ACTA ET COMMENTATIONES UNIVERSITATIS TARTUENSIS DE MATHEMATICA
Volume 20, Number 2, December 2016
Available online at http://acutm.math.ut.ee

On ϕ-pseudo symmetric LP-Sasakian manifolds with respect to quarter-symmetric non-metric connections

Santu Dey and Arindam Bhattacharyya

Abstract. The object of the present paper is to study ϕ-pseudo symmetric and ϕ-pseudo Ricci symmetric LP-Sasakian manifolds with respect to Levi–Civita connections and quarter-symmetric non-metric connections. We obtain a necessary and sufficient condition for a ϕ-pseudo symmetric LP-Sasakian manifold with respect to a quarter symmetric non-metric connection to be ϕ-pseudo symmetric LP-Sasakian manifold with respect to a Levi–Civita connection.

1. Introduction

The idea of a metric connection with torsion tensor in a Riemannian manifold was introduced by Hayden [15]. Later, Yano [37] studied some properties of semi symmetric metric connections on a Riemannian manifold. The idea of a quarter-symmetric linear connection in a differentiable manifold was introduced by Golab [14]. This generalizes the idea of a semi symmetric connection. After Golab and also Rastogi [27, 28] continued the systematic study of quarter symmetric metric connections. Pandey and Mishra [20] studied quarter symmetric metric connections in Riemannian, Kaehlerian and Sasakian manifolds. In 1982, Yano and Imai [38] studied quarter-symmetric metric connections in Hermitian and Kaehlerian manifolds. Mukhopadhyay et al. [21] studied quarter-symmetric metric connections on Riemannian manifolds with an almost complex structure.

In 2003, Sengupta and Biswas [29] defined quarter-symmetric non-metric connections in a Sasakian manifold and studied their properties. In this series, properties of quarter-symmetric non-metric connections have been

Received April 12, 2016.
2010 *Mathematics Subject Classification.* 53C25; 53C15; 53D15.
Key words and phrases. Quarter-symmetric non-metric connection; ϕ-pseudo symmetric LP-Sasakian manifold; ϕ-pseudo Ricci symmetric LP-Sasakian manifold.
http://dx.doi.org/10.12697/ACUTM.2016.20.13

studied by the authors of [1, 9, 12] and many others. Recently, Chaubey and Ojha [9] defined a quarter-symmetric non-metric connection on an almost Hermite manifold. They also have studied properties of quarter-symmetric metric connections in an Einstein manifold. In [3], Bhattacharyya and Patra studied semi-symmetric metric connections on pseudosymmetric Lorentzian α-Sasakian manifolds.

On the other hand, Matsumoto [17] introduced the notion of an LP-Sasakian manifold. Then Mihai and Roşca [19] introduced the same notion independently and obtained several results on this manifold. LP-Sasakian manifolds have also been studied by Mihai [19], Singh [31], and others.

A linear connection $\tilde{\nabla}$ on an n-dimensional Riemannian manifold (M^n, g) is said to be a *quarter-symmetric* connection (see [14]) if its torsion tensor $\tilde{T}$ defined by

$$\tilde{T}(X,Y) = \tilde{\nabla}_X Y - \tilde{\nabla}_Y X - [X,Y] \tag{1.1}$$

is of the form

$$\tilde{T}(X,Y) = \eta(Y)\phi X - \eta(X)\phi Y, \tag{1.2}$$

where η is 1-form and ϕ is a tensor of type $(1,1)$. In addition, if a quarter-symmetric linear connection $\tilde{\nabla}$ satisfies the condition

$$(\tilde{\nabla}_X g)(Y,Z) \neq 0,$$

then $\tilde{\nabla}$ is said to be a *quarter-symmetric non-metric* connection.

During the last five decades the notion of a locally symmetric manifold has been weakened by many authors in several ways to different extent such as recurrent manifolds by Walker [36], semisymmetric manifolds by Szabó [32], pseudosymmetric manifolds in the sense of Deszcz [11], and pseudosymmetric manifolds in the sense of Chaki [5]. A non-flat Riemannian manifold (M^n, g) $(n > 2)$ is said to be *pseudosymmetric* in the sense of Chaki if it satisfies the relation

$$\begin{aligned}(\nabla_W R)(X,Y,Z,U) = {} & 2A(W)R(X,Y,Z,U) + A(X)R(W,Y,Z,U) \\ & + A(Y)R(X,W,Z,U) + A(Z)R(X,Y,W,U) \\ & + A(U)R(X,Y,Z,W).\end{aligned}$$

i.e.,

$$\begin{aligned}(\nabla_W R)(X,Y)Z = {} & 2A(W)R(X,Y)Z + A(X)R(W,Y)Z \\ & + A(Y)R(X,W)Z + A(Z)R(X,Y)W \\ & + g(R(X,Y)Z,W)\rho.\end{aligned}$$

for any vector fields X, Y, Z, U, and W, where R is the Riemannian curvature tensor of the manifold, and A is a non-zero 1-form such that

$g(X,\rho) = A(X)$ for every vector field X. Such an n-dimensional manifold is denoted by $(PS)_n$.

