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On Salagean type pseudo-starlike functions

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Abstract. We construct two new subclasses of univalent functions in the open unit disk $U = \{z : |z| < 1\}$. For the first class $\pounds_\lambda(\beta)$ of Salagean type λ-pseudo-starlike functions, using the sigmoid function, we establish upper bounds for the initial coefficients of the functions in this class. Furthermore, for the second class $\pounds_\lambda(\beta, \phi)$ we obtain Fekete–Szegö inequalities. The results presented in this paper generalize the recent work of Babalola.

1. Introduction

Let A denote the class of analytic functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

in the open unit disc

$$U = \{z : z \in \mathbb{C}, \ |z| < 1\}.$$

We denote by S be the class of all functions $f \in A$ which are univalent in U.

Denote by S^* the subclass of S of starlike functions, so that $f \in S^*$ if and only if, for $z \in U$,

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > 0.$$

For $\alpha > 0$, let $B_1(\alpha)$ denote the class of Bazilevič functions defined in the open unit disc U, normalized so that $f(0) = 0$, $f'(0) > 1$, and such that, for $z \in U$,

$$\Re\left(f'(z)\left(\frac{f'(z)}{z}\right)^{\alpha-1}\right) > 0.$$

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This class of functions was studied first by Singh [12], and considered subsequently by London and Thomas [6, 13].

If the functions f and g are analytic in U, then f is said to be subordinate to g, written as

$$f(z) \prec g(z), \quad z \in U,$$

if there exists a Schwarz function $w(z)$, analytic in U, with

$$w(0) = 0 \text{ and } |w(z)| < 1, \quad z \in U,$$

such that

$$f(z) = g(w(z)), \quad z \in U.$$

The Fekete–Szegö functional $\left|a_3 - \mu a_2^2\right|$ for normalized univalent functions of the form (1.1) is well known for its rich history in the theory of geometric functions. Its origin was in the disproof by Fekete and Szegö of the 1933 conjecture of Littlewood and Paley that the coefficients of odd univalent functions are bounded by unity (see [5]).

Let $f \in A$ and $\mathbb{N} = \{1, 2, \ldots\}$. We define the differential operators $D^k,\ k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, by (see [11])

$$\begin{aligned} D^0 f(z) &= f(z), \\ D^1 f(z) &= Df(z) = zf'(z), \\ &\cdots \\ D^k f(z) &= D^1\left(D^{k-1} f(z)\right), \\ &\cdots. \end{aligned}$$

We note that

$$D^k f(z) = z + \sum_{n=2}^{\infty} n^k a_n z^n.$$

Definition 1. Let $f \in A$. Suppose that $0 \le \beta < 1$ and $\lambda \ge 1$. Then $\pounds_\lambda(\beta)$ denotes the class of Salagean type λ-*pseudo-starlike* functions if

$$\Re\left(\frac{z\left[\left(D^k f(z)\right)'\right]^\lambda}{D^k f(z)}\right) > \beta, \quad z \in U. \tag{1.2}$$

2. Preliminary considerations

Special functions can be categorized into three, namely, ramp functions, threshold functions, and sigmoid functions. The most popular among these is the sigmoid function because of its gradient descendent learning algorithm.

It can be evaluated in different ways, most especially by truncated series expansion. The sigmoid function of the form

$$h(z) = \frac{1}{1 + e^{-z}}$$

is useful because it is differentiable. The sigmoid function has the following very important properties (see [3, 4, 8, 9]):

- It outputs real numbers between 0 and 1.
- It maps a very large input domain to a small range of outputs.
- It never loses information because it is a one-to-one function.
- It increases monotonically.

In the cases when $k = 0, 1, 2$ and $\lambda = 1, 2$, Figures 1 − 6 of sigmoid function are given by Definition 1 using Maple.

