ACTA ET COMMENTATIONES UNIVERSITATIS TARTUENSIS DE MATHEMATICA
Volume 21, Number 2, December 2017
Available online at http://acutm.math.ut.ee

On basicity of the degenerate trigonometric system with excess

Aydin Sh. Shukurov

Abstract. The basis properties (completeness, minimality and Schauder basicity) of systems of the form $\{\omega(t)\varphi_n(t)\}$, where $\{\varphi_n(t)\}$ is an exponential or trigonometric (cosine or sine) systems, have been investigated in several papers. Concrete examples of the weight function $\omega(t)$ are known for which the system itself is not complete and minimal but has excess − becomes complete and minimal in corresponding L_p space only after elimination of some of its elements. The aim of this paper is to show that if $\omega(t)$ is any measurable weight function such that the system $\{\omega(t)\sin nt\}_{n\in\mathbb{N}}$ has excess, then neither this system itself, nor a system obtained from it by elimination of an element, is not a Schauder basis.

1. Introduction

The basis properties (completeness, minimality and Schauder basicity) of systems of the form $\{\omega(t)\varphi_n(t)\}$, where $\{\varphi_n(t)\}$ is an exponential or trigonometric (cosine or sine) systems have been investigated in many papers (see, for example, [1, 3−11, 13−16, 18−21, 26]). To our knowledge, the first result in this direction is [1] in which Babenko gave an example $\{|t|^\alpha \cdot e^{int}\}_{n\in\mathbb{Z}}$, where $|\alpha| < 1/2$ and $\alpha \neq 0$, answering in the affirmative a question of Bari [2] on the existence of normalized basis for $L_2(-\pi,\pi)$ that is not a Riesz basis. The result of Babenko [1] was then extended by V. F. Gaposhkin in his famous paper [7], where, in particular, some sufficient condition (on the weight function $\omega(t)$) for the system $\{\omega(t) \cdot e^{int}\}_{n\in\mathbb{Z}}$ to be a basis in $L_2(-\pi,\pi)$ was found. Eventually, a necessary and sufficient condition in terms of the weight function $\omega(t)$ which ensures the Schauder basicity of the exponential system $\{e^{int}\}_{n\in\mathbb{Z}}$ in weighted Lebesgue space $L_{p,\omega()}(-\pi,\pi)$ has

Received July 16, 2016.
2010 *Mathematics Subject Classification.* Primary 46E30, 46B15; Secondary 46B25.
Key words and phrases. Schauder bases, system of cosines with degenerate coefficients, system of sines with degenerate coefficients, Muckenhoupt condition.
http://dx.doi.org/10.12697/ACUTM.2017.21.17

been obtained (see, for example, [9, 10, 16, 20]). Such a condition is the Muckenhoupt condition with respect to the weight function $\omega(t)$:

$$\sup_I \left(\frac{1}{|I|} \int_I \omega(t)dt \right) \left(\frac{1}{|I|} \int_I \omega^{-\frac{1}{p-1}}(t)dt \right)^{p-1} < \infty,$$

where sup is taken over all intervals I and $|I|$ is the length of the interval I. Note that the study of basicity properties of a system in weighted Lebesgue spaces $L_{p,\omega()}$ is equivalent to the study of analogous properties of this system with corresponding degenerate coefficient in the "ordinary" Lebesgue space L_p. Therefore, the mentioned criterion can also be considered as a necessary and sufficient condition for the Schauder basicity in L_p of the exponential system with degenerate coefficients $\{\omega(t)e^{int}\}_{n\in\mathbb{Z}}$.

Note that similar properties also hold true for the trigonometric (sine or cosine) systems, which can be seen from the cited references.