Every recurrent manifold is pseudosymmetric in the sense of Chaki, but not conversely. Also the pseudosymmetry in the sense of Chaki is not equivalent to that in the sense of Deszcz [11]. However, pseudosymmetry by Chaki is the pseudosymmetry by Deszcz if and only if the non-zero 1-form associated with $(PS)_n$ is closed. Pseudosymmetric manifolds in the sense of Chaki have been studied by Chaki et al. [7], Özen and Altay [23, 24], Tarafdar [34, 35], and others.

A Riemannian manifold is said to be *Ricci symmetric* if its Ricci tensor S of type $(0,2)$ satisfies $\nabla S = 0$, where ∇ denotes the Riemannian connection. During the last five decades, the notion of a Ricci symmetry has been weakened by many authors in several ways to a different extent such as Ricci recurrent manifolds (see [25]), Ricci semisymmetric manifolds (see [32]), pseudo Ricci symmetric manifolds by Deszcz [10], and pseudo Ricci symmetric manifolds by Chaki [6].

A non-flat Riemannian manifold (M^n, g) is said to be *pseudo Ricci symmetric* (see [6]) if its Ricci tensor S of type $(0,2)$ is not identically zero and satisfies the condition

$$(\nabla_X S)(Y,Z) = 2A(X)S(Y,Z) + A(Y)S(X,Z) + A(Z)S(Y,X) \qquad (1.3)$$

for any vector fields X, Y, Z, where A is a nowhere vanishing 1-form and ∇ denotes the operator of covariant differentiation with respect to the metric tensor g. Such an n-dimensional manifold is denoted by $(PRS)_n$. Pseudo Ricci symmetric manifolds have been also studied by Arslan et al. [2], Chaki and Saha [8], Özen [22], and many others.

The relation (1.3) can be written as

$$(\nabla_X Q)(Y) = 2A(X)Q(Y) + A(Y)Q(X) + S(Y,X)\rho,$$

where ρ is the vector field associated to the 1-form A such that $A(X) = g(X,\rho)$, and Q is the Ricci operator, i.e., $g(QX,Y) = S(X,Y)$ for all $X, Y \in \chi(M)$, where $\chi(M)$ is the set of all differentiable vector fields on M.

The notion of a local symmetry of Riemannian manifold has been weakened by many authors in several ways to a different extent. As a weaker version of local symmetry, Takahashi [33] introduced the notion of a local ϕ-symmetry on Sasakian manifolds. In the context of contact geometry the notion of ϕ-symmetry was introduced and studied by Boeckx et al. [4] with several examples. Also in [26], Prakash studied concircularly ϕ-recurrent Kenmotsu manifolds. Shukla et al. [30] studied ϕ-Ricci symmetric Kenmotsu manifolds.

Definition 1.1 (see [16]). An LP-Sasakian manifold $(M^n, \phi, \xi, \eta, g)$ $(n > 2)$ is said to be *ϕ-pseudosymmetric* if the curvature tensor R satisfies

$$\begin{aligned}\phi^2((\nabla_W R)(X,Y)Z) = {} & 2A(W)R(X,Y)Z + A(X)R(W,Y)Z \\ & + A(Y)R(X,W)Z + A(Z)R(X,Y)W \\ & + g(R(X,Y)Z,W)\rho\end{aligned} \tag{1.4}$$

for any vector fields X, Y, Z, and W, where A is a non-zero 1-form. In particular, if $A = 0$, then the manifold is said to be *ϕ-symmetric*.

Definition 1.2 (see [16]). An LP-Sasakian manifold $(M^n, \phi, \xi, \eta, g)$ $(n > 2)$ is said to be *ϕ-pseudo Ricci symmetric* if the Ricci operator Q satisfies

$$\phi^2((\nabla_X Q)(Y)) = 2A(X)Q(Y) + A(Y)Q(X) + S(Y,X)\rho \tag{1.5}$$

for any vector fields X, Y, where A is a non-zero 1-form. In particular, if $A = 0$, then (1.5) turns into the notion of a *ϕ-Ricci symmetric* LP-Sasakian manifold.

Motivated by the above studies, the present paper deals with the study of ϕ-pseudo symmetric and ϕ-pseudo Ricci symmetric LP-Sasakian manifolds with respect to quarter symmetric non-metric connections. The paper is organized as follows. Section 2 is concerned with preliminaries. Curvature tensors of LP-Sasakian manifolds with respect to quarter-symmetric non-metric connections have been studied in Section 3. In Section 4 we study ϕ-pseudo symmetric LP-Sasakian manifolds with respect to quarter symmetric non-metric connections and obtain a necessary and sufficient condition for a ϕ-pseudo symmetric LP-Sasakian manifold with respect to a quarter-symmetric non-metric connection to be a ϕ-pseudo symmetric LP-Sasakian manifold with respect to a Levi–Civita connection. In Section 5 we have studied ϕ-pseudo Ricci symmetric LP-Sasakian manifolds with respect to quarter symmetric non-metric connections. Finally, in Section 6 we give an example of a 3-dimensional LP-Sasakian manifold with respect to a quarter-symmetric non-metric connection.

