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# A note on a new unique range set with truncated multiplicity

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Abstract. We introduce a new polynomial whose zero set forms a unique range set for meromorphic function with 11 elements under relaxed sharing hypothesis.

## 1. Introduction and definitions

Throughout the paper, $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. By a meromorphic function we shall always mean a meromorphic function in the complex plane $\mathbb{C}$. We adopt the usual notation of Nevanlinna theory as explained in [10]. By $E$ and $I$ we denote any set of finite and infinite linear measure, respectively. For any non-constant meromorphic function $h(z)$, we define $S(r,h)$ by $S(r,h) = o(T(r,h))$ where $r \to \infty,\ r \notin E$.

Let $f$ be a non-constant meromorphic function, let $a \in \overline{\mathbb{C}}$, and let $p$ be a positive integer. We denote by $E(a,f)$ the set of zeros of $f(z)-a$ (counting multiplicity) and by $E_{p)}(a,f)$ the set of zeros of $f(z)-a$ with multiplicity $\leq p$ (counting multiplicity).

Let $S \subset \mathbb{C}$. Set

$$E(S,f) = \bigcup_{a\in S} E(a,f), \quad E_{p)}(S,f) = \bigcup_{a\in S} E_{p)}(a,f).$$

Then for two non-constant meromorphic functions $f$ and $g$ we say that $f, g$ share the set $S$ truncated $p$ if $E_{p)}(S,f) = E_{p)}(S,g)$. Obviously the condition $E_{p)}(S,f) = E_{p)}(S,g)$ implies $E_{j)}(S,f) = E_{j)}(S,g)$ for all $1 \leq j \leq p$.

The inception of set sharing problem in the realm of the theory of meromorphic function was due to the famous "Gross Question" (see [8]) which is as follows.

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**Question A.** Can one find two (or possibly even one) finite sets $S_j$ ($j = 1,\ 2$) such that any two non-constant entire functions $f$ and $g$ satisfying $E(S_j, f) = E(S_j, g)$ for $j = 1,\ 2$ must be identical?

Gradually the research to find the possible answer of *Question A* corresponding to meromorphic functions has become one of the most prominent branches of the uniqueness theory. Later on many analogous questions were raised by many researchers pertinent to their investigations. It is needless to say that investigations for possible answers of these questions have enriched the uniqueness theory vis-à-vis the value distribution theory. Meanwhile, Gross and Yang [9] (see also [13]) introduced the new idea of a unique range set for meromorphic functions (URSM, in brief) in the following manner.

**Definition 1.1** (see [9])**.** A set $S$ such that for any two non-constant entire (meromorphic) functions $f$ and $g$ the condition $E(S, f) = E(S, g)$ implies $f \equiv g$ is called a unique range set of entire (meromorphic) functions. We call it URSE (URSM) in short.

Recently the definition of unique range sets have been generalized in [4] as follows.

**Definition 1.2** (see [4])**.** A set $S$ is called a $\mathrm{URSM}_{p)}$ ($\mathrm{URSE}_{p)}$) if for any two non-constant meromorphic (entire) functions $f$ and $g$ the equality $E_{p)}(S, f) = E_{p)}(S, g)$ implies $f \equiv g$.

Relevant to definition 1.1, Yi [17] introduced in 1996 a URSM with 13 elements. Two years later Frank and Reinders [6] introduced another URSM with 11 elements. Till date we have another two URSM's, one by Banerjee [3] and the other one by Alzahary [1], and both of these sets are of cardinality 11. In this paper we introduce an another $\mathrm{URSM}_{3)}$ with 11 elements. Throughout the paper we shall denote by $P(z)$ the following polynomial:

$$P(z) = z^n + 2n(n-2m)z^{n-m} + n(n-2m)(n-m)^2 z^{n-2m} + c, \qquad (1.1)$$

where $n, m \in \mathbb{N}$, $\gcd(m, n) = 1$, $c \in \mathbb{C}$ are such that $P(z)$ has no multiple zero, and

$$\beta_i = -\left(c_i^n + 2n(n-2m)c_i^{n-m} + n(n-2m)(n-m)^2 c_i^{n-2m}\right),$$

where $c_i$ are the roots of the equation $z^m + (n-m)(n-2m) = 0$ for $i = 1, 2, \ldots, m$.

