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# On the properties of $k$-balancing and $k$-Lucas-balancing numbers

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Abstract. The $k$-Lucas-balancing numbers are obtained from a special sequence of squares of $k$-balancing numbers in a natural form. In this paper, we will study some properties of $k$-Lucas-balancing numbers and establish relationship between these numbers and $k$-balancing numbers.

## 1. Introduction

Balancing numbers and Lucas-balancing numbers cover a wide range of interest for many number theorists in the recent years. Balancing numbers $B_n$ are the terms of the sequence $\{0, 1, 6, 35, 204, \dots\}$ that satisfy the recurrence relation

$$B_{n+1} = 6B_n - B_{n-1}, \quad n \geq 1,$$

beginning with the values $B_0 = 0$ and $B_1 = 1$ (see [1]). On the other hand, the numbers closely associate with the balancing numbers are the Lucas-balancing numbers $C_n$ that are the terms of the sequence

$$\{1, 3, 17, 99, 577, \dots\}.$$

Lucas-balancing numbers are recursively defined in the same way as balancing numbers but with different initials, that is,

$$C_{n+1} = 6C_n - C_{n-1}, \quad n \geq 1,$$

with initials $C_0 = 1$ and $C_1 = 3$ (see [6]). Binet's formulas for balancing and Lucas-balancing numbers are useful tools to derive identities for these sequences. They are given by the relations

$$B_n = \frac{\lambda_1^n - \lambda_2^n}{\lambda_1 - \lambda_2}, \;\; C_n = \frac{\lambda_1^n + \lambda_2^n}{2},$$

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where $\lambda_1 = 3+\sqrt{8}$ and $\lambda_2 = 3-\sqrt{8}$ (see [1, 6]).

Besides the usual balancing numbers, many kinds of generalizations of these numbers have been presented in the literature (see [2, 3, 4, 5, 7, 10]). In particular, one of the generalizations of balancing numbers, namely $k$-balancing numbers, were studied extensively in [10]. These numbers are defined recursively, depending on one real parameter $k$, by

$$B_{k,0} = 0, B_{k,1} = 1, \text{ and } B_{k,n+1} = 6kB_{k,n} - B_{k,n-1} \text{ for } k \geq 1.$$

The first few $k$-balancing numbers are

$$\begin{aligned}
B_{k,0} &= 0,\\
B_{k,1} &= 1,\\
B_{k,2} &= 6k\\
B_{k,3} &= 36k^2 - 1,\\
B_{k,4} &= 216k^3 - 12k,\\
B_{k,5} &= 1296k^4 - 108k^2 + 1,\\
B_{k,6} &= 7776k^5 - 864k^3 + 18k,\\
B_{k,7} &= 46656k^6 - 6480k^4 + 216k^2 - 1,\\
B_{k,8} &= 279936k^7 - 46656k^5 + 2160k^3 - 24k, \text{etc.}
\end{aligned}$$

It is observed that for $k = 1$, the usual sequence of balancing numbers $\{0, 1, 6, 35, 204, \ldots\}$ is obtained.

Like balancing numbers, $k$-balancing numbers are also generated through matrices which are called $k$-balancing matrices and studied in [11]. According to Ray [11], the $k$-balancing matrix denoted by $M$ is a second order matrix whose entries are the first three $k$-balancing numbers 0, 1 and $6k$, that is

$$M = \begin{pmatrix} 6k & -1 \\ 1 & 0 \end{pmatrix}.$$

He has also shown that, for any natural number $n$,

$$M^n = \begin{pmatrix} B_{k,n+1} & -B_{k,n} \\ B_{k,n} & -B_{k,n-1} \end{pmatrix}.$$

Indeed, the matrix representation is a powerful technique for proving many identities of $k$-balancing numbers.

Many important identities such as Catalan identity, Simson's identity etc. for $k$-balancing numbers are also shown in [11]. Few properties that the $k$-balancing numbers satisfy are summarized below.

- Binet's formula for $k$-balancing numbers:

$$B_{k,n} = \frac{\lambda_k^n - \lambda_k^{-n}}{\lambda_k - \lambda_k^{-1}},\ \lambda_k = 3k + \sqrt{9k^2 - 1}.$$

- Negative extension of $k$-balancing numbers: $B_{k,-n} = -B_{k,n}$.
- Catalan's identity for $k$-balancing numbers:

$$B_{k,n}^2 - B_{k,n-r}B_{k,n+r} = B_{k,r}^2.$$

- Simson's or Cassini's identity for $k$-balancing numbers:

$$B_{k,n}^2 - B_{k,n-1}B_{k,n+1} = 1.$$

- Generating function for $k$-balancing numbers:

$$f_k(x) = \frac{x}{1-6kx-x^2}.$$

- For odd $k$-balancing numbers, $B_{k,2n+1} = B_{k,n+1}^2 - B_{k,n}^2$.
- For even $k$-balancing numbers, $B_{k,2n} = \frac{1}{6k}[B_{k,n+1}^2 - B_{k,n-1}^2]$.
- First combinatorial formula for $k$-balancing numbers:

$$B_{k,n} = \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^i \binom{n-1-i}{i} (6k)^{n-2i-1}.$$

- Second combinatorial formula for $k$-balancing numbers:

$$B_{k,n} = \frac{1}{2^{n-1}} \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{2i+1} (6k)^{n-2i-1}(36k^2-4)^i.$$

An application of Binet's formula to $k$-balancing numbers gives the identity

$$B_{k,m}B_{k,n+1} - B_{k,m+1}B_{k,n} = B_{k,m-n},$$

which we call D'Ocagne's identity for $k$-balancing numbers.