It is known that there are weight functions $\omega(t)$ (not satisfying Muckenhoupt condition) for which the system $\{\omega(t)\varphi_n(t)\}$, where $\{\varphi_n(t)\}$ is an exponential or trigonometric (cosine or sine) system, has excess − becomes complete and minimal in L_p space only after elimination of some of its terms. Only some very special choices of such weight functions were found and bacisity properties of the corresponding systems with degenerate coefficients have been studied in the papers [4, 8, 13, 21]. For example, in a very special choice of the weight function $\omega(t)$ basicity properties of the system $\{\omega(t)\sin nt\}_{n\in\mathbb{N}}$ are investigated in [4]. It is proved in [4] that in this special choise of the weight function $\omega(t)$ the system obtained from the original system by elimination of an element is complete and minimal but is not a Schauder basis in $L_p(0,\pi)$ space. The aim of this paper is to show that such system − the system, obtained from $\{\omega(t)\sin nt\}_{n\in\mathbb{N}}$ by elimination of an element is not a Schauder basis for any measurable weight function $\omega(t)$.

Remark 1.1. The proof of Theorem 3.1 of the paper [4] contains a gap − the reasoning given there to prove this theorem is not sufficient to state the validity of the mentioned theorem. Nevertheless, the statement itself is true which can be seen from the following result.

Proposition 1.2 (see [26])**.** *Let $\omega(t)$ be any measurable function on $(0,\pi)$ such that* $\text{mes}\{t : \omega(t) = 0\} = 0$ *and $\omega(t)\sin nt \in L_p(0,\pi)$, $n \in \mathbb{N}$. Then the system $\{\omega(t)\sin nt\}_{n\in\mathbb{N}}$ is complete in $L_p(0,\pi)$ space.*

2. Auxiliary facts

Lemma 2.1. *Let $\omega(t)$ be any nontrivial measurable function on $[0,\pi]$ such that $\omega(t)\sin nt \in L_p(0,\pi)$, $1 \le p < \infty$, for all $n \in \mathbb{N}$. Then*

$$\inf_{n\in N} \|\omega(t)\sin nt\|_{L_p(0,\pi)} \neq 0.$$

Proof. Assume the contrary: there is a nontrivial measurable function $\omega(t)$ such that $\omega(t)\sin nt \in L_p(0,\pi)$ for all $n \in \mathbb{N}$ and $\inf_{n\in\mathbb{N}} \|\omega(t)\sin nt\|_{L_p} = 0$. Then there is a subset $E \subset [0,\pi]$ of positive measure such that $\omega(t) \neq 0$ for all $t \in E$. Moreover, our assumption implies the existence of a subsequence $\{n_k\}$ of natural numbers for which $\lim_{k\to\infty} \|\omega(t)\sin n_k t\|_{L_p(E)} = 0$. Then there is a subsequence $\{\omega(t)\sin m_k t\}$ of $\{\omega(t)\sin n_k t\}$ which converges to zero a.e. on E (see, for example, [17, p. 157]), i.e., there is a set $E' \subset E$ such that $\lim_{k\to\infty} \omega(t)\sin m_k t = 0$ on E' and $\operatorname{mes} E' = \operatorname{mes} E > 0$. We obtain from here that $\lim_{k\to\infty} \sin m_k t = 0$ for all $t \in E'$. This contradicts the Cantor–Lebesgue theorem (see, for example [17, p. 278]). The lemma is proved. □

Remark 2.2. Note that the proof of Lemma 2.1 is a modification of the proof of Lemma 2.2 from author's paper [22].

Lemma 2.3. *If the system* $\{\omega(t)sinnt\}_{n\in\mathbb{N}\setminus\{k_0\}}$ *is complete in* $L_p(0,\pi)$*, then* $\frac{t(t-\pi)}{\omega(t)} \notin L_q(0,\pi)$*, where* $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Assume the contrary: $\frac{t(t-\pi)}{\omega(t)} \in L_q(0,\pi)$. Then, $\frac{\sin k_0 t}{\omega(t)} \in L_q(0,\pi)$. The conditions posed on the function $\omega(t)$ imply that $\frac{\sin k_0 t}{\omega(t)}$ is a nontrivial function. Moreover,

$$\int_0^\pi \frac{\sin k_0 t}{\omega(t)} \cdot \omega(t)\sin nt dt = \int_0^\pi \sin k_0 t \cdot \sin nt dt = 0,$$

for all $n \neq k_0$. But this contradicts the fact that $\{\omega(t)\sin nt\}_{n\in\mathbb{N}\setminus\{k_0\}}$ is complete in $L_p(0,\pi)$. The lemma is proved. □

Lemma 2.4. *Let* ξ *be any complex number and let* n, m *be any natural numbers such that* $n \neq m$*. Then the function* $\sin nt + \xi\cdot\sin mt$ *may have only finite number of zeros in the segment* $[0,\pi]$.

