ACTA ET COMMENTATIONES UNIVERSITATIS TARTUENSIS DE MATHEMATICA
Volume 20, Number 2, December 2016
Available online at http://acutm.math.ut.ee

Certain Diophantine equations involving balancing and Lucas-balancing numbers

Prasanta Kumar Ray

Abstract. It is well known that if x is a balancing number, then the positive square root of $8x^2+1$ is a Lucas-balancing number. Thus, the totality of balancing number x and Lucas-balancing number y are seen to be the positive integral solutions of the Diophantine equation $8x^2+1=y^2$. In this article, we consider some Diophantine equations involving balancing and Lucas-balancing numbers and study their solutions.

1. Introduction

The concept of balancing numbers came into existence after the paper [2] by Behera and Panda wherein, they defined a balancing number n as a solution of the Diophantine equation $1+2+\cdots+(n-1)=(n+1)+(n+2)+\cdots+(n+r)$, calling r the balancers corresponding to n. They also proved that, x is a balancing number if and only if $8x^2+1$ is a perfect square. In a subsequent paper [7], Panda studied several fascinating properties of balancing numbers calling the positive square root of $8x^2+1$, a Lucas-balancing number for each balancing number x. In [7], Panda observed that the Lucas-balancing numbers are associated with balancing numbers in the way Lucas numbers are attached to Fibonacci numbers. Thus, all balancing numbers x and corresponding Lucas-balancing numbers y are positive integer solutions of the Diophantine equation $8x^2+1=y^2$. Though the relationship between balancing and Lucas-balancing numbers is non-linear, like Fibonacci and Lucas numbers, they share the same linear recurrence $x_{n+1}=6x_n-x_{n-1}$, while initial values of balancing numbers are $x_0=0,\ x_1=1$ and for Lucas-balancing numbers $x_0=1,\ x_1=3$. Demirtürk and Keskin [3] studied certain Diophantine equations relating to Fibonacci and Lucas numbers. Recently,

Received June 18, 2016.
2010 *Mathematics Subject Classification.* 11B39; 11B83.
Key words and phrases. Balancing numbers; Lucas-balancing numbers; Diophantine equations.
http://dx.doi.org/10.12697/ACUTM.2016.20.14

Keskin and Karaatli [4] have developed some interesting properties for balancing numbers and square triangular numbers. Alvarado et al. [1], Liptai [5, 6], and Szalay [20] studied certain Diophantine equations relating to balancing numbers. The objective of this paper is to study some Diophantine equations involving balancing and Lucas-balancing numbers. The solutions are obtained in terms of these numbers.

2. Preliminaries

In this section, we present some definitions and identities on balancing and Lucas-balancing numbers which we need in the sequel. As usual, we denote the n^{th} balancing and Lucas-balancing numbers by B_n and C_n, respectively. It is well known from [7] that $C_n = \sqrt{8B_n^2+1}$. The sequences $\{B_n\}$ and $\{C_n\}$ satisfy the recurrence relations

$$B_{n+1} = 6B_n - B_{n-1},\ B_0 = 0,\ B_1 = 1;\ C_{n+1} = 6C_n - C_{n-1},\ C_0 = 1,\ C_1 = 3.$$

The balancing numbers and Lucas balancing numbers can also be defined for negative indices by modifying their recurrences as

$$B_{n-1} = 6B_n - B_{n+1};\ C_{n-1} = 6C_n - C_{n+1},$$

and calculating backwards. It is easy to see that $B_{-1} = -1$. Because $B_0 = 0$, we can check easily that all negatively subscripted balancing numbers are negative and that $B_{-n} = -B_n$. By a similar argument, it is easy to verify that $C_{-n} = C_n$. Binet's formulas for balancing and Lucas-balancing numbers are, respectively,

$$B_n = \frac{\alpha_1^{2n} - \alpha_2^{2n}}{4\sqrt{2}} \text{ and } C_n = \frac{\alpha_1^{2n} + \alpha_2^{2n}}{2},$$

where $\alpha_1 = 1+\sqrt{2}$, $\alpha_2 = 1-\sqrt{2}$, which are units of the ring $\mathbb{Z}(\sqrt{2})$. The totality of units of $\mathbb{Z}(\sqrt{2})$ is given by

$$U = \{\alpha_1^n, \alpha_2^n, -\alpha_1^n, -\alpha_2^n\} : n \in \mathbb{Z}\}.$$