2. Preliminaries

An n-dimensional $(n = 2m + 1)$ differentiable manifold M^n is called a *Lorentzian para-Sasakian* (briefly, LP-Sasakian) manifold (see [17, 18]) if it admits a $(1,1)$-tensor field ϕ, a contravariant vector field ξ, a 1-form η, and a Lorentzian metric g which satisfy

$$\eta(\xi) = -1, \tag{2.1}$$

$$\phi^2 X = X + \eta(X)\xi, \tag{2.2}$$

$$g(\phi X, \phi Y) = g(X,Y) + \eta(X)\eta(Y),$$

$$g(X,\xi)=\eta(X), \tag{2.3}$$

$$\nabla_X\xi=\phi X, \tag{2.4}$$

$$(\nabla_X\phi)(Y)=g(X,Y)\xi+\eta(Y)X+2\eta(X)\eta(Y)\xi,$$

where ∇ denotes the covariant differentiation with respect to the Lorentzian metric g. It can be easily seen that in an LP-Sasakian manifold the following relations hold:

$$\phi\xi=0,\ \ \eta(\phi X)=0, \tag{2.5}$$

$$\operatorname{rank}(\phi)=n-1.$$

If we put

$$\Phi(X,Y)=g(X,\phi Y),$$

for any vector field X and Y, then the tensor field $\Phi(X,Y)$ is a symmetric $(0,2)$-tensor field (see [17]) and β=trace Φ. Since the 1-form η is closed in the LP-Sasakian manifold, we have (see [17])

$$(\nabla_X\eta)(Y)=\Phi(X,Y),\ \ \ \Phi(X,\xi)=0, \tag{2.6}$$

for all $X,Y\in\chi(M)$.

In an LP-Sasakian manifold, also the following relations hold (see [18]):

$$\begin{aligned}g(R(X,Y)Z,\xi)&=\eta(R(X,Y)Z)\\&=g(Y,Z)\eta(X)-g(X,Z)\eta(Y),\end{aligned}$$

$$R(\xi,X)Y=g(X,Y)\xi-\eta(Y)X, \tag{2.7}$$

$$R(X,Y)\xi=\eta(Y)X-\eta(X)Y, \tag{2.8}$$

$$R(\xi,X)\xi=X+\eta(X)\xi,$$

$$S(X,\xi)=(n-1)\eta(X), \tag{2.9}$$

$$QX=(n-1)X,\ \ r=n(n-1),$$

where Q is the Ricci operator, i.e.,

$$g(QX,Y)=S(X,Y),$$

and r is the scalar curvature with respect to the connection ∇. Moreover,

$$S(X,\phi Y)=(n-1)g(X,\phi Y) \tag{2.10}$$

and

$$S(\phi X,\phi Y)=S(X,Y)+(n-1)\eta(X)\eta(Y),$$

for any vector fields X, Y, and Z, where R and S are the Riemannian curvature tensor and the Ricci tensor of the manifold, respectively.

3. Curvature tensors of LP-Sasakian manifolds with respect to quarter-symmetric non-metric connections

In this section we express $\tilde{R}(X,Y)Z$, the curvature tensor with respect to a quarter-symmetric non-metric connection, in terms of $R(X,Y)Z$, the curvature tensor with respect to a Riemannian connection.

Here we consider a quarter symmetric non metric connection $\tilde{\nabla}$ and a Levi–Civita connection ∇ on an LP-Sasakian manifold. The relation between them is given by

$$\tilde{\nabla}_X Y = \nabla_X Y - \eta(X)\phi Y. \tag{3.1}$$

Using (3.1), we have

$$(\nabla_X g)(Y,Z) = 2\eta(X)g(\phi Y, Z). \tag{3.2}$$

The relation between $\tilde{R}(X,Y)Z$ and $R(X,Y)Z$ is also given by

$$\begin{aligned}\tilde{R}(X,Y)Z = R(X,Y)Z + g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\xi \\ + \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X.\end{aligned} \tag{3.3}$$

Moreover,

$$\tilde{R}(X,Y)\xi = 2R(X,Y)\xi = 2[\eta(Y)X - \eta(X)Y], \tag{3.4}$$

$$\tilde{R}(\xi,Y)Z = -2[\eta(Z)Y + \eta(Y)\eta(Z)\xi], \tag{3.5}$$

and

$$\tilde{R}(X,\xi)Z = 2[\eta(Z)X + \eta(X)\eta(Z)\xi]. \tag{3.6}$$

Contracting (3.3) with respect to X, we get

$$\tilde{S}(Y,Z) = S(Y,Z) - g(Y,Z) - n\eta(Y)\eta(Z), \tag{3.7}$$

where $\tilde{S}$ and S are the Ricci tensors of M^n with respect to the quarter symmetric non metric connection $\tilde{\nabla}$ and the Levi–Civita connection ∇, respectively. Also,

$$\tilde{S}(X,\xi) = 2S(X,\xi) = 2(n-1)\eta(X). \tag{3.8}$$

From (3.7) we have $\tilde{r} = r$, where $\tilde{r}$ and r are the scalar curvatures with respect to the quarter symmetric non-metric connection and the Levi–Civita connection, respectively.

Now, using (2.1)−(2.3), (2.5), (2.6), and (3.4), we get

$$\begin{aligned}(\tilde{\nabla}_W \tilde{R})(X,Y)\xi = 2[g(\phi W, Y)X - g(\phi W, X)Y] \\ - 2\eta(W)[\eta(Y)\phi X - \eta(X)\phi Y].\end{aligned} \tag{3.9}$$

Again from (2.6) and (3.4), we have

$$g((\tilde{\nabla}_W \tilde{R})(X,Y)Z, U) = g((\tilde{\nabla}_W \tilde{R})(X,Y)U, Z). \tag{3.10}$$

In the next sections we will apply the above relations.