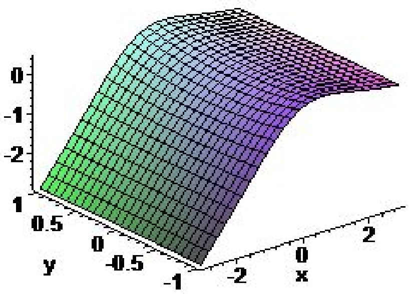

FIGURE 1. For $\lambda = 1$, $k = 0$.

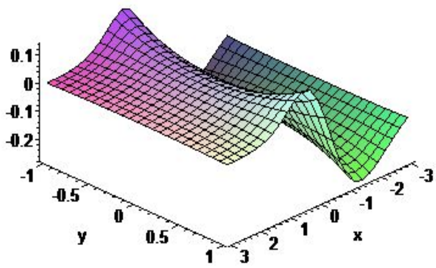

FIGURE 2. For $\lambda = 2$, $k = 0$.

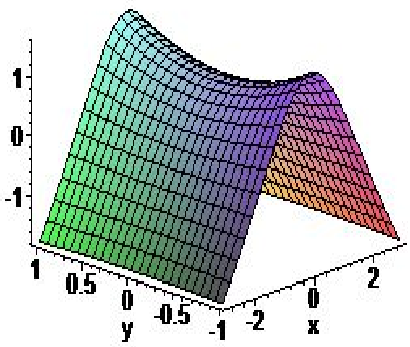

FIGURE 3. For $\lambda = 1$, $k = 1$.

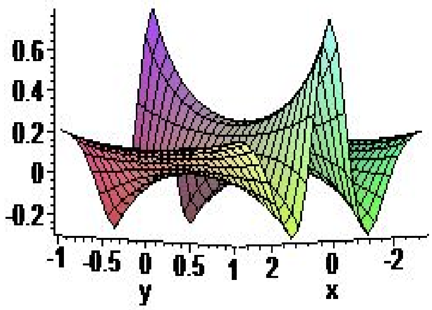

FIGURE 4. For $\lambda = 2$, $k = 1$.

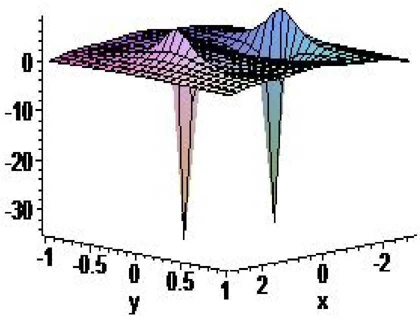

FIGURE 5. For $\lambda = 1$, $k = 2$.

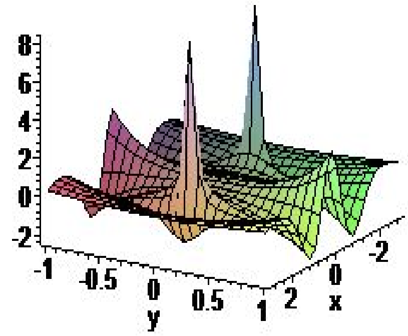

FIGURE 6. For $\lambda = 2$, $k = 2$.

Let P denote the class of functions p such that

$$p(z) = 1 + p_1 z + p_2 z^2 + \cdots = 1 + \sum_{n=1}^{\infty} p_n z^n,$$

and which are regular in the open unit disc U and satisfy $\Re(p(z)) > 0$ for any $z \in U$. Here, $p(z)$ is called a Caratheodory function [2].