The set U can be partitioned into two subsets U_1 and U_2 such that $U_1 = \{u \in U : u\overline{u} = 1\}$ and $U_2 = \{u \in U : u\overline{u} = -1\}$. Since $\overline{\alpha_1} = \alpha_2$ and $\alpha_1\alpha_2 = -1$, it follows that

$$U_1 = \{\alpha_1^{2n}, \alpha_2^{2n}, -\alpha_1^{2n}, -\alpha_2^{2n} : n \in \mathbb{Z}\},$$
$$U_2 = \{\alpha_1^{2n+1}, \alpha_2^{2n+1}, -\alpha_1^{2n+1}, -\alpha_2^{2n+1} : n \in \mathbb{Z}\}.$$

We also write $\lambda_1 = \alpha_1^2 = 3+\sqrt{8}$, $\lambda_2 = \alpha_2^2 = 3-\sqrt{8}$ and therefore, we have $\lambda_1\lambda_2 = 1$. Thus, the set U_1 can be written as

$$U_1 = \{\lambda_1^n, \lambda_2^n, -\lambda_1^n, -\lambda_2^n : n \in \mathbb{Z}\}.$$

We also need the following identities (see [9]) while establishing certain identities and solving some Diophantine equations in the subsequent section. The first identity is

$$B_n^2 = B_{n-1}B_{n+1} + 1.$$

Using the recurrence relation $B_{n+1} = 6B_n - B_{n-1}$, the identity reduces to

$$B_n^2 - 6B_nB_{n-1} + B_{n-1}^2 = 1,$$

which we may call as Cassini's formula for balancing numbers. Similar identities for Lucas-balancing numbers are

$$C_n^2 = C_{n-1}C_{n+1} - 8,\ C_n^2 - 6C_nC_{n+1} + C_{n-1}^2 = -8.$$

The idea of naming the identity as Cassini's formula comes from the literature, where this formula for Fibonacci numbers

$$F_{n-1}F_{n+1} - F_n^2 = (-1)^n \text{ or, equivalently, } F_n^2 - F_nF_{n+1} - F_{n-1}^2 = (-1)^{n-1}$$

is available. Some other important identities are found in [8, 9]:

$$\begin{aligned}
&B_{n+1} - B_{n-1} = 2C_n,\ C_{n+1} - C_{n-1} = 16B_n,\\
&B_{m+n} = B_mC_n + C_mB_n,\ B_{m-n} = B_mC_n - C_mB_n,\\
&C_{m+n} = C_mC_n + 8B_mB_n,\ C_{m-n} = C_mC_n - 8B_mB_n,\\
&C_{m+n} = B_{m+1}C_n - C_{n-1}B_m.
\end{aligned}$$

3. Some identities involving balancing and Lucas-balancing numbers

There are several known identities involving balancing, cobalancing and Lucas-balancing numbers. The interested readers are referred to [7]–[19]. In this section, we only present some new identities involving balancing and Lucas-balancing numbers.

The following two theorems are about nonlinear identities on balancing and Lucas-balancing numbers.

Theorem 3.1. *For any three integers k, m and n,*

$$C_{m+n}^2 + 16B_{k-n}C_{m+n}B_{m+k} = 8B_{m+k}^2 + C_{k-n}^2.$$

Proof. By virtue of the identities $B_{m\pm n} = B_mC_n \pm C_mB_n$ and $C_{m\pm n} = C_mC_n \pm 8B_mB_n$, we obtain

$$\begin{bmatrix} C_n & 8B_n \\ B_k & C_k \end{bmatrix} \begin{bmatrix} C_m \\ B_m \end{bmatrix} = \begin{bmatrix} C_{m+n} \\ B_{m+k} \end{bmatrix}.$$