## 2. Some identities involving $k$-Lucas-balancing numbers

Though the sequence of $k$-Lucas-balancing numbers was introduced in [7], in the present article, these numbers are studied more elaborately. In [7], the sequence of $k$-Lucas-balancing numbers is defined recursively by

$$C_{k,n+1} = 6kC_{k,n} - C_{k,n-1}, \quad n \geq 1 \tag{2.1}$$

with initial conditions $C_{k,0} = 1,\ C_{k,1} = 3k$. The first few $k$-Lucas-balancing numbers are

$$\begin{aligned}
C_{k,0} &= 1,\\
C_{k,1} &= 3k,\\
C_{k,2} &= 18k^2 - 1,\\
C_{k,3} &= 108k^3 - 9k,\\
C_{k,4} &= 648k^4 - 72k^2 + 1,\\
C_{k,5} &= 3888k^5 - 540k^3 + 15k,\\
C_{k,6} &= 23328k^6 - 3888k^4 + 162k^2 - 1, \text{etc.}
\end{aligned}$$

The present section involves some important identities concerning $k$-Lucas-balancing numbers. Before establishing the identities, we first prove the following fact.

**Lemma 2.1.** *For any integer $n$, the number $(9k^2-1)B_{k,n}^2+1$ is a perfect square.*

*Proof.* Using Binet's formula for $k$-balancing numbers, and since $\lambda_k - \lambda_k^{-1} = 2\sqrt{9k^2-1}$, we have

$$\begin{aligned}
B_{k,n}^2 &= \Big(\frac{\lambda_k^n - \lambda_k^{-n}}{\lambda_k - \lambda_k^{-1}}\Big)^2\\
&= \frac{\lambda_k^{2n} + \lambda_k^{-2n} - 2}{4(9k^2-1)}.
\end{aligned}$$

It follows that, for all integer $n$,

$$(9k^2-1)B_{k,n}^2 + 1 = \frac{[\lambda_k^n + \lambda_k^{-n}]^2}{4}$$

which is a perfect square. □

Lemma 2.1 leads to the expression

$$C_{k,n}^2 = (9k^2-1)B_{k,n}^2 + 1 \tag{2.2}$$

which yields a first kind of consequence for the generation of the $k$-Lucas-balancing numbers.

**Lemma 2.2** (Binet's formula)**.** *The closed form of $k$-Lucas-balancing numbers is given by*

$$C_{k,n} = \frac{\lambda_k^n + \lambda_k^{-n}}{2},\ \ \lambda_k = 3k + \sqrt{9k^2-1}.$$

*Proof.* The characteristic equation $\lambda^2 - 6k\lambda - 1 = 0$ of (2.1) gives the roots $\lambda_k = 3k + \sqrt{9k^2-1}$ and $\lambda_k^{-1} = 3k - \sqrt{9k^2-1}$. Therefore, the general solution of (2.1) is $C_{k,n} = A\lambda_k^n + B\lambda_k^{-n}$, where $A$ and $B$ are arbitrary constants to be determined. Applying the initial conditions $C_{k,0} = 1$ and $C_{k,1} = 3k$ for $n = 0$ and $n = 1$, we obtain $A = 1/2$ and $B = 1/2$ and hence $C_{k,n}$. □

In particular, for $k = 1$, the Binet's formula for Lucas-balancing numbers is obtained.

**Lemma 2.3** (Asymptotic behavior)**.** *If $C_{k,n}$ are the $k$-Lucas-balancing numbers, then* $\lim_{n\to\infty} C_{k,n}/C_{k,n-r} = \lambda_k^r$, *where* $\lambda_k = 3k + \sqrt{9k^2-1}$.

*Proof.* Since $\lim_{n\to\infty} B_{k,n}/B_{k,n-r} = \lambda_k^r$, using (2.2), we get the desired result. □

Using Binet's formula for $k$-Lucas-balancing numbers gives rise to certain important identities concerning these numbers.

**Theorem 2.4** (Catalan's identity)**.** *For natural numbers $n$ and $r$ with $n \geq r$,*

$$C_{k,n-r}C_{k,n+r} - C_{k,n}^2 = \frac{1}{2}\Big[C_{k,2r} - 1\Big].$$

*Proof.* Using Binet's formula, the left hand side expression reduces to

$$\frac{\lambda_k^{n-r} + \lambda_k^{-n+r}}{2} + \frac{\lambda_k^{n+r} + \lambda_k^{-n-r}}{2} - \left(\frac{\lambda_k^n + \lambda_k^{-n}}{2}\right)^2.$$

After some algebraic manipulations, it further simplifies to $\lambda_k^{2r} + \lambda_k^{-2r} - 1/2$, and the result follows. □

In particular, since $C_{k,2} = 18k^2 - 1$, the Catalan identity for $k$-Lucas-balancing numbers reduces for $r = 1$ to

$$C_{k,n-1}C_{k,n+1} - C_{k,n}^2 = 9k^2 - 1,$$

which we call Simson's or Cassini's identity for $k$-Lucas-balancing numbers. Setting $k = 1$ in Simson's identity, the Cassini formula $C_{n-1}C_{n+1} - C_n^2 = 8$ for Lucas-balancing numbers is obtained.

