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Coincidence of topological Jacobson radicals in topological algebras

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Abstract. Several classes of topological algebras for which the left topological Jacobson radical coincides with the right topological Jacobson radical are described.

1. Introduction

The notion of a *topological* (*Jacobson*) *radical* was introduced in [3] as the intersection of kernels of all continuous irreducible representations of a topological algebra A on all Hausdorff topological linear spaces X in case when the module multiplication over A on X is commutative (that is, when $ax = xa$ for all $a \in A$ and $x \in X$). In general, two different topological Jacobson radicals exist: the left topological Jacobson radical defined by the continuous irreducible representations of A on X, and the right topological Jacobson radical defined by the continuous irreducible antirepresentations of A on X. A class of topological algebras, for which the left and right topological radicals coincide, has been described in [1]. Several new classes of topological algebras, for which the left topological Jacobson radical coincides with the right topological Jacobson radical, are given in the present paper.

Let $\mathbb{K}$ denote one of the fields $\mathbb{R}$ or $\mathbb{C}$ of real or complex numbers, respectively. An associative algebra A over $\mathbb{K}$ is a *topological algebra over* $\mathbb{K}$ (shortly, a *topological algebra*) if A is a topological linear space over $\mathbb{K}$ (with respect to the same linear operations) and the multiplication in A is separately continuous. We denote by θ_A the zero element of A.

By an ideal we always mean a proper ideal. A left (right or two-sided) ideal I of A is *regular* (sometimes also called *modular*) if there exists an element $u \in A$ such that $a - au \in I$ ($a - ua \in I$ or $a - au, a - ua \in I$, respectively) for

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all $a \in A$. Such an element u is called a *right* (*left* or *two-sided*, respectively) *regular unit for* I. It is known that $u \notin I$ (see [2]).

If M is a maximal regular left (right) ideal of A with a right (left, respectively) regular unit u, then the two-sided ideal $P = \{p \in A\colon pA \subset M\}$ ($P = \{p \in A\colon Ap \subset M\}$, respectively) is called the *primitive ideal defined by* M. If M is closed, then P is closed (see [2]).

A topological algebra A is called *left* (*right*) *simplicial* if every closed regular left (right, respectively) ideal of A is contained in a closed maximal regular left (right, respectively) ideal of A, and *simplicial* if it is both left and right simplicial.

Let A be a topological algebra (or a topological ring in general). An element r of A is *left quasi-invertible* if there is an element $p \in A$ such that $p \circ r = p + r - pr = \theta_A$. An element r of A is *left topologically quasi-invertible* if there is a net $(p_\lambda)_{\lambda\in\Lambda}$ in A such that the net $(p_\lambda \circ r)_{\lambda\in\Lambda}$ converges to θ_A. We denote by $\operatorname{Qinv}_l A$ the set of all left quasi-invertible elements of A and by $\operatorname{Tqinv}_l A$ the set of all left topologically quasi-invertible elements of A. The sets of all right quasi-invertible elements $\operatorname{Qinv}_r A$ of A and all right topologically quasi-invertible elements $\operatorname{Tqinv}_r A$ of A are defined similarily. We say that a topological algebra is *advertive* if $\operatorname{Tqinv}_r A = \operatorname{Qinv}_r A$ and $\operatorname{Tqinv}_l A = \operatorname{Qinv}_l A$.

Let A be an algebra over $\mathbb{K}$ and M a linear space over $\mathbb{K}$. The space M is called a *left* (*right*) A-module if the module multiplication $(a, m) \to am$ of $A \times M$ into M ($(m, a) \to ma$ of $M \times A$ into M, respectively) is a bilinear map which satisfies the condition $a_1(a_2m) = (a_1a_2)m$ ($(ma_1)a_2 = m(a_1a_2)$, respectively) for each $a_1, a_2 \in A$ and $m \in M$. If M' is a subspace of a left (right) A-module M and for any $m \in M'$ and $a \in A$ the product am (ma, respectively) is in M', then M' is a *left* (*right*, respectively) A*-submodule* of M. In particular, when A is a topological algebra over $\mathbb{K}$ (with separately continuous multiplication) and M is a topological linear space over $\mathbb{K}$ in which the module multiplication is separately continuous, M is called a *topological left* (*right*) A*-module.* A left (right) A-module X is *nontrivial* if $AX \neq \{\theta_X\}$ ($XA \neq \{\theta_X\}$, respectively) and X is *irreducible* if it is nontrivial and the only A-submodules of X are X and $\{\theta_X\}$.