Since $\begin{vmatrix} C_n & 8B_n \\ B_k & C_k \end{vmatrix} = C_{n-k}$ which never vanishes, we have

$$\begin{bmatrix} C_m \\ B_m \end{bmatrix} = \begin{bmatrix} C_n & 8B_n \\ B_k & C_k \end{bmatrix}^{-1} \begin{bmatrix} C_{m+n} \\ B_{m+k} \end{bmatrix} = \frac{1}{C_{n-k}} \begin{bmatrix} C_k & -8B_n \\ -B_k & C_n \end{bmatrix} \begin{bmatrix} C_{m+n} \\ B_{m+k} \end{bmatrix}.$$

This implies that

$$C_m = \frac{C_k C_{m+n} - 8B_n B_{m+k}}{C_{n-k}} \text{ and } B_m = \frac{C_n B_{m+k} - 8B_k C_{m+n}}{C_{n-k}}.$$

Since $C_m^2 - 8B_m^2 = 1$, we have

$$\left[\frac{C_k C_{m+n} - 8B_n B_{m+k}}{C_{n-k}}\right]^2 - 8\left[\frac{C_n B_{m+k} - 8B_k C_{m+n}}{C_{n-k}}\right]^2 = 1,$$

from which the identity follows. □

Theorem 3.2. *If k, m and n are three integers such that $k \neq n$, then*

$$C_{m+n}^2 + C_{m+k}^2 + 8B_{k-n}^2 = 2C_{k-n}C_{m+n}C_{m+k}.$$

Proof. By virtue of the identities $B_{m\pm n} = B_m C_n \pm C_m B_n$ and $C_{m\pm n} = C_m C_n \pm 8B_m B_n$, we obtain

$$\begin{bmatrix} C_n & 8B_n \\ C_k & 8B_k \end{bmatrix} \begin{bmatrix} C_m \\ B_m \end{bmatrix} = \begin{bmatrix} C_{m+n} \\ B_{m+k} \end{bmatrix}.$$

Since $\begin{vmatrix} C_n & 8B_n \\ C_k & 8B_k \end{vmatrix} = -8B_{n-k}$, and because $k \neq n$, this determinant is non-vanishing. Therefore, we have

$$\begin{bmatrix} C_m \\ B_m \end{bmatrix} = \begin{bmatrix} C_n & 8B_n \\ C_k & 8B_k \end{bmatrix}^{-1} \begin{bmatrix} C_{m+n} \\ B_{m+k} \end{bmatrix} = \frac{1}{B_{n-k}} \begin{bmatrix} 8B_k & -8B_n \\ -C_k & -C_n \end{bmatrix} \begin{bmatrix} C_{m+n} \\ B_{m+k} \end{bmatrix},$$

which implies that

$$C_m = -\frac{B_k C_{m+n} - B_n C_{m+k}}{8B_{n-k}} \text{ and } B_m = -\frac{C_n C_{m+k} - C_k C_{m+n}}{8B_{n-k}}.$$

Since $C_m^2 - 8B_m^2 = 1$, we have

$$\left[\frac{B_k C_{m+n} - B_n C_{m+k}}{8B_{n-k}}\right]^2 - 8\left[\frac{C_n C_{m+k} - C_k C_{m+n}}{8B_{n-k}}\right]^2 = 1,$$

and the required identity follows. □

Using the matrix multiplication

$$\begin{bmatrix} B_n & C_n \\ B_k & C_k \end{bmatrix}^{-1} \begin{bmatrix} C_m \\ B_m \end{bmatrix} = \begin{bmatrix} B_{m+n} \\ B_{m+k} \end{bmatrix},$$

it is easy to prove the following theorem.

Theorem 3.3. *If k, m and n are three integers such that $k \neq n$, then*

$$B_{m+n}^2 + B_{m+k}^2 - B_{k-n}^2 = 2C_{k-n}B_{m+n}B_{m+k}.$$

4. Some Diophantine equations involving balancing and Lucas-balancing numbers

The identities of Section 3 induce the following three Diophantine equations:

$$x^2 + 16B_nxy - 8y^2 = C_n^2, \tag{4.1}$$

$$x^2 - 2C_nxy + y^2 + C_n^2 = 1, \tag{4.2}$$

$$x^2 - 2C_nxy + y^2 = B_n^2. \tag{4.3}$$

Before the study of these equations, we present the Diophantine equation

$$x^2 - 6xy + y^2 = 1 \tag{4.4}$$

resulting out of Casini's formula for balancing numbers. The following theorem shows that all solutions of (4.4) are consecutive pairs of balancing numbers only.