Expanding the Binet identity for $C_{k,n}$, and as $\lambda_k = 3k + \sqrt{9k^2-1}$, the following combinatorial formula for $k$-Lucas-balancing numbers can be easily obtained.

**Theorem 2.5** (Combinatorial formula for $k$-Lucas-balancing numbers)**.** *Let $\binom{n}{r}$ be the usual notation for binomial coefficient, then*

$$C_{k,n} = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2i} (3k)^{n-2i}(9k^2-1)^i. \tag{2.3}$$

In particular, for $k = 1$ in (2.3), the combinatorial formula

$$C_n = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2i} 8^i 3^{n-2i}$$

for Lucas-balancing numbers is obtained.

We now present some identities of $k$-Lucas-balancing numbers that are related to $k$-balancing numbers.

**Proposition 2.6.** *For $n \geq 1$, we have $B_{k,n+1} - B_{k,n-1} = 2C_{k,n}$.*

*Proof.* The method of induction will apply to prove this result. Clearly, the result holds for $n = 1$ as $B_{k,2} - B_{k,0} = 6k = 2C_{k,1}$. Assume that the result holds until $n - 1$, that is $2C_{k,n-2} = B_{k,n-1} - B_{k,n-3}$ and $2C_{k,n-1} = B_{k,n} - B_{k,n-2}$. By virtue of the recurrence relation (2.1), we have $2C_{k,n} = 6k \cdot 2C_{k,n-1} - 2C_{k,n-2}$. The desired result will be obtained by using the hypothesis and the recurrence relation for $k$-balancing numbers. □

It is observed that, for $k = 1$, one has $2C_n = B_{n+1} - B_{n-1}$ (see [6]).

A similar proof gives rise to the following identity.

**Proposition 2.7.** *For $n \geq 1$, we have $C_{k,n+1} - C_{k,n-1} = 2(9k^2 - 1)B_{k,n}$.*

**Proposition 2.8.** *For any natural number $n$, the equality $2C_{k,n}B_{k,n} = B_{k,2n}$ holds.*

*Proof.* It is enough to use Binet's formulas to prove this result. □

**Theorem 2.9** (Convolution theorem)**.** *For $m \geq 1$,*

$$C_{k,n+1}C_{k,m} - C_{k,n}C_{k,m-1} = (9k^2 - 1)B_{k,n+m}.$$

*Proof.* We use the induction on $m$. Clearly the result holds for $m = 1$, because, by virtue of Proposition 2.6 and recurrence relations for $k$-balancing and $k$-Lucas-balancing numbers, we have

$$\begin{aligned}
C_{k,n+1}C_{k,1} - C_{k,n}C_{k,0} &= 3kC_{k,n+1} - C_{k,n} \\
&= \frac{1}{2}\left[6kC_{k,n+1} - C_{k,n} - C_{k,n}\right] \\
&= \frac{1}{2}\left[C_{k,n+2} - C_{k,n}\right] \\
&= \frac{1}{4}\left[B_{k,n+3} - 2B_{k,n+1} + B_{k,n-1}\right] \\
&= \frac{1}{4}\left[6kB_{k,n+2} - 3B_{k,n+1} + 6kB_{k,n} - B_{k,n+1}\right] \\
&= \frac{1}{4}\left[6kB_{k,n+2} - 4B_{k,n+1} + 6kB_{k,n}\right] \\
&= \frac{1}{4}\left[6k(6kB_{k,n+1} - B_{k,n}) - 4B_{k,n+1} + 6kB_{k,n}\right] \\
&= (9k^2 - 1)B_{k,n+1}.
\end{aligned}$$

Assume that the formula is valid until $m-1$. Then,

$$C_{k,n+1}C_{k,m-1} - C_{k,n}C_{k,m-2} = (9k^2 - 1)B_{k,n+m-1}.$$

We proceed to show that the result is valid for $m$. It is observed that

$$\begin{aligned}
(9k^2 - 1)B_{k,n+m} &= (9k^2 - 1)[6kB_{k,n+m-1} - B_{k,n+m-2}] \\
&= 6k[C_{k,n+1}C_{k,m-1} - C_{k,n}C_{k,m-2}] \\
&\quad - C_{k,n+1}C_{k,m-2} + C_{k,n}C_{k,m-3} \\
&= C_{k,n+1}[6kC_{k,m-1} - C_{k,m-2}] - C_{k,n}[6kC_{k,m-2} - C_{k,m-3}] \\
&= C_{k,n+1}C_{k,m} - C_{k,n}C_{k,m-1},
\end{aligned}$$

and the proof completes. □

The following result is an immediate consequence of Theorem 2.9, by replacing $m$ by $n+1$.

**Corollary 2.10.** *For* $n \geq 1$, $C_{k,n+1}^2 + C_{k,n}^2 = (9k^2 - 1)B_{k,2n+1}$.

**Observation 2.11.** In particular, for $k = 1$, the identity of Theorem 2.9 reduces to a known formula concerning balancing and Lucas-balancing numbers, $C_{n+1}C_m - C_nC_{m-1} = 8B_{n+m}$ (see [6]). Further, putting $k = 1$ in the expression given in Corollary 2.10 yields $C_{n+1}^2 - C_n^2 = 8B_{2n+1}$. It is also observed that, for $m = 1$, the expression of Theorem 2.9 leads to $3kC_{k,n+1} - C_{k,n} = (9k^2 - 1)B_{k,n+1}$ which is equivalent to $C_{k,n+2} - 3kC_{k,n+1} = (9k^2 - 1)B_{k,n+1}$. Consequently, replacing $n$ by $n-1$ in the last expression gives the formula $C_{k,n+1} - 3kC_{k,n} = (9k^2 - 1)B_{k,n}$.