4. ϕ-pseudo symmetric LP-Sasakian manifolds with respect to quarter-symmetric non-metric connections

Definition 4.1. An LP-Sasakian manifold $(M^n, \phi, \xi, \eta, g)$ $(n > 2)$ is said to be *ϕ-pseudo symmetric with respect to a quarter symmetric non-metric connection* if the curvature tensor $\tilde{R}$ satisfies

$$\begin{aligned}\phi^2((\tilde{\nabla}_W \tilde{R})(X,Y)Z) = 2A(W)\tilde{R}(X,Y)Z + A(X)\tilde{R}(W,Y)Z \\ + A(Y)\tilde{R}(X,W)Z + A(Z)\tilde{R}(X,Y)W \\ + g(\tilde{R}(X,Y)Z,W)\rho\end{aligned} \tag{4.1}$$

for any vector fields X, Y, Z, and W, where A is a non-zero 1-form.

Now using (2.2) in (4.1), we have

$$\begin{aligned}&(\tilde{\nabla}_W \tilde{R})(X,Y)Z + \eta((\tilde{\nabla}_W \tilde{R})(X,Y)Z)\xi \\ &= 2A(W)\tilde{R}(X,Y)Z + A(X)\tilde{R}(W,Y)Z + A(Y)\tilde{R}(X,W)Z \\ &\quad + A(Z)\tilde{R}(X,Y)W + g(\tilde{R}(X,Y)Z,W)\rho.\end{aligned} \tag{4.2}$$

This shows that

$$\begin{aligned}&g((\tilde{\nabla}_W \tilde{R})(X,Y)Z,U) + \eta((\tilde{\nabla}_W \tilde{R})(X,Y)Z)\eta(U) \\ &= 2A(W)g(\tilde{R}(X,Y)Z,U) + A(X)g(\tilde{R}(W,Y)Z,U) \\ &\quad + A(Y)g(\tilde{R}(X,W)Z,U) + A(Z)g(\tilde{R}(X,Y)W,U) \\ &\quad + g(\tilde{R}(X,Y)Z,W)A(U).\end{aligned} \tag{4.3}$$

Let $\{e_i : i = 1,2,...,n\}$ be an orthonormal basis of the tangent space at any point of the manifold. Setting $X = U = e_i$ in (4.3) and taking summation over i, $1 \le i \le n$, and then using (2.1), (2.3), and (2.5) in (4.3), we obtain

$$\begin{aligned}&(\tilde{\nabla}_W \tilde{S})(Y,Z) + g((\tilde{\nabla}_W \tilde{R})(\xi,Y)Z,\xi) \\ &= 2A(W)\tilde{S}(Y,Z) + A(Y)\tilde{S}(W,Z) + A(Z)\tilde{S}(Y,W) \\ &\quad + A(\tilde{R}(W,Y)Z) + A(\tilde{R}(W,Z)Y).\end{aligned} \tag{4.4}$$

Because of (2.7), (3.9) and (3.10), we obtain

$$\begin{aligned}g((\tilde{\nabla}_W \tilde{R})(\xi,Y)Z,\xi) &= g((\tilde{\nabla}_W \tilde{R})(\xi,Y)\xi,Z) \\ &= 2g(\phi W,Y)\eta(Z) - 2g(\phi Y,Z)\eta(W).\end{aligned} \tag{4.5}$$

By virtue of (4.5) it follows from (4.4) that

$$\begin{aligned}(\tilde{\nabla}_W \tilde{S})(Y,Z) &= 2A(W)\tilde{S}(Y,Z) + A(Y)\tilde{S}(W,Z) \\ &\quad + A(Z)\tilde{S}(Y,W) + A(\tilde{R}(W,Y)Z) + A(\tilde{R}(W,Z)Y) \\ &\quad - 2[g(\phi W,Y)\eta(Z) - g(\phi Y,Z)\eta(W)].\end{aligned}$$

So, we have the following theorem.

Theorem 4.1. *A ϕ-pseudo symmetric LP-Sasakian manifold with respect to a quarter-symmetric non-metric connection is pseudo Ricci symmetric with respect to a quarter symmetric non-metric connection if and only if*

$$A(\tilde{R}(W,Y)Z) + A(\tilde{R}(W,Z)Y) - 2[g(\phi W,Y)\eta(Z) - g(\phi Y,Z)\eta(W)] = 0.$$

Setting $Z = \xi$ in (4.4) and using (4.5), we get

$$\begin{aligned}(\tilde{\nabla}_W\tilde{S})(Y,\xi) - 2g(\phi W,Y) = {} & 2A(W)\tilde{S}(Y,\xi) \\ & + A(Y)\tilde{S}(W,\xi) + A(\xi)\tilde{S}(Y,W) \\ & + A(\tilde{R}(W,Y)\xi) + A(\tilde{R}(W,\xi)Y).\end{aligned} \tag{4.6}$$

We know that

$$(\tilde{\nabla}_W\tilde{S})(Y,\xi) = \tilde{\nabla}_W\tilde{S}(Y,\xi) - \tilde{S}(\tilde{\nabla}_WY,\xi) - \tilde{S}(Y,\tilde{\nabla}_W\xi). \tag{4.7}$$

Thus, using (2.4), (2.9), (3.1), and (3.7) in (4.7), we get

$$(\tilde{\nabla}_W\tilde{S})(Y,\xi) = (2n-1)g(\phi Y,W) - S(\phi Y,W). \tag{4.8}$$