Theorem 4.1. *All solutions of the Diophantine equation* (4.4) *are consecutive pairs of balancing numbers only.*

Proof. After factorization, the Diophantine equation (4.4) takes the form

$$(\lambda_1 x - y)(\lambda_2 x - y) = 1,$$

where $\lambda_1 = 3+\sqrt{8}$, $\lambda_2 = 3-\sqrt{8}$. This suggests that $(\lambda_1 x - y)$ and $(\lambda_2 x - y)$ are units of $\mathbb{Z}(\sqrt{2})$, conjugate to each other, and are members of U_1. Thus for some integer n, we have the following four cases.

$$\begin{aligned}
&\text{Case 1}: \ (\lambda_1 x - y) = \lambda_1^n \text{ and } (\lambda_2 x - y) = \lambda_2^n,\\
&\text{Case 2}: \ (\lambda_1 x - y) = \lambda_2^n \text{ and } (\lambda_2 x - y) = \lambda_1^n,\\
&\text{Case 3}: \ (\lambda_1 x - y) = -\lambda_1^n \text{ and } (\lambda_2 x - y) = -\lambda_2^n,\\
&\text{Case 4}: \ (\lambda_1 x - y) = -\lambda_2^n \text{ and } (\lambda_2 x - y) = -\lambda_1^n.
\end{aligned}$$

Solving the equation for Case 1, using Cramer's rule, we find

$$x = \frac{\begin{vmatrix} \lambda_1^n & -1 \\ \lambda_2^n & -1 \end{vmatrix}}{\begin{vmatrix} \lambda_1 & -1 \\ \lambda_2 & -1 \end{vmatrix}} = \frac{\lambda_1^n - \lambda_2^n}{\lambda_1 - \lambda_2} = B_n, \ y = \frac{\begin{vmatrix} \lambda_1 & \lambda_1^n \\ \lambda_2 & \lambda_2^n \end{vmatrix}}{\begin{vmatrix} \lambda_1 & -1 \\ \lambda_2 & -1 \end{vmatrix}} = \frac{\lambda_1^{n-1} - \lambda_2^{n-1}}{\lambda_1 - \lambda_2} = B_{n-1}.$$

The solution in Case 2 is

$$x = \frac{\begin{vmatrix} \lambda_2^n & -1 \\ \lambda_1^n & -1 \end{vmatrix}}{\begin{vmatrix} \lambda_1 & -1 \\ \lambda_2 & -1 \end{vmatrix}} = -B_n = B_{-n},\ y = \frac{\begin{vmatrix} \lambda_1 & \lambda_2^n \\ \lambda_2 & \lambda_1^n \end{vmatrix}}{\begin{vmatrix} \lambda_1 & -1 \\ \lambda_2 & -1 \end{vmatrix}} = -B_{n+1} = B_{-(n+1)}.$$

Finally, the solutions in Case 3 and Case 4 are, respectively,

$$x = B_{-n},\ y = B_{-(n-1)} \text{ and } x = B_n,\ y = B_{n+1}.$$

Thus, in all cases, the solutions of the Diophantine equation (4.4) are consecutive pairs of balancing numbers only. □

Using Theorem 4.1, we can find solutions of a Diophantine equation derived from the identity of Theorem 3.1.

Theorem 4.2. *All solutions of the Diophantine equation* (4.1) *are*

$$(x, y) = (C_{m-n}, B_m), (-C_{m+n}, B_m), (-C_{m-n}, -B_m), (C_{m+n}, -B_m) \quad (4.5)$$

for $m, n \in \mathbb{Z}$.