The following theorem is the main result of this paper.

**Theorem 1.1.** *Let $S = \{z : P(z) = 0\}$. If one of the conditions*

(i) $p \geq 3$ *for* $n \geq 2m + 9$ $(n \geq 2m + 5)$,
(ii) $p = 2$ *for* $n \geq 2m + 10$ $(n \geq 2m + 5)$,
(iii) $p = 1$ *for* $n \geq 2m + 13$ $(n \geq 2m + 7)$

*holds, then $S$ is a $URSM_{p)}$ ($URSE_{p)}$).*

We have already mentioned that readers are referred to go through [10] for standard notation and definitions of the value distribution theory but below we explain some notation which is frequently used in the paper.

**Definition 1.3** (see [11]). For $a \in \mathbb{C} \cup \{\infty\}$, we denote by $N(r, a; f \mid= 1)$ the counting function of simple $a$-points of $f$. For a positive integer $m$, we denote by $N(r, a; f \mid\leq p)$ $(N(r, a; f \mid\geq p))$ the counting function of those $a$-points of $f$ whose multiplicities are not greater (less) than $p$, where each $a$-point is counted according to its multiplicity.

$\overline{N}(r, a; f \mid\leq p)$ $(\overline{N}(r, a; f \mid\geq p))$ are defined similarly, where in counting the $a$-points of $f$ we ignore the multiplicities.

Also $N(r, a; f \mid< p)$, $N(r, a; f \mid> p)$, $\overline{N}(r, a; f \mid< p)$, and $\overline{N}(r, a; f \mid> p)$ are defined analogously.

**Definition 1.4** (see [5]). Let $f$ and $g$ be two non-constant meromorphic functions and let $p$ be a positive integer such that $E_{p)}(a; f) = E_{p)}(a; g)$, where $a \in \mathbb{C} \cup \{\infty\}$. Let $z_0$ be an $a$-point of $f$ with multiplicity $s > 0$, or an $a$-point of $g$ with multiplicity $q > 0$. We denote by $\overline{N}^{p)}_L(r, a; f)$ $(\overline{N}^{p)}_L(r, a; g))$ the counting function of those $a$-points of $f$ and $g$, where $s > q$ $(q > s)$ and each $a$-point is counted only once.

**Definition 1.5** (see [5]). Let $p$ be a positive integer and, for $a \in \mathbb{C}$, let $E_{p)}(a; f) = E_{p)}(a; g)$. Let $z_0$ be a zero of $f(z) - a$ of multiplicity $s$ (respectively, a zero of $g(z) - a$ of multiplicity $q$). We denote by $\overline{N}_{f \geq p+1}(r, a; f \mid g \neq a)$ $(\overline{N}_{g \geq p+1}(r, a; g \mid f \neq a))$ the reduced counting functions of those $a$-points of $f$ and $g$ for which $s \geq p + 1$ and $q = 0$ $(q \geq p + 1$ and $s = 0)$.

**Definition 1.6** (see [5]). For $E_{p)}(1; f) = E_{p)}(1; g)$, let $z_0$ be a zero of $f(z) - 1$ with multiplicity $s(\geq 0)$ and a zero of $g(z) - 1$ with multiplicity $q(\geq 0)$. We denote by $\overline{N}_{\otimes}(r, 1; f, g)$ the reduced counting function of 1 points of $f$ and $g$ with $s \neq q$.

Clearly, we have

$$\begin{aligned}\overline{N}_{\otimes}(r, a; f, g) &= \overline{N}^{p)}_L(r, a; f) + \overline{N}^{p)}_L(r, a; g) \\ &\quad + \overline{N}_{f \geq p+1}(r, a; f \mid g \neq a) + \overline{N}_{g \geq p+1}(r, a; g \mid f \neq a) \\ &\leq \overline{N}(r, a; f \mid\geq p + 1) + \overline{N}(r, a; g \mid\geq p + 1).\end{aligned}$$

**Definition 1.7** (see [12]). Let $a, b \in \mathbb{C} \cup \{\infty\}$. We denote by $N(r, a; f \mid g = b)$ the counting function of those $a$-points of $f$, counted according to multiplicity, which are $b$-points of $g$.