In view of (3.4)$-$(3.8) and (4.8), we have from (4.6) that

$$\begin{aligned}&(2n-1)g(\phi Y,W) - S(\phi Y,W) - 2g(\phi W,Y) \\ &\qquad = 2A(W)2(n-1)\eta(Y) + 2A(Y)2(n-1)\eta(W) + A(\xi)[S(Y,W) \\ &\qquad\quad - g(Y,W) - n\eta(Y)\eta(W)] + 2[\eta(Y)A(W) - \eta(W)A(Y)] \\ &\qquad\quad + 2[\eta(Y)A(W) + \eta(Y)\eta(W)A(\xi)].\end{aligned}$$

That is,

$$\begin{aligned}A(\xi)S(Y,W) = {} & ng(W,\phi Y) - 2g(\phi W,Y) - 4(n-1)A(W)\eta(Y) \\ & - 2(n-1)A(Y)\eta(W) + A(\xi)[g(Y,W) \\ & + (n-2)\eta(Y)\eta(W)] - 4\eta(Y)A(W) + 2\eta(W)A(Y).\end{aligned} \tag{4.9}$$

Contracting (4.9) over Y and W, we obtain

$$rA(\xi) = \beta(n-2) - 6(n-1)A(\xi). \tag{4.10}$$

This leads to the following result.

Theorem 4.2. *In a ϕ-pseudo symmetric LP-Sasakian manifold with respect to a quarter-symmetric non-metric connection, the Ricci tensor and the scalar curvature are, respectively, given by* (4.9) *and* (4.10).

Using (3.10) in (4.2), we get

$$\begin{aligned}(\tilde{\nabla}_W\tilde{R})(X,Y)Z = {} & -g((\tilde{\nabla}_W\tilde{R})(X,Y)\xi,Z)\xi + 2A(W)\tilde{R}(X,Y)Z \\ & + A(X)\tilde{R}(W,Y)Z + A(Y)\tilde{R}(X,W)Z \\ & + A(Z)\tilde{R}(X,Y)W + g(\tilde{R}(X,Y)Z,W)\rho.\end{aligned} \tag{4.11}$$

In view of (3.3) and (3.9), it follows from (4.11) that

$$\begin{aligned}(\tilde{\nabla}_W\tilde{R})(X,Y)Z = &-2[g(\phi W,Y)g(X,Z) - g(\phi W,X)g(Y,Z)]\xi \\ &- 2\eta(W)[\eta(Y)g(\phi X,Z) - \eta(X)g(\phi Y,Z)]\xi \\ &+ 2A(W)[R(X,Y)Z + g(Y,Z)\eta(X)\xi \\ &- g(X,Z)\eta(Y)\xi + \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X] \\ &+ A(X)[R(W,Y)Z + g(Y,Z)\eta(W)\xi - g(W,Z)\eta(Y)\xi \\ &+ \eta(W)\eta(Z)Y - \eta(Y)\eta(Z)W] + A(Y)[R(X,W)Z \\ &+ g(W,Z)\eta(X)\xi - g(X,Z)\eta(W)\xi \\ &+ \eta(X)\eta(Z)W - \eta(W)\eta(Z)X] + A(Z)[R(X,Y)W \\ &+ g(Y,W)\eta(X)\xi - g(X,W)\eta(Y)\xi + \eta(X)\eta(W)Y \\ &- \eta(Y)\eta(W)X] + [R(X,Y,Z,W) \\ &+ g(Y,Z)\eta(X)\eta(W) - g(X,Z)\eta(Y)\eta(W) \\ &+ g(Y,W)\eta(X)\eta(Z) - g(X,W)\eta(Y)\eta(Z)]\rho\end{aligned} \tag{4.12}$$

for arbitrary vector fields X, Y, Z and W.

This leads to the following theorem.

Theorem 4.3. *An LP-Sasakian manifold is ϕ-pseudo symmetric with respect to a quarter symmetric non-metric connection if and only if the relation* (4.12) *holds.*

Let us take a ϕ-pseudo symmetric LP-Sasakian manifold with respect to a Levi–Civita connection. Then the relation (1.4) holds. Now from the equation (2.8) we have

$$(\nabla_W\tilde{R})(X,Y)\xi = g(\phi W,Y)X - g(\phi W,X)Y.$$

By virtue of (2.1)–(2.3) and the relation

$$g((\tilde{\nabla}_W\tilde{R})(X,Y)Z,U) = g((\tilde{\nabla}_W\tilde{R})(X,Y)U,Z),$$

it follows from (1.4) that

$$\begin{aligned}(\nabla_W\tilde{R})(X,Y)Z = &-[g(\phi W,Y)g(X,Z) - g(\phi W,X)g(Y,Z)]\xi \\ &+ 2A(W)R(X,Y)Z + A(X)R(W,Y)Z \\ &+ A(Y)R(X,W)Z + A(Z)R(X,Y)W \\ &+ g(R(X,Y)Z,W)\rho.\end{aligned} \tag{4.13}$$

From (4.12) and (4.13), we can state the following theorem.

