadrangular glass elements lead to economic advantages and more transparency compared to a triangular grid, translation surface are used as a basis for roofings.

Scherk's surface, obtained by H. Scherk [8], is the only non flat minimal surface, that can be represented as a translation surface.

Translation surfaces have been investigated from various viewpoints by many differential geometers. L. Verstraelen et al. [10] have investigated minimal translation surfaces in n-dimensional Euclidean spaces. H. Liu [7] has given a classification of translation surfaces with constant mean curvature or constant Gaussian curvature in the 3-dimensional Euclidean space $\mathbb{E}^3$ and the 3-dimensional Minkowski space $\mathbb{E}^3_1$. In [5], W. Goemans proved classification theorems of Weingarten translation surfaces. D. W. Yoon [11] has studied translation surfaces in the 3-dimensional Minkowski space whose Gauss map G satisfies the condition $\Delta G = AG$, $A \in Mat(3, \mathbb{R})$, where Δ denotes the Laplacian of the surface with respect to the induced metric and $Mat(3, \mathbb{R})$ is the set of 3×3 real matrices. M. I. Munteanu and A. I. Nistor [6] have studied the second fundamental form of translation surfaces in $\mathbb{E}^3$. They have given a non-existence result for polynomial translation surfaces in $\mathbb{E}^3$ with vanishing second Gaussian curvature K_{II}. They have also classified those translation surfaces for which K_{II} and H are proportional.

In this paper, we consider translation surfaces in the 4-dimensional Euclidean space $\mathbb{E}^4$. We prove that a translation surface is flat in $\mathbb{E}^4$ if and only if it is either a hyperplane or a hypercylinder. Finally, we give a necessary and sufficient condition for a quadratic triangular Bézier surface in $\mathbb{E}^4$ to become a translation surface.

2. Basic concepts

Let M be a smooth surface in $\mathbb{E}^n$ given by a patch $X(u, v)$, $(u, v) \in D \subset \mathbb{E}^2$. The tangent space to M at a point $p = X(u, v)$ of M is span $\{X_u, X_v\}$. In the chart (u, v), the coefficients of the first fundamental form of M are given by

$$E = \langle X_u, X_u \rangle, \;\; F = \langle X_u, X_v \rangle, \;\; G = \langle X_v, X_v \rangle,$$

where $\langle, \rangle$ is the Euclidean inner product. We assume that $W^2 = EG - F^2 \neq 0$, i.e., the surface patch $X(u, v)$ is regular. For each $p \in M$, consider the decomposition $T_p\mathbb{E}^n = T_pM \oplus T_p^{\perp}M$ where $T_p^{\perp}M$ is the orthogonal component of T_pM in $\mathbb{E}^n$.

Let $\chi(M)$ and $\chi^{\perp}(M)$ be the space of smooth vector fields tangent to M and the space of smooth vector fields normal to M, respectively. Given any local vector fields X_1, X_2 tangent to M, consider the second fundamental map $h : \chi(M) \times \chi(M) \rightarrow \chi^{\perp}(M)$,

$$h(X_i, X_j) = \widetilde{\nabla}_{X_i} X_j - \nabla_{X_i} X_j \quad 1 \leq i, j \leq 2, \tag{2.1}$$

where ∇ and $\widetilde{\nabla}$ are the induced connection of M and the Riemannian connection of $\mathbb{E}^n$, respectively. This map is well-defined, symmetric and bilinear.

For any orthonormal frame field $\{N_1, N_2, ..., N_{n-2}\}$ of M, the *shape operator* $A : \chi^\perp(M) \times \chi(M) \to \chi(M)$ is defined by

$$A_{N_k} X_j = -(\widetilde{\nabla}_{X_j} N_k)^T, \quad X_j \in \chi(M).$$

This operator is bilinear, self-adjoint and satisfies the condition

$$\langle A_{N_k} X_j, X_i \rangle = \langle h(X_i, X_j), N_k \rangle = c_{ij}^k, \quad 1 \le i, j \le 2,\, 1 \le k \le n - 2, \quad (2.2)$$

where c_{ij}^k are the coefficients of the second fundamental form.

