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Some inequalities associated with the Hermite–Hadamard inequalities for operator h-convex functions

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Abstract. We introduce the concept of operator h-convex functions for positive linear maps, and prove some Hermite–Hadamard type inequalities for these functions. As applications, we obtain several trace inequalities for operators.

1. Introduction and preliminaries

Let $B(H)$ stand for the C^*-algebra of all bounded linear operators on a complex separable Hilbert space H with inner product $\langle \cdot, \cdot \rangle$. An operator $A \in B(H)$ is *positive* if $\langle Ax, x \rangle \geq 0$ for all $x \in H$; in this case we write $A \geq 0$. Let $B(H)^+$ stand for the set of all positive operators in $B(H)$.

A linear map $\Phi : B(H) \to B(H)$ is *positive* if $\Phi(A) \geq 0$ whenever $A \geq 0$, and Φ is said to be *unital* if $\Phi(I) = I$.

The maps $\Phi : B(H) \to B(H)$, defined by $\Phi(A) = X^*AX$, where X is an operator in $B(H)$, and $\Phi(A) = A^*$ for $A \in B(H)$ are examples of positive linear maps.

We say that a linear map is *invertible preserving* if $\Phi(A)$ is invertible whenever A is invertible.

Let A be a self-adjoint operator in $B(H)$. The Gelfand map establishes a $*$-isometrically isomorphism Φ between the set $C(\mathrm{Sp}(A))$ of all continuous functions defined on the spectrum of A, denoted $\mathrm{Sp}(A)$, and the C^*-algebra $C^*(A)$ generated by A and the identity operator 1_H on H as follows (see, for instance, [8, page 3]).

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For any $f, g \in C(\mathrm{Sp}(A)))$ and any $\alpha, \beta \in \mathbb{C}$, we have:

- $\Phi(\alpha f + \beta g) = \alpha\Phi(f) + \beta\Phi(g)$;
- $\Phi(fg) = \Phi(f)\Phi(g)$ and $\Phi(\bar{f}) = \Phi(f)^*$;
- $\|\Phi(f)\| = \|f\| := \sup_{t\in\mathrm{Sp}(A)} |f(t)|$;
- $\Phi(f_0) = 1_H$ and $\Phi(f_1) = A$, where $f_0(t) = 1$ and $f_1(t) = t$ for $t \in \mathrm{Sp}(A)$.

With this notation, we define

$$f(A) = \Phi(f) \quad \text{for all} \quad f \in C(\mathrm{Sp}(A)),$$

and we call it the *continuous functional calculus* for a self-adjoint operator A.

If A is a self-adjoint operator and f is a real-valued continuous function on $\mathrm{Sp}(A)$, then, whenever $f(t) \geq 0$ for any $t \in \mathrm{Sp}(A)$, one has $f(A) \geq 0$, i.e., $f(A)$ is a positive operator on H. Moreover, if both f and g are real-valued functions on $\mathrm{Sp}(A)$, then the following important property holds:

$$\text{whenever} \quad f(t) \geq g(t) \quad \text{for any} \quad t \in \mathrm{Sp}(A), \quad \text{one has} \quad f(A) \geq g(A),$$

in the operator order of $B(H)$.

The following inequality holds for any convex function f defined on $\mathbb{R}$:

$$(b-a)f\left(\frac{a+b}{2}\right) \leq \int_a^b f(x)dx \leq (b-a)\frac{f(a)+f(b)}{2}, \quad a, b \in \mathbb{R}. \tag{1}$$

It was first discovered by Hermite in 1881 in the journal Mathesis (see [11]). But this result was nowhere mentioned in the mathematical literature and was not widely known as Hermite's result (see [14]).

Beckenbach [1], a leading expert on the history and the theory of convex functions, wrote that this inequality was proven by Hadamard in 1893. In 1974, Mitrinovič [11] found Hermite's note in Mathesis. Since (1) was known as Hadamard's inequality, the inequality is now commonly referred to as the Hermite–Hadamard inequality.

Let X be a vector space, $x, y \in X, x \neq y$. Define the *segment*

$$[x, y] := \{(1-t)x + ty : t \in [0, 1]\}.$$

We consider the function $f : [x, y] \to \mathbb{R}$ and the *associated function*

$$g(x, y) : [0, 1] \to \mathbb{R}, \quad g(x, y)(t) := f[(1-t)x + ty], \quad t \in [0, 1].$$

Note that f is convex on $[x, y]$ if and only if $g(x, y)$ is convex on $[0, 1]$.