**Definition 1.8** (see [12]). Let $a, b_1, b_2, \ldots, b_q \in \mathbb{C} \cup \{\infty\}$. We denote by $N(r, a; f \mid g \neq b_1, b_2, \ldots, b_q)$ the counting function of those $a$-points of $f$, counted according to multiplicity, which are not the $b_i$-points of $g$ for $i = 1, 2, \ldots, q$.

## 2. Lemmas

In this section we present some lemmas which will be needed in the sequel. Let $F$ and $G$ be two non-constant meromorphic functions defined in $\mathbb{C}$ by

$$F = \frac{f^{n-2m}(f^{2m} + 2n(n-2m)f^m + n(n-2m)(n-m)^2)}{-c}, \tag{2.1}$$

$$G = \frac{g^{n-2m}(g^{2m} + n(n-2m)g^m + n(n-2m)(n-m)^2)}{-c}, \tag{2.2}$$

and let $H$ be the function

$$H = \left(\frac{F''}{F'} - \frac{2F'}{F-1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1}\right). \tag{2.3}$$

**Lemma 2.1** (see [14]). *If $E_{p)}(1;F) = E_{p)}(1;G)$ and $H \not\equiv 0$, then*

$$N(r,1;F \mid= 1) = N(r,1;G \mid= 1) \leq N(r,H) + S(r,F) + S(r,G).$$

**Lemma 2.2** (see [15]). *Let $f$ be a non-constant meromorphic function and let*

$$R(f) = \sum_{k=0}^{n} a_k f^k \Big/ \sum_{j=0}^{m} b_j f^j$$

*be an irreducible rational function in $f$ with constant coefficients $\{a_k\}$ and $\{b_j\}$, where $a_n \neq 0$ and $b_m \neq 0$. Then*

$$T(r,R(f)) = dT(r,f) + S(r,f)$$

*with $d = \max\{n,m\}$.*

**Lemma 2.3.** *Let $f$ and $g$ be two non-constant meromorphic functions such that $E_{p)}(S,f) = E_{p)}(S,g)$, and let $F$, $G$ be given by (2.1) and (2.2) with $H \not\equiv 0$. Then*

$$\begin{aligned} N(r,H) \leq\ & \overline{N}(r,0;f) + \overline{N}(r,0;g) + \overline{N}(r,f) + \overline{N}(r,g) \\ & + \overline{N}(r,0;f^m + (n-m)(n-2m)) + \overline{N}(r,0;g^m + (n-m)(n-2m)) \\ & + N_{\otimes}(r,1;F,G) + \overline{N}_0(r,0;f') + \overline{N}_0(r,0;g'), \end{aligned}$$

*where $\overline{N}_0(r,0;f')$ is the reduced counting function of those zeros of $f'$ which are not the zeros of $f(f^m + (n-m)(n-2m))(F-1)$, and $\overline{N}_0(r,0;g')$ is similarly defined.*

*Proof.* Since $E_{p)}(S,f) = E_{p)}(S,g)$, from (2.1) and (2.2) we have $E_{p)}(1,F) = E_{p)}(1,G)$. Also

$$F' = \frac{nf^{n-2m-1}(f^m + (n-m)(n-2m))^2 f'}{-c},$$

$$G' = \frac{ng^{n-2m-1}(g^m + (n-m)(n-2m))^2 g'}{-c}.$$

Hence the result is obvious from equation (2.3). □

**Lemma 2.4** (see [5]). *Let $f$ and $g$ be two non-constant meromorphic functions such that $E_{p)}(1;f) = E_{p)}(1;g)$, where $1 \leq p < \infty$. Then*

$$\begin{aligned}
&\overline{N}(r,1;f) + \overline{N}(r,1;g) - N(r,1;f\mid=1)\\
&\quad + \left(\frac{p}{2} - \frac{1}{2}\right)\left\{\overline{N}_{f\geq p+1}(r,1;f \mid g \neq 1) + \overline{N}_{g\geq p+1}(r,1;g \mid f \neq 1)\right\}\\
&\quad + \left(p - \frac{1}{2}\right)\left\{\overline{N}_{L}^{p)}(r,1;f) + \overline{N}_{L}^{p)}(r,1;g)\right\}\\
&\leq \frac{1}{2}\left[N(r,1;f) + N(r,1;g)\right].
\end{aligned}$$