The equation (2.1) is called the Gauss formula. One has

$$h(X_i, X_j) = \sum_{k=1}^{n-2} c_{ij}^k N_k, \quad 1 \le i, j \le 2.$$

Then the Gaussian curvature K of a regular patch $X(u, v)$ is given by

$$K = \frac{1}{W^2} \sum_{k=1}^{n-2} (c_{11}^k c_{22}^k - (c_{12}^k)^2). \quad (2.3)$$

Further, the mean curvature vector of a regular patch $X(u, v)$ is given by

$$\overrightarrow{H} = \frac{1}{2W^2} \sum_{k=1}^{n-2} (c_{11}^k G + c_{22}^k E - 2c_{12}^k F) N_k. \quad (2.4)$$

The norm of the mean curvature vector $\left\| \overrightarrow{H} \right\|$ is called the *mean curvature* of M. The mean curvature H and the Gaussian curvature K play the most important roles in differential geometry for surfaces (see [1]).

Recall that a surface M is said to be *flat* (respectively *minimal*) if its Gaussian curvature (respectively mean curvature) vanishes identically (see [2]).

The kth mean curvature of M is defined by

$$H_k = \frac{1}{2W^2} (c_{11}^k G + c_{22}^k E - 2c_{12}^k F), \quad 1 \le k \le n - 2.$$

The surface M is said to be H_k-*minimal* if the kth mean curvature H_k vanishes identically.

We denote by R the *curvature tensor* associated with ∇,

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

The equations of Gauss and Ricci are given, respectively, by

$$\langle R(X, Y)Z, W \rangle = \langle h(X, W), h(Y, Z) \rangle - \langle h(X, Z)h(Y, W) \rangle,$$

$$\left\langle R^\perp(X, Y)\xi, \eta \right\rangle = \langle [A_\xi, A\eta]X, Y \rangle,$$

for vectors X, Y, Z, W tangent to M and ξ, η normal to M (see [2]).

3. Translation surfaces in $\mathbb{E}^3$

Let $\alpha, \beta : \mathbb{R} \longrightarrow \mathbb{E}^3$ be two Euclidean space curves. Put $\alpha(u) = (f_1(u), f_2(u), f_3(u))$ and $\beta(v) = (g_1(v), g_2(v), g_3(v))$. Then the sum of α and β can be considered as a surface patch $X : \mathbb{E}^2 \longrightarrow \mathbb{E}^3$,

$$X(u,v) = \alpha(u) + \beta(v), \quad u_0 < u < u_1, \ v_0 < v < v_1,$$

which is a surface in $\mathbb{E}^3$, where the tangent vectors α' and β' must be linearly independent for any u and v.

A basis for the tangent space is given by

$$X_u = (f_1{}'(u), f_2{}'(u), f_3{}'(u)),$$
$$X_v = (g_1{}'(v), g_2{}'(v), g_3{}'(v)).$$

The unit normal vector field N can be given by (see [3])

$$N = \frac{1}{\sqrt{1 - \langle \alpha', \beta' \rangle}} \left(f_2{}'g_3{}' - f_3{}'g_2{}', f_3{}'g_1{}' - f_1{}'g_3{}', f_1{}'g_2{}' - f_2{}'g_1{}' \right).$$

Definition 3.1. A surface M defined as the sum of two plane curves $\alpha(u) = (u, 0, f(u))$ and $\beta(v) = (0, v, g(v))$ is called a *translation surface* in $\mathbb{E}^3$. So, a translation surface is defined by means of the Monge patch

$$X(u,v) = (u, v, f(u) + g(v)).$$

Example 3.1. Consider the translation surfaces in $\mathbb{E}^3$ given by

a) $f(u) = \cosh(\frac{u}{3})$, $g(v) = \sin(\frac{v}{3})$ (see Figure 1(A)),

b) $f(u) = \sin(3u)$, $g(v) = \cos(3v)$ (see Figure 1(B)).

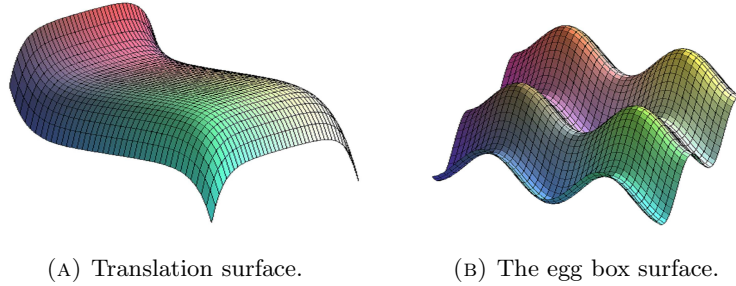(A) Translation surface.   (B) The egg box surface.

FIGURE 1. Translation surfaces in $\mathbb{E}^3$.

The following results are well known.