For any convex function defined on a segment $[x, y] \subset X$, we have the Hermite–Hadamard integral inequality (see [4, page 2], [5, page 2])

$$f\left(\frac{x+y}{2}\right) \leq \int_0^1 f[(1-t)x + ty]dt \leq \frac{f(x)+f(y)}{2}, \tag{2}$$

which can be derived from the classical Hermite–Hadamard inequality (1) for the convex function $g(x, y) : [0, 1] \to \mathbb{R}$.

Since $f(x) = \|x\|^p$ ($x \in X$ and $1 \leq p < \infty$) is a convex function, we have the following norm inequality from (2) (see [13]):

$$\left\|\frac{x+y}{2}\right\|^p \leq \int_0^1 \|(1-t)x + ty\|^p dt \leq \frac{\|x\|^p + \|y\|^p}{2} \quad \text{for any} \quad x, y \in X.$$

A real-valued continuous function f on an interval I is said to be *operator convex* (*operator concave*) if

$$f((1-\lambda)A + \lambda B) \leq (\geq)(1-\lambda)f(A) + \lambda f(B)$$

(in the operator order) for all $\lambda \in [0, 1]$ and for all self-adjoint operators A and B on a Hilbert space H, whose spectra are contained in I (see [6]).

As an example of such functions, we note that $f(t) = t^r$ is operator convex on $(0, \infty)$ if either $1 \leq r \leq 2$ or $-1 \leq r \leq 0$, and is operator concave on $(0, \infty)$ if $0 \leq r \leq 1$ (see [2, p. 147]).

Motivated by the above results, Dragomir [6] investigated the operator version of the Hermite–Hadamard inequality for operator convex functions. His result asserts that if $f : I \to \mathbb{R}$ is an operator convex function on the interval I, then, for any self-adjoint operators A and B with spectra in I, the following inequalities hold:

$$\begin{aligned} f\left(\frac{A+B}{2}\right) &\leq \frac{1}{2}\left[f\left(\frac{3A+B}{4}\right) + f\left(\frac{A+3B}{4}\right)\right] \\ &\leq \int_0^1 f\left((1-t)A + tB\right) dt \\ &\leq \frac{1}{2}\left[f\left(\frac{A+B}{2}\right) + \frac{f(A)+f(B)}{2}\right] \\ &\leq \frac{f(A)+f(B)}{2}. \end{aligned} \tag{3}$$

To prove the above inequalities, the author considered the convex function $\varphi(t) = \langle f(tA+(1-t)B)x, x\rangle$ on $[0, 1]$ for any $x \in H$ with $\|x\| = 1$, self-adjoint operators A and B with spectra in I, and an operator convex function f.

By considering $\varphi(t)$ on $[\frac{1}{4}, \frac{3}{4}]$, we can give a refinement for (3) as follows:

$$\varphi\left(\frac{\frac{1}{4}+\frac{3}{4}}{2}\right) \leq 2\int_{\frac{1}{4}}^{\frac{3}{4}} \varphi(t)dt \leq \frac{\varphi(\frac{1}{4}) + \varphi(\frac{3}{4})}{2}.$$

So,

$$\left\langle f\left(\frac{A+B}{2}\right)x, x\right\rangle \leq 2\int_{\frac{1}{4}}^{\frac{3}{4}} \langle f(tA + (1-t)B)x, x\rangle dt$$

$$\leq \frac{1}{2}\left[\left\langle f\left(\frac{A+3B}{4}\right)x, x\right\rangle + \left\langle f\left(\frac{3A+B}{4}\right)x, x\right\rangle\right].$$

The continuity of f implies that for any $x \in H$ with $\|x\| = 1$,

$$\int_{\frac{1}{4}}^{\frac{3}{4}} \langle f(tA+(1-t)B)x, x\rangle dt = \left\langle \int_{\frac{1}{4}}^{\frac{3}{4}} f(tA+(1-t)B)dtx, x\right\rangle.$$

Therefore, for self-adjoint operators A and B with spectra in I,

$$\begin{aligned} f\left(\frac{A+B}{2}\right) &\leq 2\int_{\frac{1}{4}}^{\frac{3}{4}} f(tA+(1-t)B)dt \\ &\leq \frac{1}{2}\left[f\left(\frac{A+3B}{4}\right) + f\left(\frac{3A+B}{4}\right)\right]. \end{aligned}$$