Lemma 1 (see [10]). *If $p \in P$, then*

$$|p_n| \leq 2, \quad n \in \mathbb{N},$$

and

$$\left| p_2 - \frac{p_1^2}{2} \right| \leq 2 - \frac{|p_1|^2}{2}.$$

Lemma 2 (see [7]). *If $p \in P$, then*

$$\left| p_2 - t p_1^2 \right| \leq \begin{cases} -4t + 2 & \text{if } t \leq 0, \\ 2 & \text{if } 0 \leq t \leq 1, \\ 4t - 2 & \text{if } t \geq 1. \end{cases}$$

Lemma 3 (see [4]). *Let h be a sigmoid function and let*

$$G(z) = 2h(z) = 1 + \sum_{m=1}^{\infty} \frac{(-1)^m}{2^m} \left(\sum_{n=1}^{\infty} \frac{(-1)^n}{n!} z^n \right)^m.$$

Then $G(z) \in P$, $|z| < 1$, where $G(z)$ is a modified sigmoid function.

Lemma 4 (see [4]). *Let*

$$G_{n,m}(z) = 1 + \sum_{m=1}^{\infty} \frac{(-1)^m}{2^m} \left(\sum_{n=1}^{\infty} \frac{(-1)^n}{n!} z^n \right)^m .$$

Then $|G_{n,m}(z)| < 2$.

Let P_β denote the class of functions p such that

$$p(z) = 1 + p_1 z + p_2 z^2 + \cdots = 1 + \sum_{n=1}^{\infty} p_n z^n,$$

and which are regular in the open unit disc U and satisfy $\Re(p(z)) > \beta$ for any $z \in U$.

We recall the following lemmas which are relevant for our study. They were proved in [1].

Lemma 5. *If z is complex number having positive real part, then, for any real number t such that $t \in [0,1]$, we have $\Re(z^t) \geq (\Re(z))^t$.*

Lemma 6. *Let $p \in P_\beta$. If $q(z) = [p(z)]^t$, $t \in [0,1]$, then $q(0) = 1$ and $\Re(q(z)) > \beta^t$.*

Lemma 7. *Let p be analytic in U with $p(0) = 1$, and suppose that*

$$\Re\left(1 + \frac{zp'(z)}{p(z)}\right) > \frac{3\beta - 1}{2\beta}, \quad z \in U.$$

Then

$$\Re(p(z)) > 2^{1-\frac{1}{\beta}}, \quad 1/2 \leq \beta < 1, \ z \in U,$$

and the constant $2^{1-\frac{1}{\beta}}$ is the best possible.

3. Main results

Theorem 1. *Let $0 \leq \beta < 1$ and $\lambda \geq 1$. Then*

$$\pounds_\lambda(\beta) \subset B_1\left(1 - 1/\lambda, \beta^{1/\lambda}\right).$$

Proof. Let $f \in \pounds_\lambda(\beta)$. For some $p \in P_\beta$, we have

$$\frac{z\left[\left(D^k f(z)\right)'\right]^\lambda}{D^k f(z)} = p(z),$$

which shows that

$$\frac{z^{\frac{1}{\lambda}}\left(D^k f(z)\right)'}{\left[D^k f(z)\right]^{\frac{1}{\lambda}}} = p(z)^{1/\lambda}.$$

Hence, from Lemma 6, we get that

$$\Re\left(\frac{z^{\frac{1}{\lambda}}\left(D^k f(z)\right)'}{\left[D^k f(z)\right]^{\frac{1}{\lambda}}}\right) > \beta^{1/\lambda}.$$

Taking $\alpha = 1-1/\lambda$, we have $f \in B_1\left(1-1/\lambda, \beta^{1/\lambda}\right)$. The proof is completed. □

Corollary 1. *All Salagean type λ-pseudo-starlike functions are Bazilevič functions of type $1-1/\lambda$, order $\beta^{1/\lambda}$, and univalent in U.*

Theorem 2. *Let $f \in \pounds_\lambda(\beta)$. For some $p \in P_\beta$, the differential operator $D^k f(z)$ has the integral representation*

$$D^k f(z) = \begin{cases} \left[\int_0^z \frac{\lambda-1}{\lambda}\left(\frac{p(t)}{t}\right)^{\frac{1}{\lambda}} dt\right]^{\frac{\lambda}{\lambda-1}} & \text{if } \lambda > 1, \\ \exp \int_0^z \frac{p(t)}{t} dt & \text{if } \lambda = 1. \end{cases}$$