Proof. Assume the contrary: the function $\sin nt + \xi\cdot\sin mt$ has an infinite number of zeros. Let $\{z_n\}_{n=1}^\infty \subset [0,\pi]$ be its zeros. By the Bolzano–Weierstrass theorem, the sequence $\{z_n\}_{n=1}^\infty$ has a limit point in $[0,\pi]$. Therefore, since the function $\sin nz + \xi\cdot\sin mz$ is an entire function on the whole complex plane, the uniqueness theorem for analytic functions implies that $\sin nt + \xi\cdot\sin mt \equiv 0$ on the segment $[0,\pi]$. This means that the system of functions $\{\sin nt, \sin mt\}$ is linearly dependent system. Contradiction: since the system is orthonormal, it is linearly independent. The lemma is proved. □

Lemma 2.5. *Let* $\omega(t)$ *be any measurable function on* $(0,\pi)$ *and let* N_0 *be any natural number. The relation* $\omega(t)\sin nt \in L_p(0,\pi)$, $n \geq N_0$*, is possible if and only if* $t(t-\pi)\omega(t) \in L_p(0,\pi)$.

Proof. Necessity. Denote the zeros of the function $\sin N_0 t$ lying in $(0,\pi)$ by $z_1,\dots,z_{N_0-1}$. It is easy to see that there are a natural number k_i, a small neighbourhood U_i of the point z_i and a positive number α_i such that $|\sin k_i t| > \alpha_i$ for all $t \in U_i$ and $i \in \{1,\dots,N_0-1\}$.

Take arbitrary $i \in \{1,\dots,N_0-1\}$ and write the function $\omega(t)\sin k_i t$ in the form

$$\omega(t)\sin k_i t = t(t-\pi)\omega(t)\cdot\frac{\sin k_i t}{t(t-\pi)}.$$

There is a positive number μ_i such that $|\frac{\sin k_i t}{t(t-\pi)}| > \mu_i$ for all $t \in U_i$. Therefore the estimation

$$|\omega(t)\sin k_i t| \geq \mu_i \cdot |t(t-\pi)\omega(t)|, \quad t \in U_i,$$

holds. Since $\omega(t)\sin k_i t \in L_p(U_i)$, the last relation implies that $t(t-\pi)\omega(t) \in L_p(U_i)$.

Write the function $\omega(t)\sin N_0 t$ in the following form:

$$\omega(t)\sin N_0 t = t(t-\pi)\omega(t)\cdot\frac{\sin N_0 t}{t(t-\pi)}. \tag{1}$$

Consider an auxiliary function, defined on $[0,\pi]\setminus\bigcup_{i=1}^{N_0-1}U_i$, by the formula

$$\Phi(t) = \begin{cases} \frac{\sin N_0 t}{t(t-\pi)}, & \text{if } t \in (0,\pi); \\ -\frac{N_0}{\pi}, & \text{if } t = 0; \\ \frac{(-1)^{N_0}N_0}{\pi}, & \text{if } t = \pi. \end{cases}$$

It is evident that the function $\Phi(t)$ is continuous and never vanishes at the closed set $[0,\pi]\setminus\bigcup_{i=1}^{N_0-1}U_i$. Therefore, there is a positive number m such that $|\Phi(t)| > m$ for all $t \in [0,\pi]\setminus\bigcup_{i=1}^{N_0-1}U_i$. Using these inequalities in (1), we obtain the estimation

$$|\omega(t)\sin N_0 t| \geq m\cdot|t(t-\pi)\omega(t)|, \quad t \in [0,\pi]\setminus\bigcup_{i=1}^{N_0-1}U_i.$$

This estimation implies that

$$t(t-\pi)\omega(t) \in L_p([0,\pi]\setminus\bigcup_{i=1}^{N_0-1}U_i),$$

since $\omega(t)\sin N_0 t \in L_p(0,\pi)$ by the condition of the lemma. Since we have also $t(t-\pi)\omega(t) \in L_p(U_i)$ for all $i \in \{1,\dots,N_0-1\}$, the necessity part of the lemma is proved.