Proof. The equation (4.1) can be rewritten as

$$(x + 8B_n y)^2 = C_n^2(8y^2 + 1),$$

implying that $8y^2+1$ is a perfect square. Hence, y is a balancing number. So, we may take $y = \pm B_m$ ($y = -B_m$ is equivalent to $y = B_{-m}$), $8y^2 + 1 = C_m^2$. When $y = B_m$, we have $x = -8B_mB_n \pm C_mC_n$ or $x = 8B_mB_n \pm C_mC_n$ and therefore, $x = \pm C_mC_n - 8B_mB_n$, i.e. $x = C_{m-n}$ or $-C_{m+n}$. When $y = -B_m$, we have $x = -C_{m-n}$ or C_{m+n}. Thus, the totality of solutions of (4.1) is given by (4.5) for $m, n \in \mathbb{Z}$. □

Theorem 4.3. *All solutions of the Diophantine equation* (4.2) *are given by*

$$(x, y) = (-C_{m-n}, C_m), (-C_{m+n}, C_m), (C_{m+n}, -C_m), (C_{m-n}, -C_m) \quad (4.6)$$

for $m, n \in \mathbb{Z}$.

Proof. The equation (4.2) can be rewritten as

$$(x + C_n y)^2 = (C_n^2 - 1)(y^2 - 1) = 8B_n^2(y^2 - 1),$$

implying that $8(y^2-1)$ is a perfect square. Thus, y is odd and hence $\frac{y^2-1}{8}$ is a perfect square. Since $\frac{y-1}{2}$ and $\frac{y+1}{2}$ are consecutive integers, it follows that $\frac{y^2-1}{8}$ is a square triangular number. That is, it is the square of a balancing number, say

$$\frac{y^2 - 1}{8} = B_m^2$$

and hence $y^2 = 8B_m^2 + 1$ for some m, so that $y = \pm C_m$. Consequently, $(x + C_n y)^2 = 8B_n^2(y^2 - 1)$ is equivalent to $x + C_m C_n = \pm 8B_m B_n$ if $y = C_m$, and $x - C_m C_n = \pm 8B_m B_n$ if $y = -C_m$. Thus, the totality of solutions of (4.2) is given by (4.6) for $m, n \in \mathbb{Z}$. □

In the following theorem, we consider the Diophantine equation (4.3) which may be considered as a generalization of the Diophantine equation discussed in Theorem 4.1.

Theorem 4.4. *All solutions of the Diophantine equation* (4.3) *are given by*

$$(x, y) = (B_{m-n}, B_m), (-B_{m+n}, -B_m), (-B_{m-n}, -B_m), (C_{m+n}, -B_m) \tag{4.7}$$

for $m, n \in \mathbb{Z}$.

Proof. The equation (4.3) can be rewritten as

$$(x + C_n y)^2 = (C_n^2 - 1)(y^2 - 1) = B_n^2(8y^2 + 1),$$

which suggests that $8y^2 + 1$ is a perfect square. Thus, $y = \pm B_m$ (as usual $y = -B_m$ is equivalent to $y = B_{-m}$) and hence $8y^2 + 1 = C_m^2$. Now x can be obtained from $x + C_y = \pm C_m B_n$ and therefore, the totality of solutions is given by (4.7) for $m, n \in \mathbb{Z}$. □

In the remaining theorems, we present some Diophantine equations where the proofs require certain divisibility properties of balancing and Lucas-balancing numbers.

Theorem 4.5. *The solutions of the Diophantine equation*

$$x^2 + 2C_n xy + y^2 = 1 \tag{4.8}$$

are given by

$$(x, y) = \left(\frac{-B_{(k+1)n}}{B_n}, \frac{B_{nk}}{B_n}\right), \left(\frac{-B_{(k-1)n}}{B_n}, \frac{B_{nk}}{B_n}\right) \quad (m, n \in \mathbb{Z}). \tag{4.9}$$