**Lemma 2.5.** *Let $F$, $G$ be given by (2.1). If $E_{p)}(1;F) = E_{p)}(1;G)$ and $\omega_1, \omega_2, \ldots, \omega_n$ are the distinct roots of the equation*

$$z^n + n(n-2m)z^{n-1} + bn(n-2m)(n-m)^2 + c = 0$$

*for $n \geq 3$. Then*

$$\begin{aligned}
\overline{N}_{\otimes}(r,1;F,G) \leq \frac{1}{p}\big[&\overline{N}(r,0;f) + \overline{N}(r,0;g) + \overline{N}(r,\infty;f) + \overline{N}(r,\infty;g)\\
&-N_{\oplus}(r,0;f^{'}) - N_{\oplus}(r,0;g^{'})\Big] + S(r,f) + S(r,g),
\end{aligned}$$

*where*

$$N_{\oplus}(r,0;f^{'}) = N(r,0;f^{'} \mid f \neq 0, \omega_1, \omega_2 \ldots \omega_n)$$

*and $N_{\oplus}(r,0;g^{'})$ is similarly defined.*

*Proof.* Since

$$\begin{aligned}
\overline{N}_{\otimes}(r,1;F,G) &\leq \overline{N}(r,1;F\mid\geq p+1) + \overline{N}(r,1;F\mid\geq p+1)\\
&\leq \frac{1}{p}\left[N(r,1;F) - \overline{N}(r,1;F) + N(r,1;G) - \overline{N}(r,1;G)\right],
\end{aligned}$$

the proof of the lemma can be carried out along the lines of the proof of Lemma 2.14 in [2]. □

**Lemma 2.6.** *Let $F$, $G$ be given by (2.1). If $E_{p)}(1;F) = E_{p)}(1;G)$, then*

$$\begin{aligned}
&(n+m)T(r,f) + (n+m)T(r,g)\\
&\quad\leq 2\overline{N}(r,0;f) + 2\overline{N}(r,\infty;f) + 2\overline{N}(r,0;f^m + (n-m)(n-2m))\\
&\qquad + 2\overline{N}(r,0;g) + 2\overline{N}(r,0;g^m + (n-m)(n-2m)) + 2\overline{N}(r,\infty;g)\\
&\qquad - \left(\frac{p}{2} - \frac{1}{2}\right)\left\{\overline{N}_{f\geq p+1}(r,1;f \mid g \neq 1) + \overline{N}_{g\geq p+1}(r,1;g \mid f \neq 1)\right\}\\
&\qquad - \left(p - \frac{1}{2}\right)\left\{\overline{N}_{L}^{p)}(r,1;f) + \overline{N}_{L}^{p)}(r,1;g)\right\} + \overline{N}_{\otimes}(r,1;F,G)\\
&\qquad + \frac{n}{2}[T(r,f) + T(r,g)] + S(r,f) + S(r,g).
\end{aligned}$$

*Proof.* By the second fundamental theorem, we have

$$\begin{aligned}(n+m)T(r,f)+(n+m)T(r,g)&\\ \leq \overline{N}(r,0;F-1)+\overline{N}(r,0;f)+\overline{N}(r,\infty;f)&\\ +\overline{N}(r,0;f^m+(n-m)(n-2m))+\overline{N}(r,0;G-1)+\overline{N}(r,0;g)&\\ +\overline{N}(r,0;g^m+(n-m)(n-2m))+\overline{N}(r,\infty;g)&\\ -\overline{N_0}(r,0;f^{'})-\overline{N_0}(r,0;g^{'})+S(r,f)+S(r,g).&\end{aligned}$$

Now the result immediately follows from Lemmas 2.1 – 2.4. □

**Lemma 2.7.** *Let $P(z)$ be defined by* (1.1)*. Then*

(i) $\beta_i \neq 0$,
(ii) $P(z)$ *is a critically injective polynomial for* $n > 2$.