Let X be a topological linear space over $\mathbb{K}$, and let $\mathcal{L}(X)$ be the set of all continuous linear maps on X. If we define the addition of elements and the multiplication over $\mathbb{K}$ in $\mathcal{L}(X)$ point-wise and the multiplication of elements in $\mathcal{L}(X)$ by the composition, then $\mathcal{L}(X)$ is an algebra (non-commutative in general) over $\mathbb{K}$.

Remind that an *antihomomorphism* is a map $\pi' : A \to B$ between two algebras that preserves addition and multiplication by scalars but reverses the order of multiplication, that is, $\pi'(xy) = \pi'(y)\pi'(x)$ for all $x, y \in A$. Any homomorphism π (antihomomorphism π') of an algebra A into $\mathcal{L}(X)$ is called a *representation* (*antirepresentation*, respectively) of A on X.

Every representation π (antirepresentation π') of A on X defines on X a left (right, respectively) module multiplication $\cdot_\pi$ ($\cdot_{\pi'}$, respectively) if we put $a \cdot_\pi x = \pi(a)(x)$ ($x \cdot_{\pi'} a = \pi'(a)(x)$, respectively) for each $a \in A$ and $x \in X$. In this case X becomes a left (right, respectively) A-module. We denote it by X_π ($X_{\pi'}$, respectively). A representation π (antirepresentation π', respectively) of A on X is *irreducible* if X_π ($X_{\pi'}$, respectively) is an irreducible left (right, respectively) A-module. Let $x_0 \in X$ and define $\varrho_{\pi,x_0}\colon A \to X$ by

$$\varrho_{\pi,x_0}(a) = a \cdot_\pi x_0$$

if X is a left A-module and $\varrho_{\pi',x_0}\colon A \to X$ by

$$\varrho_{\pi',x_0}(a) = x_0 \cdot_{\pi'} a$$

if X is a right A-module, for all $a \in A$. If X is a left A-module and, thus, π is a representation of A on X, then

$$\ker \varrho_{\pi,x_0} = \{a \in A\colon a \cdot_\pi x_0 = \theta_A\}$$

is a left ideal of A. If X is a right A-module and, thus, π' is an antirepresentation of A on X, then

$$\ker \varrho_{\pi',x_0} = \{a \in A\colon x_0 \cdot_{\pi'} a = \theta_A\}$$

is a right ideal of A. We say that x_0 is a *left* (*right*) *cyclic element* if $A \cdot_\pi x_0 = X$ ($x_0 \cdot_{\pi'} A = X$, respectively). Moreover, denote by

$$\mathrm{id}_r(x_0) = \{e \in A\colon e \cdot_\pi x_0 = x_0\}$$

and

$$\mathrm{id}_l(x_0) = \{e \in A\colon x_0 \cdot_{\pi'} e = x_0\}$$

the sets of one-sided units for an element x_0.

The intersection $\mathrm{Rad}(A)$ of kernels of all irreducible representations of an algebra A on linear spaces is called the *Jacobson radical of* A. If there are no irreducible representations of A, then A is called a *radical algebra* (in this case $\mathrm{Rad}(A) = A$) and if $\mathrm{Rad}(A) = \{\theta_A\}$, then A is called a *semi-simple algebra*.

We endow $\mathcal{L}(X)$ with the topology τ of simple convergence. A base of neighbourhoods of zero of τ consists of sets

$$T(S,O) = \{L \in \mathcal{L}(X) : L(S) \subset O\},$$

where S is a finite subset of X and O is a neighbourhood of zero in X. If X is a Hausdorff linear space, then $\mathcal{L}(X)$ is a Hausdorff algebra with separately continuous multiplication. We denote this algebra by $\mathcal{L}_\tau(X)$.