Another class of functions considered by Hudzik and Maligranda [10] are s-convex functions which are defined as follows. A function $f : \mathbb{R}^+ \to \mathbb{R}$, where $\mathbb{R}^+ = [0, \infty)$, is said to be *$s$-convex in the second sense* if

$$f(\lambda x + (1-\lambda)y) \leq \lambda^s f(x) + (1-\lambda)^s f(y),$$

for all $x, y \in [0, \infty)$, $\lambda \in [0, 1]$, and for a fixed $s \in (0, 1]$. The class of s-convex functions in the second sense is usually denoted by K_s^2.

In [7], Dragomir and Fitzpatrick proved the following Hermite–Hadamard type inequality for s-convex functions in the second sense. Let $f : [0, \infty) \to [0, \infty)$ be an s-convex function in the second sense, where $s \in (0, 1]$, and let $a, b \in [0, \infty)$ such that $a < b$. If $f \in L^1[a, b]$, then the following inequalities hold:

$$2^{s-1} f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{s+1}.$$

In order to extend this class of functions to operators, Ghazanfari [9] defined operator s-convex functions as follows. Let I be an interval in $[0, \infty)$. A continuous function $f : I \to \mathbb{R}$ is said to be *operator s-convex* on I for operators in $B(H)^+$ if

$$f((1-\lambda)A + \lambda B) \leq (1-\lambda)^s f(A) + \lambda^s f(B),$$

(in the operator order in $B(H)$) for all $\lambda \in [0, 1]$, for all positive operators A and B in $B(H)^+$ whose spectra are contained in I, and for a fixed $s \in (0, 1]$. The author proved that if $f : I \to \mathbb{R}$ is an operator s-convex function on the interval $I \subseteq [0, \infty)$, then the following inequalities hold:

$$2^{s-1} f\left(\frac{A+B}{2}\right) \leq \int_0^1 f((1-t)A + tB)dt \leq \frac{f(A)+f(B)}{s+1}.$$

In this paper, we introduce the concept of operator h-convex functions and obtain some Hermite–Hadamard type inequalities for this class of functions for positive linear maps. These results lead us further to obtain some inequalities for the trace functional of operators. Some of these inequalities improve recent results.

2. Inequalities for operator h-convex functions of positive linear maps

Let I, $J \subseteq \mathbb{R}$ and $(0,1) \subseteq J$. Suppose that f and h are real non-negative functions on I and J, respectively.

Definition 2.1 (see [18])**.** Let $h : J \to \mathbb{R}$ be a non-negative function with $h \not\equiv 0$. We say that $f : I \to \mathbb{R}$ is an *h-convex* function (or that f belongs to the class $SX(h, I)$) if f is non-negative, and for all $x, y \in I$ and $\lambda \in [0, 1]$ we have

$$f(\lambda x + (1 - \lambda)y) \leq h(\lambda)f(x) + h(1 - \lambda)f(y). \tag{4}$$

If inequality (4) is reversed, then f is said to be *h-concave*, and we write $f \in SV(h, I)$.

It is clear that if $h(\lambda) = \lambda$, then all non-negative convex functions belong to $SX(h, I)$, and all non-negative concave functions belong to $SV(h, I)$; if $h(\lambda) = \lambda^s$, where $s \in (0, 1]$, then $K_s^2 \subseteq SX(h, I)$.

The following inequalities due to Sarikaya [15] give Hermite–Hadamard type inequalities for h-convex functions. Let $f \in SX(h, I)$, $a, b \in I$, with $a < b$ and $f \in L^1([a, b])$. Then

$$\frac{1}{2h(\frac{1}{2})} f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq (f(a) + f(b))\int_0^1 h(t)dt. \tag{5}$$

Now, we introduce the concept of an operator h-convex function.