**Theorem 2.12** (D'Ocagne's identity)**.** *For* $m \geq n$,

$$C_{k,m}C_{k,n+1} - C_{k,m+1}C_{k,n} = -B_{k,m-n}.$$

*Proof.* By using induction on $n$, it is obvious that the identity holds for $n = 0$, because

$$C_{k,m}C_{k,1} - C_{k,m+1}C_{k,0} = -(C_{k,m+1} - 3kC_{k,m}) = -(9k^2 - 1)B_{k,m},$$

by Observation 2.10. Assume that the identity holds until $n - 1$. That is, by inductive hypothesis, we have

$$C_{k,m}C_{k,n-1} - C_{k,m+1}C_{k,n-2} = -(9k^2 - 1)B_{k,m-(n-2)}$$

and

$$C_{k,m}C_{k,n} - C_{k,m+1}C_{k,n-1} = -(9k^2 - 1)B_{k,m-(n-1)}.$$

In the inductive step, using recurrence relation for $k$-Lucas-balancing numbers, the expression $C_{k,m}C_{k,n+1} - C_{k,m+1}C_{k,n}$ reduces to

$$C_{k,m}(6kC_{k,n} - C_{k,n-1}) - C_{k,m+1}(6kC_{k,n-1} - C_{k,n-2}).$$

A further simplification leads to

$$6k(C_{k,m}C_{k,n} - C_{k,m+1}C_{k,n-1}) + (C_{k,m}C_{k,n-1} + C_{k,m+1} - C_{k,n-2}),$$

and the result follows by using the inductive hypothesis. $\square$

Cassini's formula for $k$-Lucas-balancing numbers can also be obtained from D'Ocagne's identity by setting $n = m - 1$.

The $k$-Lucas-balancing numbers can also be generated through matrices. Let

$$N = \begin{pmatrix} 3k & 9k^2 - 1 \\ 1 & 3k \end{pmatrix}$$

be the matrix representation of $k$-Lucas-balancing numbers. Indeed, the matrix $N$ is determined by taking into account the matrix $S = \begin{pmatrix} 3 & 8 \\ 1 & 3 \end{pmatrix}$ which was introduced in [9]. It is easy to verify by induction that

$$N^n = \begin{pmatrix} C_{k,n} & (9k^2 - 1)B_{k,n} \\ B_{k,n} & C_{k,n} \end{pmatrix}.$$

Consider the matrix

$$I - sN = \begin{pmatrix} 1 - 3ks & -(9k^2 - 1)s \\ -s & 1 - 3ks \end{pmatrix},$$

where $I$ denotes the identity matrix of same order as $N$. The determinant $|I - sN| = 1 - 6ks + s^2$ is nonzero and hence the matrix $I - sN$ is invertible and

$$(I - sN)^{-1} = \frac{1}{1 - 6ks + s^2} \begin{pmatrix} 1 - 3ks & (9k^2 - 1)s \\ s & 1 - 3ks \end{pmatrix}.$$

Further,

$$(I - sN)^{-1} = \sum_{n=0}^{\infty} s^n N^n,$$

which follows that

$$s^0N^0 + s^1N^1 + s^2N^2 + \ldots = \begin{pmatrix} \frac{1-3ks}{1-6ks+s^2} & \frac{(9k^2-1)s}{1-6ks+s^2} \\ \frac{s}{1-6ks+s^2} & \frac{1-3ks}{1-6ks+s^2} \end{pmatrix}.$$

Comparing the $(1,1)$ entries from the both sides, the generating function for $k$-Lucas-balancing numbers $G(s)$ is obtained as follows:

$$G(s) = C_{k,0} + sC_{k,1} + s^2C_{k,2} + \ldots = \sum_{n=0}^{\infty} s^n C_{k,n} = \frac{1-3ks}{1-6ks+s^2}.$$

An application of the generating function gives the following combinatorial identity for $k$-Lucas-balancing numbers.

**Proposition 2.13.** *One has*

$$C_{k,n} = \frac{1}{2}\sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^i \frac{n}{n-i} \binom{n-i}{i} (6k)^{n-2i}.$$

*Proof.* The Chebyshev polynomial of the second kind is given as

$$\frac{1}{1-2tz+z^2} = \sum_{n=0}^{\infty} \left( \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^i \binom{n-i}{i} (2t)^{n-2i} \right) z^n. \tag{2.4}$$

Putting $z = s$ and $t = 3k$ in (2.4), we get

$$\begin{aligned}
\sum_{n=0}^{\infty} C_{k,n}s^n &= \frac{1}{1-6ks+s^2} - \frac{3ks}{1-6ks+s^2} \\
&= \sum_{n=0}^{\infty} \left( \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^i \binom{n-i}{i} (6k)^{n-2i} \right) s^n \\
&- 3k \sum_{n=0}^{\infty} \left( \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^i \binom{n-i}{i} (6k)^{n-2i} \right) s^{n+1} \\
&= \sum_{n=0}^{\infty} \left( \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^i \binom{n-i}{i} (6k)^{n-2i} \right) s^n \\
&- \frac{1}{2} \sum_{n=0}^{\infty} \left( \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^i \binom{n-i-1}{i} (6k)^{n-2i} \right) s^n.
\end{aligned}$$

It follows that

$$C_{k,n} = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^i \left\{ \binom{n-i}{i} - \frac{1}{2} \binom{n-i-1}{i} \right\} (6k)^{n-2i}$$
$$= \frac{1}{2} \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^i \frac{n}{n-i} \binom{n-i}{i} (6k)^{n-2i},$$

which completes the proof. □

We end this section by establishing a product formula for $k$-Lucas-balancing numbers.