Let A be a topological algebra, M a closed maximal regular left (right) ideal of A, A/M the quotient space of A modulo M, and κ the canonical homomorphism from A onto A/M. It is easy to see that A/M is a Hausdorff linear space in the quotient topology if we define the addition and the multiplication over $\mathbb{K}$ in A/M as usual, and A/M is a topological left (right, respectively) A-module if we define the module multiplication in A/M over A by $ax = \kappa(aa')$ for each $a \in A$ and $x \in A/M$, where a' is an arbitrary element of the M-coset of x.

Let again A be a topological algebra. For each $a \in A$ and each closed maximal regular left (right) ideal M of A, let L_a^M (R_a^M, respectively) be the map from A/M into A/M defined by $L_a^M(x) = ax$ ($R_a^M(x) = xa$, respectively) for each $x \in A/M$, and let L_M (R_M, respectively) be the map from A into $\mathcal{L}(A/M)$ defined by $L_M(a) = L_a^M$ ($R_M(a) = R_a^M$, respectively) for each $a \in A$. Then L_M (R_M, respectively) is a representation (antirepresentation, respectively) of A on A/M, $a \cdot_{L_M} x = ax$ ($a \cdot_{R_M} x = xa$, respectively) for each $a \in A$ and $x \in A/M$, and the left (right, respectively) A-module $(A/M)_{L_M}$ ($(A/M)_{R_M}$, respectively) is irreducible (see [7], Proposition 5, page 120). Hence, L_M (R_M, respectively) is an irreducible representation (antirepresentation, respectively) of A on A/M. Thus

$$\ker L_M = \{a \in A \colon aA \subset M\} \quad (\ker R_M = \{a \in A \colon Aa \subset M\})$$

is a primitive ideal. More precisely, we call the set $\ker L_M$ ($\ker R_M$, respectively) the *primitive ideal defined by the maximal regular left* (*right*, respectively) *ideal M of A*. To show that L_M is continuous, let U be an arbitrary neighbourhood of zero in $\mathcal{L}_\tau(A/M)$. Then there exists a neighbourhood O of zero of A/M and a finite subset S of A/M such that

$$T = \{L \in \mathcal{L}_\tau(A/M) \colon L(S) \subset O\} \subset U.$$

Since the multiplication in A/M over A is separately continuous, then for each $s \in S$ there exists a neighbourhood V_s of θ_A such that $V_s s \subset O$ ($sV_s \subset O$). Let now

$$V = \bigcap_{s \in S} V_s.$$

Then $VS \subset O$ ($SV \subset O$) and V is a neighbourhood of zero in A. Since $L_M(v)(S) = L_v^M(S) = vS \subset O$ for each $v \in V$, one has $L_M(V) \subset T \subset U$. Therefore, L_M is a continuous map (because L_M is linear). Hence, L_M is a continuous irreducible representation of A on A/M and $\ker L_M$ is a closed two-sided ideal of A. Similarly, we can show that the antirepresentation R_M is continuous.

As defined in [1], the *left* (*right*) *topological radical* of a topological algebra A is the intersection $\mathrm{rad}_l(A)$ ($\mathrm{rad}_r(A)$) of kernels of all continuous irreducible representations (antirepresentations) of A on all Hausdorff linear spaces. If A has no continuous irreducible representations (antirepresentations) at all, then we say that A is a *left* (*right*, respectively) *topologically radical algebra* (in this case $\mathrm{rad}_l(A) = \mathrm{rad}_r(A) = A$). If $\mathrm{rad}_l(A) \neq A$ ($\mathrm{rad}_r(A) \neq A$), then we say that A is a *left* (*right*) *topologically nonradical algebra*, and if $\mathrm{rad}_l(A) = \{\theta_A\}$ ($\mathrm{rad}_r(A) = \{\theta_A\}$), then A is a *left* (*right*, respectively) *topologically semi-simple algebra*.

2. Description of one-sided topological radicals

The left-sided case of the following lemma is Proposition 4 in [7], page 120. Here we adopt the notation and give for completeness the proof for the right-sided case.