Sufficiency. Assume that $t(t-\pi)\omega(t) \in L_p(0,\pi)$. Take arbitrary natural number n and write the function $\omega(t)\sin nt$ in the form

$$\omega(t)\sin nt = t(t-\pi)\omega(t)\cdot\frac{\sin nt}{t(t-\pi)}. \tag{2}$$

Consider an auxiliary function

$$\Phi_n(t) = \begin{cases} \frac{\sin nt}{t(t-\pi)}, & \text{if } t \in (0,\pi); \\ -\frac{n}{\pi}, & \text{if } t = 0; \\ \frac{(-1)^n n}{\pi}, & \text{if } t = \pi. \end{cases}$$

It is evident that the function $\Phi_n(t)$ is continuous on $[0,\pi]$. Therefore, there is a number M_n such that $|\Phi_n(t)| < M_n$ for all $t \in [0,\pi]$. Using these inequalities in (2), we obtain the estimation

$$|\omega(t)\sin nt| \le M_n \cdot |t(t-\pi)\omega(t)|, \quad t \in [0,\pi].$$

Since $t(t-\pi)\omega(t) \in L_p(0,\pi)$, the last estimation implies $\omega(t)\sin nt \in L_p(0,\pi)$. The lemma is proved. □

Lemma 2.6. *Let $\omega(t)$ be a measurable function on $(0,\pi)$ such that $\omega(t)\sin nt \in L_p(0,\pi)$ for all sufficiently large values of n. Then $\omega(t)\sin nt \in L_p(0,\pi)$ for all values of n.*

Proof. The validity of this fact follows directly from Lemma 2.5. □

Lemma 2.7. *If the system $\{\omega(t)\sin nt\}_{n\in\mathbb{N}\setminus\{k_0\}}$ is minimal in $L_p(0,\pi)$, then it has a biorthogonal system $\{b_n(t)\}_{\mathbb{N}\setminus\{k_0\}}$ which is of the form*

$$b_n(t) = \frac{2}{\pi} \cdot \frac{\sin nt + \xi_n \cdot \sin k_0 t}{\omega(t)} \tag{3}$$

for all sufficiently large values of n, where ξ_n are some complex numbers.

Proof. The fact that $\{\omega(t)\sin nt\}_{n\in\mathbb{N}\setminus\{k_0\}}$ has a biorthogonal system follows from its minimality. Denote the biorthogonal system by $\{b_n(t)\}_{\mathbb{N}\setminus\{k_0\}}$. Take arbitrary sufficiently large natural number $n \ne k_0$ ($n > k_0 + 2$ is sufficient for our purposes). By the definition of the biorthogonal system,

$$\int_0^\pi b_n(t) \cdot \omega(t)\sin ktdt = 0, \quad k \ne n, k_0,$$

and

$$\int_0^\pi b_n(t) \cdot \omega(t)\sin ntdt = 1.$$

These relations imply the validity of the equality

$$\int_0^\pi b_n(t) \cdot \omega(t)\sin ktdt = \xi_{n,k} \tag{4}$$

for all $k = -1, 0, 1, 2, 3, \dots$, where numbers $\xi_{n,k}$ are defined as

$$\xi_{n,k} = \begin{cases} -\xi_n, & \text{if } k = -1; \\ \xi_n, & \text{if } k = 1; \\ 1, & \text{if } k = n; \\ 0, & \text{if } k \ne -1, 1, n; \end{cases}$$

if $k_0 = 1$, and as

$$\xi_{n,k} = \begin{cases} \xi_n, & \text{if } k = k_0; \\ 1, & \text{if } k = n; \\ 0, & \text{if } k \neq k_0, n; \end{cases}$$

if $k_0 > 1$, where ξ_n are some complex numbers.

Relations (4) imply that

$$\int_0^\pi b_n(t) \cdot \omega(t) \sin(k-1)t dt = \xi_{n,k-1},$$

$$\int_0^\pi b_n(t) \cdot \omega(t) \sin(k+1)t dt = \xi_{n,k+1}$$

for all $k = 0, 1, 2, 3, \ldots$. By subtracting the first of these equalities from the latter one, we obtain that

$$\int_0^\pi b_n(t)\omega(t) \sin t \cdot \cos kt dt = \frac{\xi_{n,k+1} - \xi_{n,k-1}}{2} \tag{5}$$

for all $k = 0, 1, 2, 3, \ldots$.