Theorem 4.4. *A ϕ-pseudo symmetric LP-Sasakian manifold is invariant under a quarter symmetric non-metric connection if and only if the relation*

$$\begin{aligned}
&[g(\phi W,X)g(Y,Z)-g(\phi W,Y)g(X,Z)]\xi-2\eta(W)[\eta(Y)g(\phi X,Z)-\eta(X)g(\phi Y,Z)]\\
&\quad+2A(W)[g(Y,Z)\eta(X)\xi-g(X,Z)\eta(Y)\xi+\eta(X)\eta(Z)Y-\eta(Y)\eta(Z)X]\\
&\quad+A(X)[g(Y,Z)\eta(W)\xi-g(W,Z)\eta(Y)\xi+\eta(W)\eta(Z)Y-\eta(Y)\eta(Z)W]\\
&\quad+A(Y)[g(W,Z)\eta(X)\xi-g(X,Z)\eta(W)\xi+\eta(X)\eta(Z)W-\eta(W)\eta(Z)X]\\
&\quad+A(Z)[g(Y,W)\eta(X)\xi-g(X,W)\eta(Y)\xi+\eta(X)\eta(W)Y-\eta(Y)\eta(W)X]\\
&\quad+[g(Y,Z)\eta(X)\eta(W)-g(X,Z)\eta(Y)\eta(W)+g(Y,W)\eta(X)\eta(Z)\\
&\quad-g(X,W)\eta(Y)\eta(Z)]\rho=0
\end{aligned}$$

holds for arbitrary vector fields X, Y, Z and W.

5. ϕ-pseudo Ricci symmetric LP-Sasakian manifolds with respect to quarter-symmetric non-metric connections

Definition 5.1. An LP-Sasakian manifold (M^n,ϕ,ξ,η,g) $(n>2)$ is said to be *ϕ-pseudo Ricci symmetric with respect to a quarter symmetric non-metric connection* if the Ricci tensor $\tilde{S}$ satisfies

$$\phi^2((\tilde{\nabla}_X\tilde{Q})(Y))=2A(X)\tilde{Q}(Y)+A(Y)\tilde{Q}(X)+\tilde{S}(Y,X)\rho \qquad (5.1)$$

for any vector fields X,Y, where A is a non-zero 1-form. In particular, if $A=0$, then (5.1) turns into the notion of a *ϕ-Ricci symmetric LP-Sasakian manifold with respect to quarter-symmetric non-metric connection.*

Let us take LP-Sasakian manifold (M^n,ϕ,ξ,η,g) $(n>2)$ which is ϕ-pseudo Ricci symmetric with respect to quarter symmetric non-metric connection. Then by virtue of (2.2) it follows from (5.1) that

$$g(\tilde{\nabla}_X\tilde{Q}(Y),Z)+\eta((\tilde{\nabla}_X\tilde{Q}(Y))\xi=2A(X)\tilde{Q}(Y)+A(Y)\tilde{Q}(X)+\tilde{S}(Y,X)\rho$$

from which it follows that

$$\begin{aligned}
g(\tilde{\nabla}_X\tilde{Q}(Y),Z)+\tilde{S}(\tilde{\nabla}_XY,Z)+\eta((\tilde{\nabla}_X\tilde{Q})(Y))\eta(Z)&\\
=2A(X)\tilde{S}(Y,Z)+A(Y)\tilde{S}(X,Z)&\qquad (5.2)\\
+A(Z)\tilde{S}(Y,X).&
\end{aligned}$$

Putting $Y=\xi$ in (5.2) and using (2.4), (2.9), (3.1), (3.7) and (3.8), we get

$$\begin{aligned}
2(n-1)g(\phi X,Z)-S(\phi X,Z)+g(\phi X,Z)+2(n-1)g(\phi X,\xi)\eta(Z)&\\
=4A(X)(n-1)\eta(Z)+2A(Z)(n-1)\eta(X)&\qquad (5.3)\\
+A(\xi)[S(X,Z)-g(X,Z)-n\eta(X)\eta(Z)].&
\end{aligned}$$

In view of (2.5) and (2.10), the equation (5.3) becomes

$$\begin{aligned} A(\xi)S(X,Z) &= ng(\phi X,Z) - 4(n-1)A(X)\eta(Z) \\ &\quad + 2(n-1)A(Z)\eta(X) + A(\xi)[g(X,Z) + n\eta(X)\eta(Z)]. \end{aligned} \tag{5.4}$$

This leads to the following theorem.

Theorem 5.1. *In a ϕ-pseudo Ricci symmetric LP-Sasakian manifold admitting a quarter symmetric non-metric connection, the Ricci tensor is of the form* (5.4).

6. Example of a 3-dimensional LP-Sasakian manifold with respect to a quarter-symmetric non-metric connection

We consider a 3-dimensional manifold $M = \{(x,y,z) \in R^3\}$, where (x,y,z) are the standard coordinates of R^3. Let e_1, e_2, e_3 be the vector fields on M^3 given by

$$e_1 = -2\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial z},\ e_2 = \frac{\partial}{\partial y},\ e_3 = \frac{\partial}{\partial z}.$$

Clearly, $\{e_1, e_2, e_3\}$ is a set of linearly independent vectors for each point of M and hence a basis of $\chi(M)$. The Lorentzian metric g is defined by

$$g(e_1,e_2) = g(e_2,e_3) = g(e_1,e_3) = 0,$$

$$g(e_1,e_1) = 1,\ g(e_2,e_2) = 1,\ g(e_3,e_3) = -1.$$

Let η be the 1-form defined by $\eta(Z) = g(Z, e_3)$ for any $Z \in \chi(M)$. The $(1,1)$-tensor field ϕ is defined by

$$\phi e_1 = e_2,\ \phi e_2 = e_1,\ \phi e_3 = 0.$$

From the linearity of ϕ and g, we have

$$\eta(e_3) = -1,$$

$$\phi^2 X = X + \eta(X)e_3,$$

and

$$g(\phi X, \phi Y) = g(X,Y) + \eta(X)\eta(Y)$$

for any $X \in \chi(M)$. Then, for $e_3 = \xi$, the structure (ϕ, ξ, η, g) defines a Lorentzian paracontact structure on M.