**Theorem 2.14.** *For $n \geq 2$, we have*

$$\prod_{i=1}^{n-1} C_{k,2^i} = \frac{1}{6k2^{n-1}} B_{k,2^n}. \tag{2.5}$$

*Proof.* Once again the method of induction is used to prove the result. Clearly the result is true for $n = 2$, because

$$\prod_{i=1}^{1} C_{k,2} = 18k^2 - 1 = \frac{1}{12k} B_{k,4}.$$

Assume that the result holds for $n$. Then, by inductive hypothesis, (2.5) holds. In the inductive step, $\prod_{i=1}^{n} C_{k,2^i}$ splits into the expression $C_{k,2^n} \prod_{i=1}^{n-1} C_{k,2^i}$. Using (2.5), a further simplification leads to

$$\prod_{i=1}^{n} C_{k,2^i} = \frac{1}{6k2^{n-1}} B_{k,2^n} C_{k,2^n}.$$

By virtue of Proposition 2.7, we have $\prod_{i=1}^{n} C_{k,2^i} = \frac{1}{6k2^n} B_{k,2^{n+1}}$, which completes the proof. □

In particular, for $k = 1$, for both balancing and Lucas-balancing numbers, we have $\prod_{i=1}^{n-1} C_{2^i} = \frac{1}{6 \cdot 2^{n-1}} B_{2^n}$.

## 3. Some identities involving common factors of $k$-balancing and $k$-Lucas-balancing numbers

In this section, we present some generalized identities involving common factors of $k$-balancing and $k$-Lucas-balancing numbers. For the derivation of the identities we shall use Binet's formulas for both these numbers. Recall that Binet's formulas for $k$-balancing and $k$-Lucas-balancing numbers are, respectively,

$$\frac{\lambda_{k_1}^n - \lambda_{k_2}^n}{\lambda_{k_1} - \lambda_{k_2}} \text{ and } \frac{\lambda_{k_1}^n + \lambda_{k_2}^n}{2},$$

where $\lambda_{k_1} = 3k + \sqrt{9k^2-1}$ and $\lambda_{k_2} = 3k - \sqrt{9k^2-1}$. It is observed that

$$\lambda_{k_1} + \lambda_{k_2} = 6k,\ \lambda_{k_1} - \lambda_{k_2} = 2\sqrt{9k^2-1} \text{ and } \lambda_{\mathrm{k}_1}\lambda_{\mathrm{k}_2} = 1.$$

**Proposition 3.1.** *For $n \geq m+1$, we have $B_{k,n+m}B_{k,n-m} - B_{k,n}^2 = -B_{k,m}^2$.*

*Proof.* Using Binet's formula for $k$-balancing numbers, the left hand side expression reduces to

$$\frac{1}{(\lambda_{k_1} - \lambda_{k_2})^2}\left[-\lambda_{k_1}^{n+m}\lambda_{k_2}^{n-m} - \lambda_{k_2}^{n+m}\lambda_{k_1}^{n-m} + 2\right].$$

Since $\lambda_{k_1}\lambda_{k_2} = 1$, the expression simplifies to

$$-\frac{1}{(\lambda_{k_1} - \lambda_{k_2})^2}\left[\lambda_{k_1}^{2m} + \lambda_{k_2}^{2m} - 2\right]$$

which is $B_{k,m}^2$. □

The following result can be proved analogously.

**Proposition 3.2.** *For $n \geq m+1$, $C_{k,n+m}C_{k,n-m} - C_{k,n}^2 = (9k^2-1)$.*

**Proposition 3.3.** *For $n \geq 1$ and $p \geq 0$, $B_{k,4n+p} - B_{k,p} = 2B_{k,2n}C_{k,2n+p}$.*

*Proof.* For $n \geq 1$ and $p \geq 0$,

$$\begin{aligned}
2B_{k,2n}C_{k,2n+p} &= \left(\frac{\lambda_{k_1}^{2n} - \lambda_{k_2}^{2n}}{\lambda_{k_1} - \lambda_{k_2}}\right)\left(\frac{\lambda_{k_1}^{n} + \lambda_{k_2}^{n}}{2}\right)\\
&= \left(\frac{\lambda_{k_1}^{4n+p} - \lambda_{k_2}^{4n+p}}{\lambda_{k_1} - \lambda_{k_2}}\right) - \left(\frac{\lambda_{k_1}^{2n+p}\lambda_{k_2}^{2n} - \lambda_{k_2}^{2n+p}\lambda_{k_1}^{2n}}{\lambda_{k_1} - \lambda_{k_2}}\right)\\
&= B_{k,4n+p} - \frac{\lambda_{k_1}^{p} - \lambda_{k_2}^{p}}{\lambda_{k_1} - \lambda_{k_2}}\\
&= B_{k,4n+p} - B_{k,p},
\end{aligned}$$