Since $\omega(t) \sin t \in L_p(0, \pi)$ (apply Lemma 2.6 if $k_0 = 1$), $b_n(t)\omega(t) \sin t \in L_1(0, \pi)$. Therefore, relations (5), along with the fact that the Fourier coefficients of a summable function with respect to the cosine system are unique, imply that

$$b_n(t)\omega(t) \sin t = \frac{1}{\pi} \cdot \frac{\xi_{n,1} - \xi_{n,-1}}{2} + \frac{2}{\pi} \sum_{k=1}^{\infty} \frac{\xi_{n,k+1} - \xi_{n,k-1}}{2} \cos kt.$$

Taking into account the definition of numbers $\xi_{n,k}$ given above, we find from here the validity of the statement of the lemma. □

For simplicity, let us denote

$$\Phi_n(t) = \sin nt + \xi_n \sin k_0 t.$$

Lemma 2.8. *If* $\lim_{n\to\infty} \xi_n = 0$*, then:*

1) *there is a natural number* n_0 *such that* $\Phi'_{n_0}(0) \neq 0$ *and* $\Phi'_{n_0}(\pi) \neq 0$*;*

2) *given any number* $t_0 \in (0, \pi)$*, there is a natural number* n_0 *such that* $\Phi_{n_0}(t_0) \neq 0$*.*

Proof. Part 1) follows from the fact that $|\xi_n| \neq n/k_0$ for all sufficiently large values of n. Part 2) of the lemma follows from the fact that $\lim_{n\to\infty} \sin nt_0 = 0$ is possible if and only if $\sin t_0 = 0$. The lemma is proved. □

Lemma 2.9. *If the system* $\{\omega(t) \sin nt\}_{n\in\mathbb{N}\setminus\{k_0\}}$ *is complete and minimal in* $L_p(0, \pi)$*, then the numbers* ξ_n *in* (3) *are uniquely determined and* $\lim_{n\to\infty} \xi_n \neq 0$ *(if this limit exists).*

Proof. The uniqueness of the numbers ξ_n follows from the fact that a biorthogonal system of the complete and minimal system is unique.

To prove the second part, assume the contrary: $\lim_{n\to\infty}\xi_n = 0$.

By Lemma 2.8, there is a natural number n_0 for which $\Phi'_{n_0}(0) \neq 0$ and $\Phi'_{n_0}(\pi) \neq 0$. The function $\sin n_0 t + \xi_{n_0}\sin k_0 t$ may have (see Lemma 2.4) a finite number of zeros on $[0,\pi]$. If it has no zeros, then the relation $b_{n_0}(t) \in L_q(0,\pi)$ implies that $\frac{1}{\omega(t)} \in L_q(0,\pi)$ which is impossible by Lemma 2.3. Now, assume that $\sin n_0 t + \xi_{n_0}\sin k_0 t$ has m zeros on $[0,\pi]$ and let $z_1 = 0, z_2 = \pi, \ldots, z_m$ be its zeros. It is easy to see (Lemma 2.8) that for every $i \neq 1,2$ there are a natural number n_i, a neighbourhood U_i of the point z_i and a positive number α_i such that $|\sin n_i t + \xi_{n_i}\sin k_0 t| > \alpha_i$ for all $t \in U_i$. Therefore, the relation $b_n(t) \in L_q(0,\pi)$ implies that

$$\frac{\sin n_i t + \xi_{n_i}\sin k_0 t}{\omega(t)} \in L_q(U_i),$$

and thus

$$\frac{1}{\omega(t)} \in L_q(U_i), \quad i \neq 1,2. \tag{6}$$

Write the function $b_{n_0}(t)$ in the following form:

$$b_{n_0}(t) = \frac{2}{\pi}\cdot\frac{t(t-\pi)}{\omega(t)}\cdot g_0(t), \tag{7}$$