Lemma 1. *Let X be a left (right) A-module, let π (π', respectively) be a representation (antirepresentation, respectively) of A on X, and let $x_0 \in X \setminus \{\theta_X\}$.*

a) *Each element of* $\mathrm{id}_r(x_0)$ ($\mathrm{id}_l(x_0)$) *is a right (left, respectively) regular unit for the left ideal* $\ker \varrho_{\pi,x_0}$ *(right ideal* $\ker \varrho_{\pi',x_0}$*, respectively).*

b) *If x_0 is a left (right) cyclic element, then* $\mathrm{id}_r(x_0)$ ($\mathrm{id}_l(x_0)$) *is non-void and* $\ker \varrho_{\pi,x_0}$ *is a regular left* ($\ker \varrho_{\pi',x_0}$ *is a regular right, respectively) ideal of A.*

c) *If X is irreducible, then x_0 is a left (right, respectively) cyclic element and* $\ker \varrho_{\pi,x_0}$ ($\ker \varrho_{\pi',x_0}$*, respectively) is a maximal regular left (right, respectively) ideal.*

Proof. a) Let $e \in \mathrm{id}_l(x_0)$. Then, for all $a \in A$,

$$x_0 \cdot_{\pi'} (a - ea) = x_0 \cdot_{\pi'} a - x_0 \cdot_{\pi'} a = \theta_A$$

and so $a - ea \in \ker \varrho_{\pi',x_0}$.

b) Since x_0 is a right cyclic element, $x_0 \cdot_{\pi'} A = X$. So there is an element $e \in A$ such that $x_0 \cdot_{\pi'} e = x_0$, that is, $e \in \mathrm{id}_l(x_0)$. By the statement a), e is a left regular unit for $\ker \varrho_{\pi',x_0}$ and therefore $\ker \varrho_{\pi',x_0}$ is a regular right ideal of A.

c) Let X be irreducible. Since $x_0 \cdot_{\pi'} A$ is an A-submodule, either $x_0 \cdot_{\pi'} A = \{\theta_A\}$ or $x_0 \cdot_{\pi'} A = A$. If $x_0 \cdot_{\pi'} A = \{\theta_A\}$, then $N = \{x \in X \colon x \cdot_{\pi'} A = \{\theta_A\}\}$ is a non-zero A-submodule, and so $N = X$ because X is irreducible. In this case $X \cdot_{\pi'} A = \{\theta_A\}$, but this is impossible since X is non-trivial. Consequently, $x_0 \cdot_{\pi'} A = X$ and x_0 is right cyclic. By the statement b), $\ker \varrho_{\pi',x_0}$ is a regular right ideal. Let J be a right ideal of A with $\ker \varrho_{\pi',x_0} \subsetneqq J$. Now $x_0 \cdot_{\pi'} J$ is an A-submodule and for every $j \in J \setminus \ker \varrho_{\pi',x_0}$ we have $x_0 \cdot_{\pi'} j \neq \theta_A$ by the definition of $\ker \varrho_{\pi',x_0}$, which means that $x_0 \cdot_{\pi'} J \neq \{\theta_A\}$. Therefore $x_0 \cdot_{\pi'} J = X$. So there exists an element $e \in J \cap \mathrm{id}_l(x_0)$. By the statement a), e is a left regular unit for $\ker \varrho_{\pi',x_0}$ and, thus, also for J. But then $J = A$ since $e \in J$. The proof for the left-sided case is similar. □

Theorem 1 in [4] considers the case when the A-module X is commutative. Here we examine the case when the multiplication over A is not necessarily commutative.

We denote by $m_l(A)$ ($m_r(A)$ or $m_t(A)$) the set of all closed maximal regular left (right or two-sided, respectively) ideals of A. In addition, $\pi_l(A)$

$(\pi_r(A))$ is the set of all closed primitive ideals of A defined by closed maximal regular left (right, respectively) ideals.

Theorem 2. *Let A be a right (left) topologically nonradical algebra. Then* $\mathrm{rad}_r(A)$ $(\mathrm{rad}_l(A)$, *respectively) is equal to*

a) *the intersection of all closed maximal regular right (left, respectively) ideals of A;*
b) *the intersection of all primitive ideals of A which are defined by closed maximal regular right (left, respectively) ideals of A.*

That is,

$$\mathrm{rad}_k(A) = \bigcap_{P\in\pi_k(A)} P = \bigcap_{M\in m_k(A)} M$$

where $k = r$ ($k = l$, respectively).