which completes the proof. □

**Proposition 3.4.** *For $n \geq 1$ and $p \geq 0$, $B_{k,4n+p} + B_{k,p} = 2B_{k,2n+p}C_{k,2n}$.*

*Proof.* The proof of this result is analogous to Proposition 3.3. □

**Proposition 3.5.** *For $n \geq 1$ and $p \geq 0$, $C_{k,4n+p} + C_{k,p} = 2C_{k,2n+p}C_{k,2n}$.*

*Proof.* For $n \geq 1$ and $p \geq 0$,

$$\begin{aligned}
2C_{k,2n}C_{k,2n+p} &= \left(\lambda_{k_1}^{2n} + \lambda_{k_2}^{2n}\right)\left(\frac{\lambda_{k_1}^{2n+p} + \lambda_{k_2}^{2n+p}}{2}\right) \\
&= \left(\frac{\lambda_{k_1}^{4n+p} + \lambda_{k_2}^{4n+p}}{2}\right) - \left(\frac{\lambda_{k_1}^{2n+p}\lambda_{k_2}^{2n} + \lambda_{k_2}^{2n+p}\lambda_{k_1}^{2n}}{2}\right) \\
&= C_{k,4n+p} + \frac{\lambda_{k_1}^{p} + \lambda_{k_2}^{p}}{2} \\
&= C_{k,4n+p} + C_{k,p}.
\end{aligned}$$

This ends the proof. □

The following result can be shown similarly.

**Proposition 3.6.** *For $n \geq 1$ and $p \geq 0$,*

$$C_{k,4n+p} - C_{k,p} = (18k^2 - 2)B_{k,2n+p}B_{k,2n}.$$

**Proposition 3.7.** *For $n \geq 1$ and $p \geq 0$,*

$$B_{k,4n+p} - B_{k,p} = 4B_{k,n}C_{k,n}C_{k,2n+p}.$$

*Proof.* For $n \geq 1$ and $p \geq 0$,

$$\begin{aligned}
4B_{k,n}C_{k,n}C_{k,2n+p} &= 4\left(\frac{\lambda_{k_1}^{n} - \lambda_{k_2}^{n}}{\lambda_{k_1} - \lambda_{k_2}}\right)\left(\frac{\lambda_{k_1}^{n} + \lambda_{k_2}^{n}}{2}\right)\left(\frac{\lambda_{k_1}^{2n+p} + \lambda_{k_2}^{2n+p}}{2}\right) \\
&= \left(\frac{\lambda_{k_1}^{n} - \lambda_{k_2}^{n}}{\lambda_{k_1} - \lambda_{k_2}}\right)(\lambda_{k_1}^{3n+p} + \lambda_{k_2}^{3n+p} + \lambda_{k_1}^{n}\lambda_{k_2}^{2n+p} + \lambda_{k_2}^{n}\lambda_{k_1}^{2n+p}) \\
&= \left(\frac{\lambda_{k_1}^{4n+p} - \lambda_{k_2}^{4n+p}}{\lambda_{k_1} - \lambda_{k_2}}\right) - \frac{\lambda_{k_1}^{p} - \lambda_{k_2}^{p}}{\lambda_{k_1} - \lambda_{k_2}} \\
&= B_{k,4n+p} - B_{k,p},
\end{aligned}$$

as desired. □

The proof of the following proposition is analogous to Proposition 3.5.

**Proposition 3.8.** *For $n \geq 1$ and $p \geq 0$,*

$$C_{k,4n+p} - C_{k,p} = (36k^2 - 4)B_{k,n}C_{k,n}B_{k,2n+p}.$$

**Proposition 3.9.** *For $n \geq 1$ and $p \geq 0$, $B_{k,3n+p} - B_{k,n+p} = 2B_{k,n}C_{k,2n+p}$.*

*Proof.* For $n \geq 1$ and $p \geq 0$,

$$\begin{aligned}
2B_{k,n}C_{k,2n+p} &= 2\left(\frac{\lambda_{k_1}^n - \lambda_{k_2}^n}{\lambda_{k_1} - \lambda_{k_2}}\right)\left(\frac{\lambda_{k_1}^{2n+p} + \lambda_{k_2}^{2n+p}}{2}\right) \\
&= \left(\frac{\lambda_{k_1}^{3n+p} - \lambda_{k_2}^{3n+p}}{\lambda_{k_1} - \lambda_{k_2}}\right) - \left(\frac{\lambda_{k_1}^{2n+p}\lambda_{k_2}^n - \lambda_{k_2}^{2n+p}\lambda_{k_1}^n}{\lambda_{k_1} - \lambda_{k_2}}\right) \\
&= B_{k,3n+p} - \frac{\lambda_{k_1}^{n+p} - \lambda_{k_2}^{n+p}}{\lambda_{k_1} - \lambda_{k_2}} \\
&= B_{k,3n+p} - B_{k,n+p},
\end{aligned}$$