*Proof.* Clearly we have

$$P^{'}(z) = nz^{n-2m-1}(z^m+(n-m)(n-2m))^2.$$

Let $c_i$ to be the roots of the equation $z^m+(n-m)(n-2m)=0$ for $i = 1,2,\ldots,m$, i.e., $c_i^m = -(n-m)(n-2m)$. Therefore,

$$c_i^{2m}+2n(n-2m)c_i^m+n(n-2m)(n-m)^2 = 2(n-m)(n-2m)m^2.$$

(i) Since $\gcd(m,n)=1$ and $n>2$, we have

$$\begin{aligned}\beta_i &= -\left(c_i^n+2n(n-2m)c_i^{n-m}+n(n-2m)(n-m)^2c_i^{n-2m}\right)\\ &= -2c_i^{(n-2m)}(n-m)(n-2m)m^2 \neq 0.\end{aligned}$$

(ii) We have

$$\begin{aligned}P(c_i) &= c_i^{(n-2m)}(c_i^{2m}+2n(n-2m)c_i^m+n(n-2m)(n-m)^2)+c\\ &= 2c_i^{(n-2m)}(n-m)(n-2m)m^2+c.\end{aligned}$$

So it is obvious that when $c_i \neq c_j$, $P(c_i)=P(c_j)$ implies

$$c_i^{(n-2m)} = c_j^{(n-2m)}. \tag{2.4}$$

Since $c_i^m = -(n-m)(n-2m) = c_j^m$, the equality (2.4) implies $c_i^n = c_j^n$, i.e., $(c_i/c_j)^n = 1$. Also from $c_i^m = c_j^m$, we have $(c_i/c_j)^m = 1$. Since $\gcd(m,n)=1$, we get $c_i/c_j = 1$. Thus $c_i = c_j$ which is a contradiction. Hence $P(c_i) \neq P(c_j)$ for $c_i \neq c_j$. Also,

$$P(0) = c \neq 2c_i^{(n-2m)}(n-m)(n-2m)m^2+c = P(c_i).$$

Thus $P(z)$ is critically injective. □

**Lemma 2.8** (see [7]). *Suppose that $P(z)$ is a monic polynomial without multiple zeros, whose derivatives have mutually distinct $k$ zeros given by $d_1, d_2, \ldots, d_k$ with multiplicities $q_1, q_2, \ldots, q_k$, respectively. Also suppose that $P(z)$ is critically injective. Then $P(z)$ is a uniqueness polynomial if and only if*

$$\sum_{1\le l<m\le k} q_l q_m > \sum_{l=1}^{k} q_l.$$

*In particular, the above inequality is always satisfied whenever $k \ge 4$. When $k = 3$ and $\max\{q_1, q_2, q_3\} \ge 2$, or when $k = 2$, $\min\{q_1, q_2\} \ge 2$, and $q_1+q_2 \ge 5$, the above inequality also holds.*

**Lemma 2.9.** *Let $F$, $G$ be defined by* (2.1) *and* (2.2). *Then $F \equiv G$ implies $f \equiv g$ for $n \ge 2m + 4$.*

*Proof.* Since $F \equiv G$, we have $P(f) \equiv P(g)$. By Lemma 2.7 we know that $P(z)$ is critically injective. Also, we have

$$P^{'}(z) = nz^{n-2m-1}(z^m + (n-m)(n-2m))^2,$$

which implies $k = m + 1$. Since $n - 2m - 1 \ge 3$, by Lemma 2.8 we get that $P(z)$ is a UPM and hence $f \equiv g$. □

## 3. Proof of Theorem 1.1

*Proof of Theorem 1.1.* Let us consider the following cases.

**Case 1.** Let $H \not\equiv 0$.

Let $f$ and $g$ be two non-constant meromorphic functions such that $E_{p)}(S, f) = E_{p)}(S, g)$. Then from (2.1) and (2.2) we get $E_{p)}(1, F) = E_{p)}(1, G)$. Now we consider the following subcases.

**Subcase 1.1.** Let $p \ge 3$. Then, using Lemma 2.6 for $p \ge 3$, we get

$$\begin{aligned}
&(n+m)T(r,f) + (n+m)T(r,g)\\
&\quad\le 2\overline{N}(r,0;f) + 2\overline{N}(r,0;g) + 2\overline{N}(r,f) + 2\overline{N}(r,g)\\
&\qquad + 2\overline{N}(r,0;f^m + (n-m)(n-2m)) + 2\overline{N}(r,0;g^m + (n-m)(n-2m))\\
&\qquad + \frac{n}{2}(T(r,f) + T(r,g) + S(r,f) + S(r,g)
\end{aligned}$$

which is a contradiction for $n \ge 2m + 9$.