Definition 2.2. Let A, $B \in B(H)$ be two self-adjoint operators whose spectra are contained in I. A continuous function $f : I \to \mathbb{R}$ is said to be *operator h-convex* on I if, for all $\lambda \in [0, 1]$,

$$f(\lambda A + (1 - \lambda)B) \leq h(\lambda)f(A) + h(1 - \lambda)f(B).$$

Lemma 2.3. *If f is an operator h-convex function, then for any $x \in H$ with $\|x\| = 1$, the function*

$$\varphi_{x,A,B}(t) = \langle f(tA + (1 - t)B)x, x\rangle$$

is an h-convex function on $[0, 1]$*.*

Proof. Let f be an operator h-convex function. Then for $u, v \in [0, 1]$ we have

$$\begin{aligned}\varphi_{x,A,B}(tu + (1 - t)v) &= \langle f\,[(tu + (1 - t)v)A + (1 - (tu + (1 - t)v)B)]x, x\rangle \\ &= \langle f(t[uA + (1-u)B] + (1-t)[(vA + (1-v)B)]x, x\rangle \\ &\leq h(t)\langle f(uA + (1 - u)B)x, x\rangle\end{aligned}$$

$$\begin{aligned} &+ h(1-t)\langle f(vA+(1-v)B)x, x\rangle \\ &= h(t)\varphi_{x,A,B}(u) + h(1-t)\varphi_{x,A,B}(v). \end{aligned}$$

So, $\varphi_{x,A,B}$ is an h-convex function on $[0,1]$. □

Theorem 2.4. *Let f be an operator h-convex function. Then*

$$\frac{1}{2h(\frac{1}{2})} f\left(\frac{A+B}{2}\right) \le \int_0^1 f(tA+(1-t)B)dt \le (f(A)+f(B)) \int_0^1 h(t)dt,$$

for self-adjoint operators $A, B \in B(H)$ whose spectra are contained in I.

Proof. Since f is operator h-convex function, by Lemma 2.3 we have that

$$\varphi_{x,A,B}(t) = \langle f(tA+(1-t)B)x, x\rangle$$

is h-convex function on $[0,1]$. So, by (5) we obtain

$$\begin{aligned} \frac{\varphi_{x,A,B}(\frac{1}{2})}{2h(\frac{1}{2})} &\le \int_0^1 \varphi_{x,A,B}(t)dt \\ &\le (\varphi_{x,A,B}(0) + \varphi_{x,A,B}(1)) \int_0^1 h(t)dt. \end{aligned}$$

Hence,

$$\begin{aligned} \frac{1}{2h(\frac{1}{2})} \left\langle f\left(\frac{A+B}{2}\right) x, x\right\rangle &\le \int_0^1 \langle f(tA+(1-t)B)x, x\rangle dt \\ &\le (\langle f(A)x, x\rangle + \langle f(B)x, x\rangle) \int_0^1 h(t)dt \end{aligned}$$

for any $x \in H$ with $\|x\| = 1$ and self-adjoint operators A and B with spectra in I.

By the continuity of f, we have

$$\int_0^1 \langle f(tA+(1-t)B)x, x\rangle dt = \left\langle \int_0^1 f(tA+(1-t)B)dtx, x\right\rangle.$$

The proof is complete. □

Let $I = [0,\infty)$ in the above theorem. Since Φ is a positive linear map and the spectrum of a positive operator is in $[0,\infty)$, we have the following result.

Corollary 2.5. *Let $f : [0,\infty) \to \mathbb{R}$ be an operator h-convex function for operators in $B(H)^+$, and let $\Phi : B(H) \to B(H)$ be a positive linear map. Then*

$$\begin{aligned} \frac{1}{2h(\frac{1}{2})} f\left(\Phi\left(\frac{A+B}{2}\right)\right) &\le \int_0^1 f(\Phi(tA+(1-t)B))dt \\ &\le [f(\Phi(A)) + f(\Phi(B))] \int_0^1 h(t)dt. \end{aligned}$$

We can obtain some results for positive linear operators when $h(t) = t^s$ and $h(t) = t$.

If $h(t) = t^s$ for some $s \in (0, 1]$ in Corollary 2.5, then we have

$$\begin{aligned} 2^{s-1} f\left(\Phi\left(\frac{A+B}{2}\right)\right) &\leq \int_0^1 f(\Phi(tA + (1-t)B))dt \\ &\leq \frac{f(\Phi(A)) + f(\Phi(B))}{s+1}. \end{aligned} \tag{6}$$

If $h(t) = t$ in Corollary 2.5, then

$$\begin{aligned} f\left(\Phi\left(\frac{A+B}{2}\right)\right) &\leq \int_0^1 f(\Phi(tA + (1-t)B))dt \\ &\leq \left(\frac{f(\Phi(A)) + f(\Phi(B))}{2}\right). \end{aligned} \tag{7}$$

Recall that $AB + BA$ is called the *symmetrized product* of A and B.