where

$$g_0(t) = \frac{\sin n_0 t + \xi_{n_0}\sin k_0 t}{t(t-\pi)}.$$

If we extend the definition of the function $g_0(t)$ to the points 0 and π by assigning values $-\Phi'_{n_0}(0)/\pi$ and $\Phi'_{n_0}(\pi)/\pi$ at the points 0 and π, respectively, it is easy to see that the obtained function is continuous and never vanishes at the compact set $[0,\pi]\setminus\bigcup_{i\neq 1,2}^m U_i$. Then, according to the Weierstass theorem, there is a positive number α such that $|g_0(t)| > \alpha$ for all $t \in [0,\pi]\setminus\bigcup_{i\neq 1,2}^m U_i$. Therefore, the relation $b_{n_0}(t) \in L_q(0,\pi)$ and representation (7) imply that $\frac{t(t-\pi)}{\omega(t)} \in L_q\left([0,\pi]\setminus\bigcup_{i\neq 1,2}^m U_i\right)$. This relation and (6) imply that $\frac{t(t-\pi)}{\omega(t)} \in L_q(0,\pi)$ which contradicts Lemma 2.3. The lemma is proved. □

3. Main result and its proof

Theorem 3.1. *Let $\omega(t)$ be a measurable function and let k_0 be a natural number. Then the system $\{\omega(t)\sin nt\}_{n\in\mathbb{N}\setminus\{k_0\}}$ is not a Schauder basis in the $L_p(0,\pi)$ space.*

Proof. Assume the contrary: there is a measurable function $\omega(t)$ and a natural number k_0 such that the system $\{\omega(t)\sin nt\}_{n\in\mathbb{N}\setminus\{k_0\}}$ is a Schauder basis in $L_p(0,\pi)$. Then $\omega(t)\sin k_0 t \in L_p(0,\pi)$ by Lemma 2.6. Therefore, it has an expansion (in the $L_p(0,\pi)$ norm) of the form

$$\omega(t)\sin k_0 t = \sum_{n=1, n\neq k_0}^{\infty} c_n \cdot \omega(t)\sin nt. \tag{8}$$

Take an arbitrary sufficiently large natural number $n \neq k_0$ (the relation $n > k_0+2$ is sufficient for our reasonings to hold). Applying the biorthogonal system (3) to both sides of (8), we obtain that $c_n = \xi_n$. Thus, the series $\sum_{n=1, n\neq k_o}^{\infty} \xi_n \cdot \omega(t)\sin nt$ is convergent. Therefore, by a necessary condition for the convergence of the series,

$$\lim_{n\to\infty} \|\xi_n \cdot \omega(t)\sin nt\|_{L_p(0,\pi)} = 0.$$

This equality and Lemma 2.1 imply that $\lim_{n\to\infty} \xi_n = 0$. But this is impossible by Lemma 2.9. The theorem is proved. □

Note that negative results on Schauder basicity of some systems of a certain form were also studied earlier in papers [12, 22−25].

Acknowledgements

The author is grateful to Professor B. T. Bilalov for encouraging discussion. The author also thanks A. A. Huseynli for discussion of the proof of Lemma 2.1 and N. J. Guliyev for his useful assistance.

This work was supported by the Science Development Foundation under the President of the Republic of Azerbaijan − Grant N EIF/GAM-3-2014-6(21)-24/05/1.

References

[1] K. I. Babenko, *On conjugate functions*, Doklady Akad. Nauk SSSR (N. S.) **62** (1948). 157–160. (Russian)

[2] N. K. Bari, *Biorthogonal systems and bases in Hilbert space*, Moskov. Gos. Univ. Učenye Zapiski Matematika **148**(4) (1951). 69–107. (Russian)

[3] B. T. Bilalov and Z. A. Gasimov, *On completeness of exponent system with complex coefficients in weight spaces*, Trans. NAS Azerb. Ser. Phys.-Tech. Math. Sci. **25**(7) (2005), 9–14.

[4] B. Bilalov and F. Guliyeva, *On the frame properties of degenerate system of sines*, J. Funct. Spaces Appl. **2012**, Art. ID 184186, 12 pp.