which completes the proof. □

**Proposition 3.10.** *For $n \geq 1$ and $p \geq 0$,*

$$B_{k,3n+p} + B_{k,n+p} = 2B_{k,2n+p}C_{k,n}.$$

*Proof.* The proof of this result is analogous to Proposition 3.9. □

**Proposition 3.11.** *For $n \geq 1$ and $p \geq 0$,*

$$C_{k,3n+p} - C_{k,n+p} = (18k^2 - 2)B_{k,n}B_{k,2n+p}.$$

*Proof.* For $n \geq 1$ and $p \geq 0$,

$$\begin{aligned}
(18k^2 - 2)B_{k,n}B_{k,2n+p} &= \frac{1}{2}(\lambda_{k_1} - \lambda_{k_2})^2 B_{k,n}B_{k,2n+p} \\
&= \frac{1}{2}(\lambda_{k_1} - \lambda_{k_2})^2\left(\frac{\lambda_{k_1}^n - \lambda_{k_2}^n}{\lambda_{k_1} - \lambda_{k_2}}\right)\left(\frac{\lambda_{k_1}^{2n+p} - \lambda_{k_2}^{2n+p}}{\lambda_{k_1} - \lambda_{k_2}}\right) \\
&= \left(\frac{\lambda_{k_1}^{3n+p} + \lambda_{k_2}^{3n+p}}{2}\right) - \left(\frac{\lambda_{k_1}^{2n+p}\lambda_{k_2}^n + \lambda_{k_2}^{2n+p}\lambda_{k_1}^n}{2}\right) \\
&= C_{k,3n+p} - \frac{\lambda_{k_1}^{n+p} + \lambda_{k_2}^{n+p}}{2} \\
&= C_{k,3n+p} - C_{k,n+p},
\end{aligned}$$

as required. □

The following result can be shown analogously.

**Proposition 3.12.** *For $n \geq 1$ and $p \geq 0$,*

$$C_{k,3n+p} + C_{k,n+p} = 2C_{k,n}C_{k,2n+p}.$$

**Proposition 3.13.** *For any natural numbers $n, m$ and $r$,*

$$B_{k,n+m}B_{k,r+m} - B_{k,n}B_{k,r} = B_{k,m}B_{k,m+n+r}.$$

*Proof.* For any natural numbers $n, m$ and $r$,

$$\begin{aligned}
&B_{k,n+m}B_{k,r+m} - B_{k,n}B_{k,r}\\
&= \frac{\lambda_{k_1}^{n+m} - \lambda_{k_2}^{n+m}}{\lambda_{k_1} - \lambda_{k_2}} \frac{\lambda_{k_1}^{r+m} - \lambda_{k_2}^{r+m}}{\lambda_{k_1} - \lambda_{k_2}} - \frac{\lambda_{k_1}^{n} - \lambda_{k_2}^{n}}{\lambda_{k_1} - \lambda_{k_2}} \frac{\lambda_{k_1}^{r} - \lambda_{k_2}^{r}}{\lambda_{k_1} - \lambda_{k_2}}\\
&= \frac{1}{(\lambda_{k_1} - \lambda_{k_2})^2} \left[\lambda_{k_1}^{n+2m+r} - \lambda_{k_1}^{n+r} + \lambda_{k_2}^{n+2m+r} - \lambda_{k_2}^{n+r}\right]\\
&= \frac{1}{(\lambda_{k_1} - \lambda_{k_2})^2} \left[\lambda_{k_1}^{n+m+r}\lambda_{k_1}^{m} - \lambda_{k_1}^{n+m+r}\lambda_{k_2}^{m}\right.\\
&\quad \left.+\lambda_{k_2}^{n+m+r}\lambda_{k_2}^{m} - \lambda_{k_2}^{n+m+r}\lambda_{k_1}^{m}\right]\\
&= \frac{\lambda_{k_1}^{n+m+r} - \lambda_{k_2}^{n+m+r}}{\lambda_{k_1} - \lambda_{k_2}} \frac{\lambda_{k_1}^{m} - \lambda_{k_2}^{m}}{\lambda_{k_1} - \lambda_{k_2}}\\
&= B_{k,m+n+r}B_{k,m},
\end{aligned}$$

which completes the proof. □

Applications of Binet's formulas for $k$-balancing and $k$-Lucas-balancing numbers give rise to the following identities.

**Proposition 3.14.** *For any natural numbers $n$ and $m$,*

$$C_{k,nm}C_{k,n} + (9k^2 - 1)B_{k,nm}B_{k,n} = C_{k,(m+1)n}.$$

**Proposition 3.15.** *For any natural numbers $n$ and $m$,*

$$B_{k,nm}C_{k,n} + C_{k,nm}B_{k,n} = B_{k,(m+1)n}.$$

**Proposition 3.16.** *For any natural numbers $n$ and $m$,*

$$C_{k,m+n}^2 - C_{k,m}^2 = 4(9k^2 - 1)B_{k,2n+m}B_{k,m}.$$

**Proposition 3.17.** *For any natural numbers $n$ and $m$,*

$$C_{k,n+m}C_{k,n} - C_{k,n-m}C_{k,n} = (9k^2 - 1)B_{k,2n}B_{k,m}.$$

**Proposition 3.18.** *For any natural numbers $n, m$ and $r$,*

$$B_{k,m+2rn}B_{k,2n+m} - B_{k,2rn}B_{k,2n} = B_{k,2(r+1)n+m}B_{k,m}.$$

## 4. Generalized identities on the products of $k$-balancing and $k$-Lucas-balancing numbers

In this section, some generalized identities concerning the products of $k$-balancing and $k$-Lucas-balancing numbers are presented. Once again Binet's formulas play the key role to derive such identities.