Proposition 3.1 (see [7])**.** *Let M be a translation surface in $\mathbb{E}^3$. Then the Gaussian and the mean curvature of M can be given by*

$$K = \frac{f''g''}{(1 + (f')^2 + (g')^2)^2}$$

and

$$H = \frac{f''(1 + (g')^2) + g''(1 + (f')^2)}{2(1 + (f')^2 + (g')^2)^{\frac{3}{2}}}.$$

From the previous proposition, one can get the following results.

Theorem 3.1. *Let M be a translation surface in $\mathbb{E}^3$. Then M has vanishing Gaussian curvature if and only if either M is a plane or a part of a cylinder with the axis parallel to $(1, 0, a)$ or $(0, 1, c)$, where a, c are real constants.*

Theorem 3.2 (see [8])**.** *Let M be a translation surface in $\mathbb{E}^3$. If M has constant Gaussian curvature, then M is congruent to a cylinder. So, $K = 0$.*

Corollary 3.1. *Let M be a translation surface in $\mathbb{E}^3$. Then M is a minimal surface if and only if*

$$\frac{f''}{1 + (f')^2} = -\frac{g''}{1 + (g')^2} = a,$$

where a is a non-zero constant.

Theorem 3.3 (see [8])**.** *Let M be a translation surface in $\mathbb{E}^3$. Then M is minimal if and only if M is a surface of Scherk given by the parametrization*

$$f(u) = \frac{1}{a} \log |\cos(au)|,$$
$$g(v) = -\frac{1}{a} \log |\cos(av)|,$$

where a is a non-zero constant.

Theorem 3.4 (see [7])**.** *Let M be a translation surface with constant mean curvature $H \neq 0$ in the 3-dimensional Euclidean space $\mathbb{E}^3$. Then M is congruent to a surface given by the parametrization*

$$f(u) = \frac{-\sqrt{1 - a^2}}{2H} \sqrt{1 - 4H^2 u^2},$$
$$g(v) = -av,$$

where $a < 1$ is a non-zero positive constant.

4. Translation surfaces in $\mathbb{E}^4$

Let $\alpha, \beta : \mathbb{R} \longrightarrow \mathbb{E}^4$ be two curves in $\mathbb{E}^4$. Put $\alpha(u) = (f_1(u), f_2(u), f_3(u), f_4(u))$ and $\beta(v) = (g_1(v), g_2(v), g_3(v), g_4(v))$. Then the sum of α and β can be considered as a surface patch $X : \mathbb{E}^2 \longrightarrow \mathbb{E}^4$,

$$X(u,v) = \alpha(u) + \beta(v), \quad u_0 < u < u_1,\ v_0 < v < v_1,$$

which is a surface in $\mathbb{E}^4$.

Definition 4.1. A surface M defined as the sum of two space curves $\alpha(u) = (u, 0, f_3(u), f_4(u))$ and $\beta(v) = (0, v, g_3(v), g_4(v))$ is called a *translation surface* in $\mathbb{E}^4$. So, a translation surface is defined by a patch

$$X(u,v) = (u, v, f_3(u) + g_3(v), f_4(u) + g_4(v)). \tag{4.1}$$

The tangent space of M is spanned by the vector fields

$$X_u = (1, 0, f_3{}'(u), f_4{}'(u)),$$
$$X_v = (0, 1, g_3{}'(v), g_4{}'(v)).$$

Hence the coefficients of the first fundamental form of the surface are

$$E = \langle X_u, X_u \rangle = 1 + (f_3{}')^2 + (f_4{}')^2,$$
$$F = \langle X_u, X_v \rangle = f_3{}'g_3{}' + f_4{}'g_4{}',$$
$$G = \langle X_v, X_v \rangle = 1 + (g_3{}')^2 + (g_4{}')^2,$$

where $\langle\ ,\ \rangle$ is the standard scalar product in $\mathbb{E}^4$. Since the surface M is non-degenerate, $\|X_u \times X_v\| = \sqrt{EG - F^2} \neq 0$. For the later use we define a smooth function W as $W = \|X_u \times X_v\|$.