Let ∇ be the Levi–Civita connection with respect to the Lorentzian metric g. Then we have

$$[e_1, e_2] = -2e_3,\ [e_1, e_3] = 0,\ [e_2, e_3] = 0.$$

Using Koszul's formula

$$\begin{aligned} 2g(\nabla_X Y, Z) &= Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) \\ &\quad - g(X,[Y,Z]) - g(Y,[X,Z]) + g(Z,[X,Y]), \end{aligned}$$

we can calculate that

$$\nabla_{e_1}e_1 = 0,\ \nabla_{e_1}e_2 = -e_3,\ \nabla_{e_1}e_3 = e_2,$$
$$\nabla_{e_2}e_1 = e_3,\ \nabla_{e_2}e_2 = 0,\ \nabla_{e_2}e_3 = e_1,$$
$$\nabla_{e_3}e_1 = e_2,\ \nabla_{e_3}e_2 = e_1,\ \nabla_{e_3}e_3 = 0.$$

From these calculations we see that the manifold under consideration satisfies $\eta(\xi) = -1$ and $\nabla_X\xi = \phi X$. Hence the structure (ϕ, ξ, η, g) is an LP-Sasakian manifold.

Using (3.1), we find $\tilde{\nabla}$, the quarter-symmetric connection on M:

$$\tilde{\nabla}_{e_1}e_1 = 0,\ \tilde{\nabla}_{e_1}e_2 = -e_3,\ \tilde{\nabla}_{e_1}e_3 = e_2,$$
$$\tilde{\nabla}_{e_2}e_1 = e_3,\ \tilde{\nabla}_{e_2}e_2 = 0,\ \tilde{\nabla}_{e_2}e_3 = e_1,$$
$$\tilde{\nabla}_{e_3}e_1 = 2e_2,\ \tilde{\nabla}_{e_3}e_2 = 2e_1,\ \tilde{\nabla}_{e_3}e_3 = 0.$$

By (1.2) we see that the torson tensor T with respect to the quarter-symmetric connection $\tilde{\nabla}$, satisfies the relations

$$\tilde{T}(e_i, e_i) = 0, \quad i = 1, 2, 3,$$
$$\tilde{T}(e_1, e_2) = 0,\ \tilde{T}(e_2, e_3) = -e_1,\ \tilde{T}(e_3, e_1) = e_2.$$

Now, using (3.2), we calculate the metric g with respect to the quarter-symmetric connection $\tilde{\nabla}$ as follows:

$$(\tilde{\nabla}_{e_1}g)(e_2, e_3) = 0,\ (\tilde{\nabla}_{e_2}g)(e_3, e_1) = 0,\ (\tilde{\nabla}_{e_3}g)(e_1, e_2) = -2 \neq 0.$$

From this we can conclude that $(\tilde{\nabla}_X g)(Y, Z) \neq 0$, where X, Y, Z are any vector fields in $\chi(M)$. Thus M is an LP-Sasakian manifold with respect to the quarter-symmetric non-metric connection $\tilde{\nabla}$.

Acknowledgements

The first author is supported by DST/INSPIRE Fellowship/2013/1041, Govt. of India. The authors wish to express their sincere thanks and gratitude to the referee for valuable suggestions towards the improvement of the paper.

References

[1] M. Ahmad, A. Haseeb, and C. Özgür, *Hypersurfaces of an almost r-paracontact Riemannian manifold endowed with a quarter-symmetric non-metric connection*, Kyungpook Math. J. **49** (2009), 533–543.

[2] K. Arslan, R. Ezentas, C. Murathan, and C. Özgür, *On pseudo Ricci symmetric manifolds*, Balkan J. Geom. Appl. **6** (2001), 1–5.

[3] A. Bhattacharyya and C. Patra, *Semi-symmetric metric connections on pseudosymmetric Lorentzian α-Sasakian manifolds*, Acta Comment. Univ. Tartu. Math. **18** (2014), 197–210.

[4] E. Boeckx, P. Buecken, and L. Vanhecke, *ϕ-symmetric contact metric spaces*, Glasgow Math. J. **41** (1999), 409–416.

[5] M. C. Chaki, *On pseudo symmetric manifolds,* An. Ştiinţ. Univ. Al. I. Cuza Iaşi Sect. I a Mat. **33** (1987), 53–58.

[6] M. C. Chaki, *On pseudo Ricci symmetric manifolds*, Bulg. J. Phys. **15** (1988), 526–531.

[7] M. C. Chaki and B. Chaki, *On pseudosymmetric manifolds admitting a type of semi-symmetric connection*, Soochow J. Math. **13** (1987), 1–7.

[8] M. C. Chaki and S. K. Saha, *On pseudo-projective Ricci symmetric manifolds*, Bulg. J. Phys. **21** (1994), 1–7.