Proof. Since $f \in \pounds_\lambda(\beta)$, there exist a function $p \in P_\beta$ such that

$$\frac{z\left[\left(D^k f(z)\right)'\right]^\lambda}{D^k f(z)} = p(z)$$

and, thereby,

$$\frac{z^{\frac{1}{\lambda}}\left(D^k f(z)\right)'}{\left[D^k f(z)\right]^{\frac{1}{\lambda}}} = p(z)^{\frac{1}{\lambda}}.$$

Then, taking $\alpha = 1-1/\lambda$, we have

$$\frac{z^{1-\alpha}\left(D^k f(z)\right)'}{\left[D^k f(z)\right]^{1-\alpha}} = p(z)^{1-\alpha}$$

so that

$$\left[\left(D^k f(z)\right)^\alpha\right]' = \alpha z^{\alpha-1} p(z)^{1-\alpha}.$$

Hence

$$D^k f(z) = \left\{\int_0^z \alpha t^{\alpha-1} p(t)^{1-\alpha} dt\right\}^{\frac{1}{\alpha}},$$

which gives the desired representation. □

Corollary 2. *Let $f \in \pounds_2(\beta)$. Then $D^k f(z)$ has the integral representation*

$$D^k f(z) = \left\{\int_0^z \frac{1}{2} t^{-\frac{1}{2}} p(t)^{\frac{1}{2}} dt\right\}^2$$

for some $p \in P_\beta$.

Theorem 3. *If $f \in A$ satisfies*

$$\Re\left\{\lambda\frac{z\left[D^k f(z)\right]''}{\left[D^k f(z)\right]'} - \frac{z\left[D^k f(z)\right]'}{D^k f(z)}\right\} > -\frac{1+\beta}{2\beta}, \;\; z \in U,$$

then $f \in B_2\left(2^{1-\frac{1}{\beta}}\right)$, $1/2 \le \beta < 1$. The constant $2^{1-\frac{1}{\beta}}$ is the best possible.

Proof. For $z \in U$, define

$$p(z) = \frac{z\left[\left(D^k f(z)\right)'\right]^{\lambda}}{D^k f(z)}.$$

Then

$$\begin{aligned}\frac{zp'(z)}{p(z)} &= \frac{D^k f(z)}{\left[\left(D^k f(z)\right)'\right]^{\lambda}} p'(z) \\ &= 1 + \lambda\frac{z\left(D^k f(z)\right)''}{\left(D^k f(z)\right)'} - \frac{z\left(D^k f(z)\right)'}{D^k f(z)}\end{aligned}$$

and thus

$$\begin{aligned}\Re\left(1 + \frac{zp'(z)}{p(z)}\right) &= \Re\left\{2 + \lambda\frac{z\left(D^k f(z)\right)''}{\left(D^k f(z)\right)'} - \frac{z\left(D^k f(z)\right)'}{D^k f(z)}\right\} \\ &> \frac{3\beta - 1}{2\beta}.\end{aligned}$$

This yields that

$$\Re\left\{\lambda\frac{z\left(D^k f(z)\right)''}{\left(D^k f(z)\right)'} - \frac{z\left(D^k f(z)\right)'}{D^k f(z)}\right\} > -\frac{1+\beta}{2\beta},$$

which, by Lemma 7, implies

$$\Re(p(z)) = \Re\left(\frac{z\left[\left(D^k f(z)\right)'\right]^{\lambda}}{D^k f(z)}\right) > 2^{1-\frac{1}{\beta}}, \;\; 1/2 \le \beta < 1,$$

as required. □

Corollary 3. *If $f \in A$ satisfies*

$$\Re\left\{\lambda\frac{z\left[D^k f(z)\right]''}{\left[D^k f(z)\right]'} - \frac{z\left[D^k f(z)\right]'}{D^k f(z)}\right\} > -\frac{3}{2}, \;\; z \in U,$$

then

$$\Re\left\{\frac{z\left[D^k f(z)\right]'}{D^k f(z)}\right\} > \frac{1}{2}.$$

That is, f is starlike of order $1/2$ in U.