**Proposition 4.1.** *For $n \geq m + 1$, $B_{k,n+m} - B_{k,n-m} = 2B_{k,m}C_{k,n}$.*

*Proof.* For $n \geq m+1$,

$$\begin{aligned} 2B_{k,m}C_{k,n} &= 2\left(\frac{\lambda_{k_1}^m - \lambda_{k_2}^m}{\lambda_{k_1} - \lambda_{k_2}}\right)\left(\frac{\lambda_{k_1}^n + \lambda_{k_2}^n}{2}\right) \\ &= \left(\frac{\lambda_{k_1}^{m+n} - \lambda_{k_2}^{m+n}}{\lambda_{k_1} - \lambda_{k_2}}\right) + \left(\frac{\lambda_{k_1}^m\lambda_{k_2}^n - \lambda_{k_2}^m\lambda_{k_1}^n}{\lambda_{k_1} - \lambda_{k_2}}\right) \\ &= B_{k,m+n} - (\lambda_{k_1}\lambda_{k_2})^m\left(\frac{\lambda_{k_1}^{n-m} - \lambda_{k_2}^{n-m}}{\lambda_{k_1} - \lambda_{k_2}}\right) \\ &= B_{k,m+n} - B_{k,n-m}, \end{aligned}$$

which ends the proof. □

**Observation 4.2.** For $m = 0$, the left hand side expression given in Proposition 4.1 is zero for $n \geq 1$, while for $m = 1$ and $m = 2$, it reduces to the identities $2C_{k,n} = B_{k,n+1} - B_{k,n-1}$ for $n \geq 2$ and $6nC_{k,n} = B_{k,n+2} - B_{k,n-2}$ for $n \geq 3$ and so on.

The following result can be shown similarly.

**Proposition 4.3.** *For $n \geq m+1$, $C_{k,n+m} - C_{k,n-m} = 2(9k^2-1)B_{k,m}B_{k,n}$.*

**Proposition 4.4.** *For $n \geq 1$ and $m \geq 0$,*

$$2B_{k,n}C_{k,2n+m} = B_{k,3n+m} - B_{k,n+m}.$$

*Proof.* For $n \geq 1$ and $m \geq 0$,

$$\begin{aligned} 2B_{k,n}C_{k,2n+m} &= 2\left(\frac{\lambda_{k_1}^n - \lambda_{k_2}^n}{\lambda_{k_1} - \lambda_{k_2}}\right)\left(\frac{\lambda_{k_1}^{2n+m} + \lambda_{k_2}^{2n+m}}{2}\right) \\ &= \left(\frac{\lambda_{k_1}^{3n+m} - \lambda_{k_2}^{3n+m}}{\lambda_{k_1} - \lambda_{k_2}}\right) + (\lambda_{k_1}\lambda_{k_2})^n\left(\frac{\lambda_{k_2}^{n+m} - \lambda_{k_1}^{n+m}}{\lambda_{k_1} - \lambda_{k_2}}\right) \\ &= B_{k,3n+m} - B_{k,n+m}, \end{aligned}$$

which completes the proof. □

**Observation 4.5.** For $m = 0$, the expression given in Proposition 4.3 leads to $2B_{k,n}C_{k,2n} = B_{k,3n} - B_{k,n}$ for $n \geq 1$, while for $m = 1$, it reduces to the identity $2B_{k,n}C_{k,2n+1} = B_{k,3n+1} - B_{k,n+1}$ for $n \geq 1$ and so on.

**Proposition 4.6.** *For $n \geq 1$ and $m \geq 0$,*

$$2B_{k,2n+m}C_{k,n} = B_{k,3n+m} + B_{k,n+m}.$$

*Proof.* The proof of this result is similar to Proposition 4.4. □

**Proposition 4.7.** *For $n \geq 1$ and $m \geq 0$,*

$$2B_{k,2n}C_{k,2n+m} = B_{k,4n+m} - B_{k,m}.$$

*Proof.* For $n \geq 1$ and $m \geq 0$,

$$\begin{aligned} 2B_{k,2n}C_{k,2n+m} &= 2\left(\frac{\lambda_{k_1}^{2n}-\lambda_{k_2}^{2n}}{\lambda_{k_1}-\lambda_{k_2}}\right)\left(\frac{\lambda_{k_1}^{2n+m}+\lambda_{k_2}^{2n+m}}{2}\right) \\ &= \left(\frac{\lambda_{k_1}^{4n+m}-\lambda_{k_2}^{4n+m}}{\lambda_{k_1}-\lambda_{k_2}}\right) + (\lambda_{k_1}\lambda_{k_2})^{2n}\left(\frac{\lambda_{k_2}^{m}-\lambda_{k_1}^{m}}{\lambda_{k_1}-\lambda_{k_2}}\right) \\ &= B_{k,4n+m} - B_{k,m}, \end{aligned}$$

which ends the proof. □

The following result can be shown similarly.

**Proposition 4.8.** *For $n \geq 1$ and $m \geq 0$,*

$$2B_{k,2n+m}C_{k,2n} = B_{k,4n+m} + B_{k,m}.$$

**Observation 4.9.** For $m = 0$, the expressions given in Proposition 4.6 and Proposition 4.7 lead to $2B_{k,2n}C_{k,2n} = B_{k,4n}$ for $n \geq 1$, while for $m = 1$, these expressions reduce to the identities $2B_{k,2n}C_{k,2n+1} = B_{k,4n+1} - 1$ and $2B_{k,2n+1}C_{k,2n} = B_{k,4n+1} + 1$ for $n \geq 1$ and so on.

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