The second partial derivatives of $X(u,v)$ are given by

$$\begin{aligned}
X_{uu} &= (0, 0, f_3{}''(u), f_4{}''(u)), \\
X_{uv} &= (0, 0, 0, 0), \\
X_{vv} &= (0, 0, g_3{}''(v), g_4{}''(v)).
\end{aligned} \tag{4.2}$$

Further, the *normal space* of M is spanned by the orthonormal vector fields

$$\begin{aligned}
N_1 &= \frac{1}{\sqrt{\widetilde{E}}}(-f_3{}'(u), -g_3{}'(v), 1, 0), \\
N_2 &= \frac{1}{\sqrt{\widetilde{E}\widetilde{W}}}(\widetilde{F}f_3{}'(u) - \widetilde{E}f_4{}'(u), \widetilde{F}g_3{}'(v) - \widetilde{E}g_4{}'(v), -\widetilde{F}, \widetilde{E}),
\end{aligned} \tag{4.3}$$

where

$$\widetilde{E} = 1 + (f_3{}')^2 + (g_3{}')^2,$$
$$\widetilde{F} = f_3{}'f_4{}' + g_3{}'g_4{}',$$
$$\widetilde{G} = 1 + (f_4{}')^2 + (g_4{}')^2$$
$$\widetilde{W} = \sqrt{\widetilde{E}\widetilde{G} - \widetilde{F}^2}.$$

Using (4.2) and (4.3), we can calculate the coefficients of the second fundamental form as follows:

$$c_{11}^1 = \frac{f_3{}''}{\sqrt{\widetilde{E}}}, c_{22}^1 = \frac{g_3{}''}{\sqrt{\widetilde{E}}},$$
$$c_{12}^1 = c_{12}^2 = 0,$$
$$c_{11}^2 = \frac{\widetilde{E}f_4{}'' - \widetilde{F}f_3{}''}{\sqrt{\widetilde{E}}\widetilde{W}}, \tag{4.4}$$
$$c_{22}^2 = \frac{\widetilde{E}g_4{}'' - \widetilde{F}g_3{}''}{\sqrt{\widetilde{E}}\widetilde{W}}.$$

Using (4.4) and (2.2), the second fundamental tensors A_{N_α} become

$$A_{N_1} = \frac{1}{W^2}\begin{bmatrix} \frac{f_3{}''}{\sqrt{\widetilde{E}}} & 0 \\ 0 & \frac{g_3{}''}{\sqrt{\widetilde{E}}} \end{bmatrix}, \quad A_{N_2} = \frac{1}{W^2}\begin{bmatrix} \frac{\widetilde{E}f_4{}''-\widetilde{F}f_3{}''}{\sqrt{\widetilde{E}}\widetilde{W}} & 0 \\ 0 & \frac{\widetilde{E}g_4{}''-\widetilde{F}g_3{}''}{\sqrt{\widetilde{E}}\widetilde{W}} \end{bmatrix}.$$

By (4.4) together with (2.3) and (2.4), we get the following result.

Proposition 4.1. *Let M be a translation surface in $\mathbb{E}^4$. Then the Gaussian curvature and mean curvature vector field of M can be given by*

$$K = \frac{f_3{}''g_3{}''\widetilde{G} - (f_3{}''g_4{}'' + g_3{}''f_4{}'')\widetilde{F} + f_4{}''g_4{}''\widetilde{E}}{\widetilde{W}^2 W^2}$$

and

$$\overrightarrow{H} = \frac{f_3{}''G + g_3{}''E}{2\sqrt{\widetilde{E}}W^2}N_1 + \frac{G(f_4{}''\widetilde{E} - f_3{}''\widetilde{F}) + E(g_4{}''\widetilde{E} - g_3{}''\widetilde{F})}{2\sqrt{\widetilde{E}}\widetilde{W}W^2}N_2.$$

From this proposition, one can get the following results.

Theorem 4.1. *Let M be a translation surface in $\mathbb{E}^4$. Then M has vanishing Gaussian curvature if and only if either M is a plane or a part of a hyper-cylinder of the form*

$$X(u,v) = (0, v, b_3 + g_3(v), b_4 + g_4(v)) + u(1, 0, a_3, a_4)$$

or

$$X(u,v) = (u, 0, d_3 + f_3(u), d_4 + f_4(u)) + v(0, 1, c_3, c_4),$$

where a_i, b_i, c_i, d_i $(i = 3, 4)$ are real constants, and b_3, b_4, d_3, and d_4 can be taken to be 0.

Theorem 4.2 (see [3]). *Let M be a translation surface in $\mathbb{E}^4$. Then M is minimal if and only if either M is a plane or*

$$f_k(u) = \frac{c_k}{c_3^2 + c_4^2} \left(\log \left| \cos(\sqrt{a}u) \right| + cu \right) + e_k u,$$

$$g_k(v) = \frac{c_k}{c_3^2 + c_4^2} \left(-\log \left| \cos(\sqrt{b}v) \right| + dv \right) + p_k v, \quad k = 3, 4$$

where c_k, e_k, p_k, a, b, c, d are real constants with $a > 0$ and $b > 0$.