4. Coefficient inequalities for the function class $\pounds_{\lambda}(\beta)$

Theorem 4. *Let a function f given by* (1.1) *be in the class $\pounds_{\lambda}(\beta)$. Then*

$$|a_2| \leq \frac{1-\beta}{(2\lambda-1)\,2^{k+1}},$$

$$|a_3| \leq \frac{\left(4\lambda-2\lambda^2-1\right)(1-\beta)^2}{4\,(2\lambda-1)^2\,(3\lambda-1)\,3^k},$$

and

$$|a_4| \leq \frac{\left(24\lambda^4-80\lambda^3+84\lambda^2-28\lambda+3\right)(1-\beta)^3}{24\,(2\lambda-1)^3\,(3\lambda-1)\,(4\lambda-1)\,4^k} + \frac{1-\beta}{24\,(4\lambda-1)\,4^k}.$$

Proof. Let $f \in \pounds_{\lambda}(\beta)$. Then there exists a $G(z) \in P$ such that

$$\frac{z\left[\left(D^k f(z)\right)'\right]^{\lambda}}{D^k f(z)} = \beta + (1-\beta)\,G(z), \tag{4.1}$$

where the function $G(z)$ is a modified sigmoid function given by

$$G(z) = 1 + \frac{1}{2}z - \frac{1}{24}z^3 + \frac{1}{240}z^5 - \frac{1}{64}z^6 + \frac{779}{20160}z^7 - \cdots.$$

Furthermore,

$$z\left[\left(D^k f(z)\right)'\right]^{\lambda} = D^k f(z)\left[\beta + (1-\beta)\,G(z)\right]. \tag{4.2}$$

Equating coefficients in (4.2) yields

$$\lambda 2^{k+1} a_2 = \frac{1-\beta}{2} + 2^k a_2, \tag{4.3}$$

$$\lambda 3^{k+1} a_3 + 2\lambda(\lambda-1)4^k a_2^2 = \frac{(1-\beta)\,2^k}{2} a_2 + 3^k a_3, \tag{4.4}$$

and

$$\begin{aligned} \lambda 4^{k+1} a_4 + 6\lambda(\lambda-1)6^k a_2 a_3 + \frac{4\lambda(\lambda-1)(\lambda-2)8^k}{3} a_2^3 \\ = \frac{(1-\beta)\,3^k}{2} a_3 - \frac{1-\beta}{24} + 4^k a_4, \end{aligned} \tag{4.5}$$

and the desired inequalities follow from (4.3)−(4.5). □

Taking $\lambda = 1$ and $k = 0$ in Theorem 4, we have the following corollary.

Corollary 4. *Let a function f given by* (1.1) *be in the class $\pounds_1(\beta)$. Then*

$$|a_2| \leq \frac{1-\beta}{2},$$

$$|a_3| \leq \frac{(1-\beta)^2}{8},$$

and

$$|a_4| \le \tfrac{(1-\beta)^3}{48} + \tfrac{1-\beta}{72}.$$

Taking $\lambda = 1$, $\beta = 0$, and $k = 0$ in Theorem 4, we have the following corollary.

Corollary 5. *Let a function* f *given by* (1.1) *be in the class* $\pounds_1 = S^*$. *Then*

$$|a_2| \le \frac{1}{2},$$

$$|a_3| \le \frac{1}{8},$$

and

$$|a_4| \le \frac{1}{144}.$$

Remark 1. Corollary 5 is an improvement of the estimates given by Babalola [1, page 145, Corollary 5].