If $f$ and $g$ are entire functions, then setting $\overline{N}(r,\infty;f) = \overline{N}(r,\infty;g) = 0$, we get a contradiction for $n \ge 2m + 5$.

**Subcase 1.2.** Let $p = 2$. Then, by Lemmas 2.6 and 2.5, we have

$$\begin{aligned}
&(n+m)T(r,f) + (n+m)T(r,g)\\
&\quad\le 2\overline{N}(r,0;f) + 2\overline{N}(r,f) + 2\overline{N}(r,0;f^m + (n-m)(n-2m))\\
&\qquad + 2\overline{N}(r,0;g) + 2\overline{N}(r,0;g^m + (n-m)(n-2m)) + 2\overline{N}(r,g)
\end{aligned}$$

$$
\begin{aligned}
&- \frac{1}{2}\left\{\overline{N}_{f\geq 3}(r,1;f \mid g\neq 1) + \overline{N}_{g\geq 3}(r,1;g \mid f\neq 1)\right\}\\
&- \frac{3}{2}\left\{\overline{N}_{L}^{2)}(r,1;f) + \overline{N}_{L}^{2)}(r,1;g)\right\} + \overline{N}_{\oplus}(r,1;F,G)\\
&+ \frac{n}{2}[T(r,f)+T(r,g)] + S(r,f) + S(r,g)\\
\leq\ & (2m+4+\frac{n}{2})[T(r,f)+T(r,g)] + \frac{1}{2}\overline{N}_{\oplus}(r,1;F,G) + S(r,f) + S(r,g)\\
\leq\ & (2m+4+\frac{n}{2}+\frac{1}{2})[T(r,f)+T(r,g)] + S(r,f) + S(r,g),
\end{aligned}
$$

which is a contradiction for $n \geq 2m+10$.

If $f$ and $g$ are entire functions, then setting $\overline{N}(r,\infty;f) = \overline{N}(r,\infty;g) = 0$ we get a contradiction for $n \geq 2m+5$.

**Subcase 1.3.** Let $p = 1$. Then, again from Lemmas 2.6 and 2.5, we have

$$
\begin{aligned}
&(n+m)T(r,f) + (n+m)T(r,g)\\
&\leq \overline{N}(r,0;f) + 2\overline{N}(r,f) + 2\overline{N}(r,0;f^m + (n-m)(n-2m))\\
&\quad + 2\overline{N}(r,0;g) + 2\overline{N}(r,0;g^m + (n-m)(n-2m)) + 2\overline{N}(r,g)\\
&\quad - \frac{1}{2}\left\{\overline{N}_{L}^{1)}(r,1;f) + \overline{N}_{L}^{1)}(r,1;g)\right\} + \overline{N}_{\oplus}(r,1;F,G)\\
&\quad + \frac{n}{2}[T(r,f)+T(r,g)] + S(r,f) + S(r,g)\\
&\leq (2m+4+\frac{n}{2})[T(r,f)+T(r,g)] + \overline{N}_{\oplus}(r,1;F,G) + S(r,f) + S(r,g)\\
&\leq (2m+4+\frac{n}{2}+2)[T(r,f)+T(r,g)] + S(r,f) + S(r,g),
\end{aligned}
$$

which is a contradiction for $n \geq 2m+13$.

If $f$ and $g$ are entire functions, then setting $\overline{N}(r,\infty;f) = \overline{N}(r,\infty;g) = 0$, we get a contradiction for $n \geq 2m+7$.

**Case 2.** Let $H \equiv 0$. Then from (2.3) we get

$$\frac{1}{F-1} \equiv \frac{A}{G-1} + B, \tag{3.1}$$

where $A(\neq 0)$ and $B$ are two constants. So in view of Lemma 2.2, from (3.1) we get

$$T(r,f) = T(r,g) + O(1). \tag{3.2}$$

**Subcase 2.1.** Suppose that $B \neq 0$. Then from (3.1) we get

$$F - 1 \equiv \frac{G-1}{BG + A - B}. \tag{3.3}$$

**Subcase 2.1.1.** If $A - B \neq 0$, then noting that $\frac{B-A}{B} \neq 1$, from (3.3) we get

$$\overline{N}(r, \frac{B-A}{B}; G) = \overline{N}(r,\infty;F).$$

Therefore, in view of Lemma 2.2 and equation (3.2), using the second fundamental theorem, we have

$$\begin{aligned} nT(r,g) &\leq \overline{N}(r,0;G)+\overline{N}(r,\infty;G)+\overline{N}(r,\frac{B-A}{B};G)+S(r,G)\\ &\leq (2m+1)T(r,g)+\overline{N}(r,\infty;g)+\overline{N}(r,\infty;f)+S(r,g), \end{aligned}$$

which is a contradiction for $n \geq 2m+4$.