Example 2.6 (see [9])**.** Let $\mathcal{S} = \{A, B \in B(H)^+; AB + BA \geq 0\}$. Then the continuous function $f(t) = t^s$, $0 < s \leq 1$, is an operator s-convex function on $[0, \infty)$ for operators in $\mathcal{S}$.

It should be mentioned here that $f(t) = t^s$ is not necessarily operator s-convex function for $s \in (0, 1]$ without the additional condition $AB+BA \geq 0$. For showing this, let

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix}.$$

One can check that $AB + BA \ngeq 0$ and $(A + B)^{\frac{1}{2}} \nleq A^{\frac{1}{2}} + B^{\frac{1}{2}}$.

It is obvious that $f(t) = t^s$, for $s \in (0, 1]$, is an operator s-convex function on $[0, \infty)$ if the C^*-algebra $B(H)$ is commutative. Uchiyama [17, Theorem 1] showed the relation between commutativity and symmetrized product of A and B. Later Nagisa et al. [12, Theorem 2] gave a weaker condition for two commuting operators. They proved that an unital C^*-algebra $\mathcal{A}$ is commutative if and only if positive operators A and B in $\mathcal{A}$ satisfy $AB + BA \geq 0$ and $AB^2 + B^2A \geq 0$.

By inequalities (6) and Example 2.6, we have that

$$2^{s-1}\left(\Phi\left(\frac{A+B}{2}\right)\right)^s \leq \int_0^1 (\Phi(tA + (1-t)B))^s dt \leq \frac{(\Phi(A))^s + (\Phi(B))^s}{s+1}$$

for $A, B \in B(H)^+$ such that $AB + BA \geq 0$.

The following Jensen type inequality is due to Davis [3].

Lemma 2.7. *Let $\Phi : B(H) \to B(H)$ be a unital positive linear map, and let f be an operator convex function on $[0, \infty)$. Then, for every $A \geq 0$,*

$$f(\Phi(A)) \leq \Phi(f(A)).$$

Now, by applying the above lemma and inequalities (7), we obtain the following inequalities for a unital positive linear map Φ:

$$f\left(\Phi\left(\frac{A+B}{2}\right)\right) \le \int_0^1 f(\Phi(tA+(1-t)B))dt \le \Phi\left(\frac{f(A)+f(B)}{2}\right),$$

where f is an operator convex function on $[0,\infty)$.

Also, by making use of the following lemma, we can improve the above inequalities for a specific interval $I \subseteq [0,\infty)$. The following lemma is well known. However, for reader's convenience, we provide a short proof.

Lemma 2.8. *Let $\Phi : B(H) \to B(H)$ be a unital linear invertible preserving map. Then Φ is spectrum compressing* (*i.e.,* $\mathrm{Sp}(\Phi(A)) \subseteq \mathrm{Sp}(A)$).

Proof. If $A \in B(H)$ and $\lambda \in \mathbb{C}$, then $\Phi(\lambda I - A) = \lambda I - \Phi(A)$. Since Φ is invertible preserving map, $\lambda \notin \mathrm{Sp}(\Phi(A))$ whenever $\lambda \notin \mathrm{Sp}(A)$. Hence, $\mathrm{Sp}(\Phi(A)) \subseteq \mathrm{Sp}(A)$. □

By Corollary 2.5 and the above lemma, we have the following corollary.

Corollary 2.9. *Let $f : I \subseteq [0,\infty) \to \mathbb{R}$ be an operator s-convex function for operators in $B(H)^+$ with $\mathrm{Sp}(A), \mathrm{Sp}(B) \subseteq I$, and let $\Phi : B(H) \to B(H)$ be a unital positive linear invertible preserving map. Then*

$$2^{s-1} f\left(\Phi\left(\frac{A+B}{2}\right)\right) \le \int_0^1 f(\Phi(tA+(1-t)B))dt \le \frac{f(\Phi(A))+f(\Phi(B))}{s+1}.$$

If $s = 1$ in the above corollary, then we have

$$f\left(\Phi\left(\frac{A+B}{2}\right)\right) \le \int_0^1 f(\Phi(tA+(1-t)B))dt \le \Phi\left(\frac{f(A)+f(B)}{2}\right)$$

for an operator convex function f.

3. Some trace inequalities for operators

In this section, by applying inequalities from Section 2, we obtain some trace inequalities.