Taking $\lambda = 2$, $\beta = 0$, and $k = 0$ in Theorem 4, we have the following corollary.

Corollary 6. *Let a function* f *given by* (1.1) *be in the class* $\pounds_2$. *Then*

$$|a_2| \le \frac{1}{6},$$

$$|a_3| \le \frac{1}{180},$$

and

$$|a_4| \le \frac{1}{210}.$$

Remark 2. Corollary 6 is an improvement of the estimates given by Babalola [1, page 145, Corollary 6].

5. Fekete–Szegö inequalities for the function class $\pounds_\lambda(\beta, \phi)$

In the following, let ϕ be an analytic function with positive real part in U, $\phi(0) = 1$, and $\phi'(0) > 0$. Also, let $\phi(U)$ be starlike with respect to 1 and symmetric with respect to the real axis. Thus, ϕ has the Taylor series expansion

$$\phi(z) = 1 + C_1 z + C_2 z^2 + C_3 z^3 + \cdots \quad (C_1 > 0). \tag{5.1}$$

Definition 2. A function $f \in A$ is said to be in $\pounds_\lambda(\beta, \phi)$, $0 \le \beta < 1$, $\lambda \ge 1$, if the following subordination holds:

$$\frac{z\left[\left(D^k f(z)\right)'\right]^\lambda}{D^k f(z)} \prec \phi(z).$$

Theorem 5. *Let a function f given by* (1.1) *be in the class* $\pounds_\lambda(\beta,\phi)$, $\mu\in\mathbb{R}$, *and let*

$$M=\frac{(2\lambda-1)^2 4^k (C_2-C_1)+4^k\left(4\lambda-2\lambda^2-1\right)C_1^2}{3^k(3\lambda-1)C_1^2},\quad N=\frac{(2\lambda-1)^2 4^k (C_2+C_1)+4^k\left(4\lambda-2\lambda^2-1\right)C_1^2}{3^k(3\lambda-1)C_1^2}.$$

Then

$$\left|a_3-\mu a_2^2\right|\le\begin{cases}\frac{C_2}{(3\lambda-1)3^k}-\frac{\mu C_1^2}{(2\lambda-1)^2 4^k}+\frac{\left(4\lambda-2\lambda^2-1\right)C_1^2}{(2\lambda-1)^2(3\lambda-1)3^k} & \text{for } \mu\le M,\\ \frac{C_1}{(3\lambda-1)3^k} & \text{for } M\le\mu\le N,\\ -\frac{C_2}{(3\lambda-1)3^k}+\frac{\mu C_1^2}{(2\lambda-1)^2 4^k}-\frac{\left(4\lambda-2\lambda^2-1\right)C_1^2}{(2\lambda-1)^2(3\lambda-1)3^k} & \text{for } \mu\ge N.\end{cases}$$

These inequalities are sharp.

Proof. Let $f\in\pounds_\lambda(\beta,\phi)$. Then there exist a function u, analytic in U with $u(0)=0$ and $|u(z)|<1$, $z\in U$, such that

$$\frac{z\left[\left(D^k f(z)\right)'\right]^\lambda}{D^k f(z)}=\phi(u(z)),\quad z\in U. \tag{5.2}$$

Next, define a function p by

$$p(z)=\frac{1+u(z)}{1-u(z)}=1+p_1z+p_2z^2+\cdots. \tag{5.3}$$

Clearly, $\Re(p(z))>0$. From (5.3), one has

$$u(z)=\frac{p(z)-1}{p(z)+1}=\frac{1}{2}p_1z+\frac{1}{2}\left(p_2-\frac{1}{2}p_1^2\right)z^2+\cdots. \tag{5.4}$$

Combining (5.1), (5.2) and (5.4), we get

$$\frac{z\left[\left(D^k f(z)\right)'\right]^\lambda}{D^k f(z)}=1+\frac{1}{2}C_1p_1z+\left(\frac{1}{4}C_2p_1^2+\frac{1}{2}C_1\left(p_2-\frac{1}{2}p_1^2\right)\right)z^2+\cdots. \tag{5.5}$$