Proof. The equation (4.8) can be rewritten as

$$(x + C_n y)^2 = (C_n^2 - 1)(y^2 - 1) = 8B_n^2 y^2 + 1,$$

which suggests that $B_n y$ is a balancing number. Letting $B_n y = B_m$, we have $y = \frac{B_m}{B_n}$, and since y is an integer, it follows that B_n divides B_m. Hence by Theorem 2.8 in [7] (see also [4]), n divides m. Thus, $m = nk$ for some integer k and $y = \frac{B_{nk}}{B_n}$. Further,

$$(x + C_n y)^2 = 8B_n^2 y^2 + 1 = 8B_{nk}^2 + 1 = C_{nk}^2,$$

and hence $x + C_n y = \pm C_{nk}$. It follows that,

$$x = -\frac{C_n B_{nk}}{B_n} \pm C_{nk} = \frac{-B_{(k+1)n}}{B_n}, \ \frac{-B_{(k-1)n}}{B_n}.$$

Thus, the totality of solutions of (4.8) is given by (4.9). □

The following theorem that resembles Theorem 4.2 deals with a Diophantine equation whose proof requires conditions under which a Lucas-balancing numbers divides balancing and Lucas-balancing numbers.

Theorem 4.6. *The solutions of the Diophantine equation* $x^2 + 16B_nxy - 8y^2 = 1$ *are given by*

$$(x, y) = \left(\frac{-C_{(k+1)n}}{C_n}, \frac{B_{(2k+1)n}}{C_n}\right), \left(\frac{C_{2kn}}{C_n}, \frac{B_{(2k+1)n}}{B_n}\right) \quad (m, n \in \mathbb{Z}). \quad (4.10)$$

Proof. The equation $x^2 + 16B_nxy - 8y^2 = 1$ can be rewritten as

$$(x + 8B_ny)^2 = 8C_n^2y^2 + 1,$$

implying that $8C_n^2y^2 + 1$ is a perfect square and C_ny is a balancing number, say $C_ny = B_m$, hence C_n divides B_m. It is easy to see that this is possible if and only if m is an even multiple of n, and hence $m = 2kn$ for some integer k. Thus, $y = \frac{B_{2kn}}{C_n}$. Further,

$$(x + 8B_ny)^2 = 8B_{2kn}^2 + 1 = C_{2kn}^2,$$

and hence

$$x = -8B_ny \pm C_{2kn} = \frac{-8B_nB_{2kn}}{C_n} \pm C_{2kn}.$$

Therefore, the totality of solutions is given by (4.10). □

Lastly, we present a theorem which is a variant of Theorems 3.1 and 4.5.

Theorem 4.7. *The solutions of the Diophantine equation*

$$x^2 - 2C_nxy + y^2 = -8B_n^2 \quad (4.11)$$

are given by

$$(x, y) = (C_{m-n}, C_m), (C_{m+n}, C_m), (-C_{m+n}, -C_m), (-C_{m-n}, -C_m) \quad (4.12)$$

for $m, n \in \mathbb{Z}$.

Proof. The equation (4.11) can be rewritten as $(x - C_ny)^2 = 8B_n^2(y^2 - 1)$, which suggests that $8(y^2 - 1)$ is a perfect square. Hence y is odd and

$$8(y^2 - 1) = 64 \cdot \frac{1}{2} \cdot \frac{y-1}{2} \cdot \frac{y+1}{2}.$$

Since $\frac{y-1}{2}$ and $\frac{y+1}{2}$ are consecutive integers, it follows that $\frac{1}{2} \cdot \frac{y-1}{2} \cdot \frac{y+1}{2}$ is a square triangular number and hence is equal to the square of a balancing number (see [2]), say,

$$\frac{1}{2} \cdot \frac{y-1}{2} \cdot \frac{y+1}{2} = B_m^2$$

for some m and we have $8(y^2 - 1) = 64B_m^2$. Thus, $y^2 = 8B_m^2 + 1 = C_m^2$ implying that $y = \pm C_m$, and the equation $(x - C_ny)^2 = 8B_n^2(y^2 - 1)$ is reduced to $(x - C_n y)^2 = 64B_m^2 B_n^2$. Therefore, $x - C_n y = \pm 8B_m B_n$ and the solutions of the equation (4.11) are given by (4.12) for $m, n \in \mathbb{Z}$. $\square$

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Veer Surendra Sai University of Technology, Odisha, Burla-768018, India
E-mail address: `rayprasanta2008@gmail.com`