[9] S. K. Chaubey and R. H. Ojha, *On a semi-symmetric non-metric and quarter-symmetric metric connections*, Tensor (N. S.) **70** (2008), 202–213.

[10] R. Deszcz, *On Ricci-pseudosymmetric warped products*, Demonstratio Math. **22** (1989), 1053–1065.

[11] R. Deszcz, *On pseudosymmetric spaces*, Bull. Soc. Math. Belg. Sér. A **44** (1992), 1–34.

[12] A. K. Dubey, R. H. Ojha, and S. K. Chaubey, *Some properties of quarter-symmetric non-metric connection in a Kähler manifold*, Int. J. Contemp. Math. Sci. **5** (2010), 1001–1007.

[13] A. Friedmann and J. A. Schouten. *Über die Geometrie der halbsymmetrischen Übertragung*, Math. Z. **21** (1924), 211–223.

[14] S. Golab. *On semi-symmetric and quarter-symmetric linear connections*, Tensor (N. S.) **29** (1975), 249–254.

[15] H. A. Hayden. *Subspaces of a space with torsion*, Proc. London Math. Soc. **34** (1932), 27–50.

[16] S. K. Hui, *On ϕ-pseudo symmetric para Sasakian manifolds*, Acta Univ. Apulensis Math. Inform. **39** (2014), 161–178.

[17] K. Matsumoto, *On Lorentzian Para-contact manifold*, Bull. Yamagata Univ. Natur. Sci. **12** (1989), no. 2, 1–7.

[18] K. Matsumoto and I. Mihai. *On a certain transformation in a Lorentzian para-Sasakian manifold*, Tensor (N. S.), **47**(1988), 189–197.

[19] I. Mihai and R. Roşca, *On Lorentzian P-Sasakian manifolds*, in: Classical analysis (Kazimierz Dolny, 1991), World Sci. Publ., River Edge, NJ, 1992, pp. 155–169.

[20] R. S. Mishra and S. N. Pandey, *On quarter-symmetric metric F-connection*, Tensor (N. S.) **34** (1980), 1-7.

[21] S. Mukhopadhyay, A. K. Roy, and B. Barua, *Some properties of a quartersymmetric metric connection on a Riemannian manifold*, Soochow J. Math. **17** (1991), 205–211.

[22] F. Özen, *On pseudo M-projective Ricci symmetric manifolds*, Int. J. Pure Appl. Math. **72** (2011), 249–258.

[23] F. Özen and S. Altay, *On weakly and pseudo symmetric Riemannian spaces*, Indian J. Pure Appl. Math. **33** (2001), 1477–1488.

[24] F. Özen and S. Altay, *On weakly and pseudo concircular symmetric structures on a Riemannian manifold*, Acta Univ. Palack. Olomuc. Fac. Rerm Natur. Math. **47** (2008), 129–138.

[25] E. M. Patterson, *Some theorems on Ricci-recurrent spaces*, J. London Math. Soc. **27** (1952), 287–295.

[26] A. Prakash, *On concircularly ϕ-recurrent Kenmotsu manifolds*, Bull. Math. Anal. Appl. **3** (2011), no. 2, 83–88.

[27] S. C. Rastogi, *On quarter-symmetric metric connection*, C. R. Acad. Bulgare Sci. **31** (1978), 811–814.

[28] S. C. Rastogi, *On quarter-symmetric metric connection*, Tensor (N. S.) **44** (1987), 133–141.

[29] J. Sengupta and B. Biswas, *Quarter-symmetric non-metric connection on a Sasakian manifold*, Bull. Calcutta Math. Soc. **95** (2003), 169–176.

[30] A. S. Shukla and M. K. Shukla, *On ϕ-Ricci symmetric Kenmotsu manifolds*, Novi Sad J. Math. **39** (2009), 89–95.

[31] J. P. Singh, *M-projective curvature tensor on LP-Sasakian manifolds*, J. Progressive Sci. **3** (2012), 73–76.

[32] Z. I. Szabó, *Structure theorems on Riemannian spaces satisfying $R(X, Y).R = 0$. I. The local version*, J. Differential Geom. **17** (1982), 531–582.

[33] T. Takahashi, *Sasakian ϕ-symmetric spaces*, Tôhoku Math. J. **29** (1977), 91–113.

[34] M. Tarafdar, *On pseudo symmetric and pseudo Ricci symmetric Sasakian manifolds*, Period. Math. Hungar. **22** (1991), 125–129.

[35] M. Tarafdar, *On conformally flat pseudo symmetric manifolds*, An. Ştiinţ. Univ. Al. I. Cuza Iaşi Sect. I a Mat. **41**(1995), 237–241.

[36] A. G. Walker, *On Ruses spaces of recurrent curvature*, Proc. London Math. Soc. **52** (1950), 36–64.

[37] K. Yano. *On semi-symmetric metric connections*, Rev. Roumaine Math. Pures Appl. **15**(1970), 1579–1586.

[38] K. Yano and T. Imai, *Quarter-symmetric metric connections and their curvature tensors*, Tensor (N. S.) **38** (1982), 13–18.

Faculty of Science, Department of Mathematics, Jadavpur University, Kolkata 700032, India

E-mail address: santu.mathju@gmail.com

E-mail address: bhattachar1968@yahoo.co.in