From (5.5), we deduce that

$$2^k(2\lambda-1)a_2=\frac{1}{2}C_1p_1, \tag{5.6}$$

$$3^k(3\lambda-1)a_3-4^k\left(4\lambda-2\lambda^2-1\right)a_2^2=\frac{1}{4}C_2p_1^2+\frac{1}{2}C_1\left(p_2-\frac{1}{2}p_1^2\right). \tag{5.7}$$

Now, from (5.6) and (5.7), it follows that

$$\begin{aligned}a_3-\mu a_2^2&=\frac{C_1}{2(3\lambda-1)3^k}\left\{p_2-\frac{p_1^2}{2}\left[1-\frac{C_2}{C_1}+\frac{\mu(3\lambda-1)3^k-\left(4\lambda-2\lambda^2-1\right)4^k}{(2\lambda-1)^2 4^k}C_1\right]\right\}\\&=\frac{C_1}{2(3\lambda-1)3^k}\left(p_2-tp_1^2\right),\end{aligned}$$

where

$$t=\frac{1}{2}\left[1-\frac{C_2}{C_1}+\frac{\mu(3\lambda-1)3^k-\left(4\lambda-2\lambda^2-1\right)4^k}{(2\lambda-1)^2 4^k}C_1\right].$$

By applying Lemma 2, the proof is completed. □

References

[1] K. O. Babalola, *On λ-pseudo-starlike functions*, J. Class. Anal. **3** (2013), 137–147.
[2] P. L. Duren, *Univalent Functions*, Grundlehren der Mathematischen Wissenschaften **259**, Springer-Verlag, New York, 1983.
[3] O. A. Fadipe-Joseph, B. O. Moses, and M. O. Oluwayemi, *Certain new classes of analytic functions defined by using sigmoid function*, Adv. Math. Sci. J. **5** (2016), 83–89.
[4] O. A. Fadipe-Joseph, A. T. Oladipo, and U. A. Ezeafulukwe, *Modified sigmoid function in univalent function theory*, Int. J. Math. Sci. Eng. Appl. **7** (2013), 313–317.
[5] M. Fekete and G. Szegö, *Eine Bemerkung über ungerade schlichte Funktionen*, J. London Math. Soc. **2** (1933), 85–89.
[6] R. R. London and D. K. Thomas, *The derivative of Bazilevič functions*, Proc. Amer. Math. Soc. **104** (1988), 235–238.
[7] W. Ma and D. Minda, *A unified treatment of some special classes of univalent functions*, in: Proceedings of the Conference on Complex Analysis, Int. Press, Cambridge, 1994, pp. 157–169.
[8] G. Murugusundaramoorthy and T. Janani, *Sigmoid function in the space of univalent λ-pseudo starlike functions*, Int. J. Pure Appl. Math. **101** (2015), 33–41.
[9] S. O. Olatunji, *Sigmoid function in the space of univalent lambda pseudo starlike function with Sakaguchi type functions*, J. Progr. Res. Math. **7**(4) (2016), 1164–1172.
[10] C. Pommerenke, *Univalent Functions*, Vandenhoeck & Ruprecht, Göttingen, 1975.
[11] G. S. Salagean, *Subclasses of univalent functions*, in: Complex Analysis–Fifth Romanian- Finnish Seminar, Part 1 (Bucharest, 1981), Lecture Notes in Mathematics **1013**, Springer, Berlin, 1983, pp. 362–372.
[12] R. Singh, *On Bazilevič functions*, Proc. Amer. Math. Soc. **38** (1973), 261–271.
[13] D. K. Thomas, *On a subclass of Bazilevič functions*, Internat. J. Math. Math. Sci. **8** (1985), 779–783.

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