Hence, $\mathrm{rad}_r(A)$ $(\mathrm{rad}_l(A)$, *respectively) is closed in A.*

Proof. Since $\ker R_M \subset M$ for each $M \in m_r(A)$, we have

$$\mathrm{rad}_r(A) \subset \bigcap_{M\in m_r(A)} \ker R_M \subset \bigcap_{M\in m_r(A)} M.$$

To show the converse inclusion, let $a \in \cap\{M\colon M \in m_r(A)\}$. If $a \notin \mathrm{rad}_r(A)$, then there is a continuous irreducible antirepresentation π' of A on some Hausdorff linear space X such that $a \notin \ker \pi'$. Therefore there exists an element $x_0 \in X$ such that $\pi'(a)(x_0) \neq \theta_X$. Hence $a \notin I = \ker \varrho_{\pi',x_0}$. By Lemma 1, $\ker \varrho_{\pi',x_0}$ is a maximal regular right ideal of A. To show that $\ker \varrho_{\pi',x_0}$ is closed in A, we show that the map ϱ_{π',x_0} is continuous on A. To this end, let a_0 be an arbitrary element in A, let O be a neighbourhood of $\pi'(a_0)(x_0)$ in X, let O_X be a neighbourhood of zero in X such that $O = \pi'(a_0)(x_0) + O_X$, and let

$$O(\pi'(a_0)) = \{L \in \mathcal{L}_\tau(X)\colon (L - \pi'(a_0))(x_0) \in O_X\}.$$

Since $O(\pi'(a_0))$ is a neighbourhood of $\pi'(a_0)$ in $\mathcal{L}_\tau(X)$ and π' is a continuous antirepresentation of A on X, there is a neighbourhood $O(a_0)$ of a_0 in A such that $\pi'(O(a_0)) \subset O(\pi'(a_0))$. If now $b \in O(a_0)$, then

$$\varrho_{\pi',x_0}(b) = \pi'(b)(x_0) \in \pi'(a_0)(x_0) + O_X = O.$$

Therefore ϱ_{π',x_0} is continuous. It means that I is a closed maximal regular right ideal of A which does not contain a, but this is not possible. Consequently, $a \in \mathrm{rad}_r(A)$. Hence the statement a) holds. Since $\ker R_M$ is a closed primitive ideal of A for each $M \in m_r(A)$, the statement b) holds too.

The proof for left topological radicals is similar. □

Note that $\mathrm{Rad}(A) \subset \mathrm{rad}_l(A) \cap \mathrm{rad}_r(A)$.

3. Coincidence of left and right topological radicals

The description of the class of unital topological algebras in which all maximal one-sided ideals are closed has been given in [5], and the class in which all maximal two-sided ideals are closed is described in [6] (for the nonunital case, see [3]). In these cases $\mathrm{rad}_l(A) = \mathrm{rad}_r(A)$ by Theorem 2. This is the case when A is, e.g., a Q-algebra (that is, the set of all left quasi-invertible elements and the set of all right quasi-invertible elements of A are open). Here we consider the case when maximal regular ideals are not necessarily closed. A class of topological algebras for which one-sided topological radicals coincide has been described in [1].

Theorem 3. *Let A be a topological algebra and $\kappa_P\colon A \to A/P$ be the quotient map for every $P \in \pi(A)$. Then $\mathrm{rad}_l(A) = \mathrm{rad}_r(A)$ if and only if*

$$\bigcap_{P\in\pi_l(A)} \kappa_P^{-1}(\mathrm{rad}_l(A/P)) = \bigcap_{Q\in\pi_r(A)} \kappa_Q^{-1}(\mathrm{rad}_r(A/Q)). \tag{1}$$