**Subcase 2.1.2.** If $A-B=0$, then from (3.3) we have

$$\begin{aligned} \frac{G-1}{F-1} &\equiv BG\\ &= B\frac{g^{n-2m}(g^{2m}+2n(n-2m)g^m+n(n-2m)(n-m)^2}{-c}, \end{aligned} \tag{3.4}$$

i.e., 0's of $g$ and $(g^{2m}+2n(n-2m)g^m+n(n-2m)(n-m)^2)$ are poles of $F$. It can be easily proved that all the zeros $\xi_i$, $i \in \{1,2,\ldots,2m\}$, of $w^{2m}+2n(n-2m)w^m+n(n-2m)(n-m)^2$ are simple. If each $\xi_i$-point of $g$ is of multiplicity $p$, then it is a pole of $F$ of multiplicity $q$ for some $q \geq 1$. Thus from (3.4) we get $p=nq$, i.e., $p \geq n$. Similarly any zero of $g$ of multiplicity $r$ it is a pole of $F$ of multiplicity $s$ for some $s \geq 1$, i.e., $r(n-2m)=sn$. Hence $r=\frac{sn}{n-2m} \geq 2$ as $n > n-2m$. Now, using the second fundamental theorem, we get

$$\begin{aligned} (2m-1)T(r,g) &\leq \sum_{i=1}^{2m}\overline{N}(r,\xi_i;g)+\overline{N}(r,0;g)+S(r,g)\\ &\leq (\frac{2m}{n}+\frac{1}{2})T(r,g)+S(r,g), \end{aligned}$$

which is a contradiction for $n \geq 2m+3$.

**Subcase 2.2.** Suppose that $B=0$. Then from (3.1) we get

$$G-1=A(F-1),$$

i.e.,

$$\begin{aligned} &g^n+2n(n-2m)g^{n-m}+n(n-2m)(n-m)^2g^{n-2m}\\ &\equiv A\left(f^n+2n(n-2m)f^{n-m}+n(n-2m)(n-m)^2f^{n-2m}+c\frac{A-1}{A}\right). \end{aligned} \tag{3.5}$$

Now we consider the following subcases.

**Subcase 2.2.1.** Let $A \neq 1$. Since $P(z)$ has no multiple zeros, $c \neq 0$. Hence $c\frac{A-1}{A} \neq 0$ and, as $\beta_i \neq 0$, we consider the following subcases.

**Subcase 2.2.1.1.** Suppose that

$$c\frac{A-1}{A}=\beta_i.$$

From Lemma 2.3, we know that

$$F' = n\frac{f^{n-2m-1}(f^m + (n-m)(n-2m))^2}{-c}f'.$$

Also by Lemma 2.7, we get $\beta_i \neq 0$ and that $P(z)$ is critically injective. Since any critically injective polynomial can have at most one multiple zero, we have

$$f^n + 2n(n-2m)f^{n-m} + n(n-2m)(n-m)^2 f^{n-2m} + \beta_i = (f-c_i)^3 \prod_{j=1}^{n-3}(f-\eta_j),$$

where $\eta_j$'s are $(n-3)$ distinct zeros of $z^n + 2n(n-2m)z^{n-m} + n(n-2m)(n-m)^2 z^{n-2m} + \beta_i$ such that $\eta_j \neq c_i, 0$. Then from (3.5), we get

$$\begin{aligned} &g^{n-2m}(g^{2m} + 2n(n-2m)g^m + n(n-2m)(n-m)^2) \\ &\quad = A(f-c_i)^3 \prod_{j=1}^{n-3}(f-\eta_j). \end{aligned}$$

Therefore, using the second fundamental theorem and (3.2), we get

$$\begin{aligned} (n-3)T(r,f) &\leq \sum_{j=1}^{n-3}\overline{N}(r,\eta_j;f) + \overline{N}(r,c_i;f) + \overline{N}(r,\infty;f) + S(r,f) \\ &\leq (2m+1)T(r,g) + T(r,f) + S(r,f), \end{aligned}$$

which is a contradiction for $n \geq 2m+6$.