[5] B. T. Bilalov and Z. V. Mamedova, *On the frame properties of some degenerate trigonometric systems*, Dokl. Nats. Akad. Nauk Azerb. **68**(5) (2012), 14–18. (Russian)

[6] B. T. Bilalov and S. G. Veliyev, *Bases of eigenfunctions of two discontinuous differential operators*, Diff. Uravn. **42**(9) (2006), 190–192. (Russian)

[7] V. F. Gaposhkin, *A generalization of the theorem of M. Riesz on conjugate functions*, Mat. Sb. N. S. **46** (1958), 359–372. (Russian)

[8] E. S. Golubeva, *The system of weighted exponentials with power weights*, in: Vestnik Sam. GU Estestvenno-Nauchnaya Ser. 2011, No. 2 (83), pp. 15–25. (Russian).

[9] R. Hunt, B. Muckenhoupt, and R. Wheeden, *Weighted norm inequalities for the conjugate function and Hilbert transform*, Trans. Amer. Math. Soc. **176** (1973), 227–251.

[10] R. Hunt and W. Young, *A weighted norm inequality for Fourier series*, Bull. Amer. Math. Soc. **80** (1974), 274–277.

[11] K. S. Kazaryan and P. I. Lizorkin, *Multipliers, bases and unconditional bases of the weighted spaces B and SB*, Trudy Mat. Inst. Steklov. **187** (1989), 98–115. (Russian)

[12] L. V. Kritskov, *On the basis property of the system of functions* $\{e^{i\alpha nt}\sin(nt)\}$, Dokl. Akad. Nauk **346**(3) (1996), 297–298. (Russian)

[13] Z. V. Mamedova, *On basis properties of degenerate exponential system*, Appl. Math. **3** (2012), 1963–1966.

[14] E. I. Moiseev, *On the basis property of sine and cosine systems in a weighted space*, Differ. Uravn. **34**(1) (1998), 40–44. (Russian)

[15] E. I. Moiseev, *The basis property of a system of eigenfunctions of a differential operator in a weighted space*, Differ. Uravn. **35**(2) (1999), 200–205. (Russian)

[16] B. Muckenhoupt, *Weighted norm inequalities for the Hardy maximal function*, Trans. Amer. Math. Soc. **165** (1972), 207–226.

[17] I. P. Natanson. *Theory of Functions of a Real Variable*, Moscow, Nauka, 1974. (Russian)

[18] S. S. Pukhov, *Bases of exponentials, sines, and cosines in weighted spaces on a finite interval*, Izv. Math. **75**(2) (2011), 413–443.

[19] S. S. Pukhov, *Bases of exponentials in weighted spaces generated by zeros of the Fourier-Stieltjes transform*, Math. Notes **95**(5-6) (2014), 820–832.

[20] S. S. Pukhov and A. M. Sedletskiĭ, *Bases of exponentials, sines, and cosines in weighted spaces on a finite interval*, Dokl. Akad. Nauk **425**(4) (2009), 452–455. (Russian)

[21] S. R. Sadigova and Z. V .Mamedova, *Frames from cosines with the degenerate coefficients*, Amer. J. Appl. Math. Statist. **1**(3) (2013), 36–40.

[22] A. Sh. Shukurov, *Necessary condition for Kostyuchenko type systems to be a basis in Lebesgue spaces*, Colloq. Math. **127**(1) (2012), 105–109.

[23] A. Sh. Shukurov, *Addendum to "Necessary condition for Kostyuchenko type systems to be a basis in Lebesgue spaces"*, Colloq. Math. **137**(2) (2014), 297–298.

[24] A. Sh. Shukurov, *The power system is never a basis in the space of continuous functions*, Amer. Math. Monthly **122**(2) (2015), 137.

[25] A. Sh. Shukurov, *Impossibility of power series expansion for continuous functions*, Azerb. J. Math. **6**(1) (2016), 122–125.

[26] A. Sh. Shukurov, *Comment on "On the Frame Properties of Degenerate System of Sines"*, J. Funct. Spaces **2017**, Art. ID 9257076, 4 pp.

Institute of Mathematics and Mechanics NAS of Azerbaijan, B. Vahabzade 9, Az1141, Baku, Azerbaijan

E-mail address: ashshukurov@gmail.com