For the general case of the previous theorem see [10].

Proposition 4.2. *Let M be a translation surface in $\mathbb{E}^4$ given by the surface patch (4.1). If the functions $f_3(u)$ and $g_3(v)$ are linear polynomials, then M is H_1-minimal.*

Proof. The first mean curvature of the translation surface M is

$$H_1 = \frac{f_3'' G + g_3'' E}{2\sqrt{\widetilde{E}} W^2}.$$

Suppose that $f_3(u)$ and $g_3(v)$ are linear polynomials of the form

$$f_3(u) = a_1 u + a_2, \ g_3(v) = b_1 v + b_2.$$

Then the first mean curvature of the translation surface M vanishes identically. $\square$

5. Bézier translation surfaces in $\mathbb{E}^4$

Quadratic triangular Bézier surfaces in $\mathbb{E}^4$ can be parametrized with the help of barycentric coordinates u, v, and $t = 1 - u - v$ as follows:

$$s(u, v, t) = \sum_{i+j+k=2} B_{ijk}^2(u, v, t) b_{ijk},$$

where

$$B_{ijk}^2 = \frac{2!}{i!j!k!} u^i v^j t^k$$

are basis functions and b_{ijk} are control points (see [9]).

A quadratic triangular Bézier surface $M \subset \mathbb{E}^4$ can be parametrized with the help of affine parameters u, v as follows:

$$X(u, v) = \frac{1}{2} x u^2 + y u v + \frac{1}{2} z v^2 + w u + c v + d, \qquad (5.1)$$

where x, y, z, w, c, d are constant vectors in $\mathbb{E}^4$.

Furthermore, a quadratic triangular Bézier surface can be considered as the sum of two curves

$$\alpha(u) = \sum_{i=1}^{4} \frac{1}{2} x_i u^2 + w_i u + a_i,$$

$$\beta(v) = \sum_{i=1}^{4} \frac{1}{2} z_i v^2 + c_i v + b_i.$$

Corollary 5.1. *Let M be a quadratic triangular Bézier surface in $\mathbb{E}^4$ given by (5.1). If*

$$
\begin{aligned}
w_1 &= 1, \ x_1 = z_1 = c_1 = d_1 = 0, \\
c_2 &= 1, \ x_2 = z_2 = w_2 = d_2 = 0, \\
y &= 0,
\end{aligned}
\tag{5.2}
$$

then M is a translation surface.

Proof. If the equalities (5.2) hold, then

$$X(u,v) = (u, v, r(u,v), s(u,v)),$$

where

$$r(u,v) = \frac{1}{2} x_3 u^2 + \frac{1}{2} z_3 v^2 + w_3 u + c_3 v + d_3,$$

$$s(u,v) = \frac{1}{2} x_4 u^2 + \frac{1}{2} z_4 v^2 + w_4 u + c_4 v + d_4.$$

So the Bézier surface becomes a translation surface of the form

$$
\begin{aligned}
f_3(u) &= \frac{1}{2} x_3 u^2 + w_3 u + a_3; \quad g_3(v) = \frac{1}{2} z_3 v^2 + c_3 v + b_3, \\
f_4(u) &= \frac{1}{2} x_4 u^2 + w_4 u + a_4; \quad g_4(v) = \frac{1}{2} z_4 v^2 + c_4 v + b_4,
\end{aligned}
\tag{5.3}
$$

where $d_i = a_i + b_i, \ i = 3, 4.$ $\qquad\square$

Example 5.1. We construct a $3D$ geometric shape model in $\mathbb{E}^3$ by using the projection of the Bézier translation surface in equation (5.3), which is given by

$$X(u,v) = (u, v, -\frac{u^2}{2}, -\frac{v^2}{2}),$$

where $x_3 = -1, \ z_4 = -1, \ w_3 = a_3 = z_3 = c_3 = b_3 = x_4 = w_4 = a_4 = c_4 = b_4 = 0.$

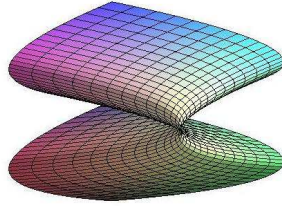Furthermore, we plot the graph (see Figure 2) of the given surface by using the *Maple* plotting command

$$\text{plot}3d([x+y,z,w], x=a..b, y=c..d).$$

FIGURE 2. The projection of Bézier translation surface in $\mathbb{E}^3$.

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