**Subcase 2.2.1.2.** Suppose that

$$c\frac{(A-1)}{A} \neq \beta_i$$

for all $i \in \{1, 2, \ldots, m\}$. So,

$$w^n + 2n(n-2m)w^{n-m} + n(n-2m)(n-m)^2 w^{n-2m} + c\frac{(A-1)}{A} = 0$$

has only simple roots, say $\alpha_i$ for $i = 1, 2, \ldots, n$. Therefore, from (3.5) we have

$$g^{n-2m}(g^{2m} + ag^m + b) \equiv A\prod_{i=1}^{n}(f-\alpha_i).$$

Again, applying (3.2) and the second fundamental theorem, we get

$$\begin{aligned} (n-2)T(r,f) &\leq \sum_{i=1}^{n}\overline{N}(r,\alpha_i;f) + S(r,f) \\ &\leq (2m+1)T(r,g) + S(r,f), \end{aligned}$$

which is a contradiction for $n \geq 2m+4$.

**Subcase 2.2.2.** If $A = 1$, then we have $F \equiv G$. Therefore, by Lemma 2.9, we have $f \equiv g$ for $n \geq 2m+4$.

□

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## References

[1] T. C. Alzahary, *Meromorphic functions with weighted sharing of one set*, Kyungpook Math. J. **47** (2007), 57–68.
[2] A. Banerjee, *Some uniqueness results on meromorphic functions sharing three sets*, Ann. Polon. Math. **92**(3) (2007), 261–274.
[3] A. Banerjee, *A new class of strong uniqueness polynomials satisfying Fujimoto's condition*, Annal. Acad. Sci. Fenn. Math. **40** (2015), 465-474.
[4] A. Banerjee and I. Lahiri, *A uniqueness polynomial generating a unique range set and vise versa*, Comput. Method Funct. Theory **12**(2) (2012), 527–539.
[5] A. Banerjee and S. Mukherjee, *Some uniqueness results on meromorphic functions sharing two or three sets*, Sarajevo J. Math. **5**(18) (2009), 169–189
[6] G. Frank and M. Reinders, *A unique range set for meromorphic functions with 11 elements*, Complex Var. Theory Appl. **37**(1) (1998), 185–193.
[7] H. Fujimoto, *On uniqueness of meromorphic functions sharing finite sets*, Amer. J. Math. **122** (2000), 1175–1203.
[8] F. Gross, *Factorization of meromorphic functions and some open problems*, in: Proc. Conf. Univ. Kentucky, Leixngton, Ky., 1976, pp. 51–67; Lecture Notes in Mathematics **599**, Springer, Berlin, 1977.
[9] F. Gross and C. C. Yang, *On preimage and range sets of meromorphic functions*, Proc. Japan Acad. **58** (1982), 17–20.
[10] W. K. Hayman, *Meromorphic Functions*, The Clarendon Press, Oxford, 1964.
[11] I. Lahiri, *Value distribution of certain differential polynomials*, Int. J. Math. Math. Sci. **28**(2) (2001), 83–91.
[12] I. Lahiri and A. Banerjee, *Weighted sharing of two sets*, Kyungpook Math. J. **46**(1) (2006), 79–87.
[13] P. Li and C. C. Yang, *Some further results on the unique range sets for meromorphic functions*, Kodai Math. J. **18** (1995), 437–450.
[14] W. C. Lin and H. X. Yi, *Some further results on meromorphic functions that share two sets*, Kyungpook Math. J. **43** (2003), 73–85.
[15] A. Z. Mokhon'ko, *The Nevanlinna characteristics of certain meromorphic functions*, Teor. Funkciĭ Funkcional. Anal. i Prilożen. **14**(1971), 83–87. (Russian)
[16] H. X. Yi, *Meromorphic functions that share one or two values*, Complex Variables Theory Appl. **28** (1995), 1–11.
[17] H. X. Yi, *Unicity theorems for meromorphic or entire functions. III*, Bull. Austral. Math. Soc. **53** (1996), 71–82.

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