Proof. Note that

$$\begin{aligned}\mathrm{rad}_l(A) &= \bigcap_{M\in m_l(A)} M = \bigcap_{P\in\pi_l(A)} \bigcap_{\mathcal{M}\in m_l(A/P)} \kappa_P^{-1}(\mathcal{M}) \\ &= \bigcap_{P\in\pi_l(A)} \kappa_P^{-1}\Big(\bigcap_{\mathcal{M}\in m_l(A/P)} \mathcal{M}\Big) \\ &= \bigcap_{P\in\pi_l(A)} \kappa_P^{-1}(\mathrm{rad}_l(A/P))\end{aligned}$$

by Theorem 2 and Theorem 12 in [2]. Similarly,

$$\mathrm{rad}_r(A) = \bigcap_{Q\in\pi_r(A)} \kappa_Q^{-1}(\mathrm{rad}_r(A/Q))$$

and, therefore, $\mathrm{rad}_l(A) = \mathrm{rad}_r(A)$ if and only if (1) holds. □

Corollary 4. *Let A be a topological algebra. If $\pi_l(A) = \pi_r(A)$ and $\mathrm{rad}_l(A/P) = \mathrm{rad}_r(A/P)$ for every $P \in \pi_l(A)$, then $\mathrm{rad}_l(A) = \mathrm{rad}_r(A)$.*

Proof. If $\pi_l(A) = \pi_r(A)$ and $\mathrm{rad}_l(A/P) = \mathrm{rad}_r(A/P)$ for every $P \in \pi_l(A)$, then (1) holds and so, by Theorem 3, $\mathrm{rad}_l(A) = \mathrm{rad}_r(A)$. □

Theorem 5. *Let A be a simplicial topological algebra and I a closed two-sided ideal of A. Then A/I is simplicial.*

Proof. Let $\mathcal{J}$ be a closed left (right) ideal of A/I. Then $\kappa_I^{-1}(\mathcal{J})$ is a closed left (respectively, right) ideal of A (by Corollary 2 in [2]). Since A is simplicial, there is $M \in m_l(A)$ ($M \in m_r(A)$, respectively) such that $\kappa_I^{-1}(\mathcal{J}) \subset M$. Therefore, $\mathcal{J} = \kappa_I(\kappa_I^{-1}(\mathcal{J})) \subset \kappa_I(M)$. Since $I \subset M$, then $\kappa_I(M)$ is a closed maximal regular left (right, respectively) ideal of A/I, by the statement a) of Corollary 2 in [2]. Hence, every closed left (right, respectively) ideal of A/I is contained in a closed maximal regular left (right, respectively) ideal of A/I. □

If all closed maximal regular one-sided ideals in a topological algebra A are two-sided ideals, then $\pi_l(A) = \pi_r(A)$.

Corollary 6. *Let A be a topological algebra. If $\pi_l(A) = \pi_r(A)$ and A/P is simplicial and advertive for every $P \in \pi_l(A)$, then* $\mathrm{rad}_l(A) = \mathrm{rad}_r(A)$.

Proof. By the assumption, A/P is simplicial and advertive for every $P \in \pi_l(A)$, and therefore, by Theorem 1 in [1], $\mathrm{rad}_l(A/P) = \mathrm{rad}_r(A/P)$ for every $P \in \pi_l(A)$. Since also $\pi_l(A) = \pi_r(A)$, by Corollary 4, $\mathrm{rad}_l(A) = \mathrm{rad}_r(A)$. □

Corollary 7. *Let A be a simplicial topological algebra. If $\pi_l(A) = \pi_r(A)$ and A/P is advertive for every $P \in \pi_l(A)$, then* $\mathrm{rad}_l(A) = \mathrm{rad}_r(A)$.

Proof. By Theorem 5, A/P is simplicial for every $P \in \pi_l(A)$, since A is simplicial. Now, by Corollary 6, $\mathrm{rad}_l(A) = \mathrm{rad}_r(A)$. □

Corollary 8. *Let A be a simplicial topological algebra. If $\pi_l(A) = \pi_r(A)$ and for every $P \in \pi_l(A)$ there exists an advertive subalgebra $B_P \subset A$ such that $A = P \oplus B_P$, then* $\mathrm{rad}_l(A) = \mathrm{rad}_r(A)$.

Proof. Let A be a simplicial topological algebra. If $A = P \oplus B_P$ for every $P \in \pi_l(A)$, then $A/P \cong B_P$ for every $P \in \pi_l(A)$ and, by Corollary 7, $\mathrm{rad}_l(A) = \mathrm{rad}_r(A)$. □

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