We begin with some basic properties of the trace for operators. Let $\{e_i\}_{i\in I}$ be an orthonormal basis of H. We say that $A \in B(H)$ is *trace class* if

$$\|A\|_1 := \sum_{i\in I} \langle |A| e_i, e_i\rangle < \infty.$$

The definition of $\|A\|_1$ does not depend on the choice of the orthonormal basis $\{e_i\}_{i\in I}$. We denote by $B_1(H)$ the set of trace class operators in $B(H)$.

We define the *trace* of a trace class operator $A \in B_1(H)$ to be

$$\mathrm{Tr}(A) := \sum_{i\in I} \langle A e_i, e_i\rangle, \tag{8}$$

where $\{e_i\}_{i\in I}$ is an orthonormal basis of H. Note that this definition coincides with the usual definition of the trace if H is finite-dimensional. We observe that the series (8) converges absolutely and it is independent from the choice of basis.

The following result collects some properties of the trace.

Theorem 3.1. *The following statements hold:*
(i) *If* $A \in B_1(H)$, *then* $A^* \in B_1(H)$ *and* $\operatorname{Tr}(A^*) = \overline{\operatorname{Tr}(A)}$;
(ii) *If* $A \in B_1(H)$ *and* $T \in B(H)$, *then* $AT, TA \in B_1(H)$, $\operatorname{Tr}(AT) = \operatorname{Tr}(TA)$, *and* $|\operatorname{Tr}(AT)| \leq \|A\|_1 \|T\|$;
(iii) $\operatorname{Tr}(\cdot)$ *is a bounded linear functional on* $B_1(H)$ *with* $\|\operatorname{Tr}\| = 1$;
(iv) *If* $A, B \in B_1(H)$, *then* $\operatorname{Tr}(AB) = \operatorname{Tr}(BA)$.

For the theory of trace functionals and their applications, the reader is referred to [16].

Example 3.2. Let $\Phi : B_1(H) \to \mathbb{R}^+$, where $\Phi(A) = \operatorname{Tr}(A)$. Then Φ is a positive linear map which preserves invertibility. Moreover, for the finite-dimensional case, $\Phi(A) = \frac{\operatorname{Tr}(A)}{n}$ is also unital, where n is the size of the matrix A.

If $\Phi(A) = \operatorname{Tr}(A)$ in inequalities (6), then we have

$$\frac{(\operatorname{Tr}(A+B))^s}{2} \leq \int_0^1 (\operatorname{Tr}(tA + (1-t)B))^s dt \leq \frac{(\operatorname{Tr}(A))^s + (\operatorname{Tr}(B))^s}{s+1}$$

for $s \in (0, 1]$ and $A, B \in B_1(H)^+$.

Similarly, by making use of (7) for operator convex functions, we obtain the inequalities

$$f\left(\operatorname{Tr}\left(\frac{A+B}{2}\right)\right) \leq \int_0^1 f(\operatorname{Tr}(tA + (1-t)B))dt \leq \frac{f(\operatorname{Tr}(A)) + f(\operatorname{Tr}(B))}{2}. \tag{9}$$

Since $f(t) = t^r$ is an operator convex function on $(0, \infty)$ for $-1 \leq r \leq 0$ and $1 \leq r \leq 2$, inequalities (9) imply

$$\left(\operatorname{Tr}\left(\frac{A+B}{2}\right)\right)^r \leq \int_0^1 (\operatorname{Tr}(tA+(1-t)B))^r dt \leq \frac{(\operatorname{Tr}(A))^r + (\operatorname{Tr}(B))^r}{2}. \tag{10}$$

If $r = 2$ in inequalities (10), then we have

$$\left(\operatorname{Tr}\left(\frac{A+B}{2}\right)\right)^2 \leq \int_0^1 (\operatorname{Tr}(tA+(1-t)B))^2 dt \leq \frac{(\operatorname{Tr}(A))^2 + (\operatorname{Tr}(B))^2}{2}. \tag{11}$$

It is well known that the trace functional is sub-multiplicative, that is, for operators A and B in $B_1(H)^+$,

$$0 \leq \operatorname{Tr}(AB) \leq \operatorname{Tr}(A) \operatorname{Tr}(B).$$

So, by the above inequality we have, for all $A \in B_1(H)^+$,

$$\operatorname{Tr}(A^2) \leq (\operatorname{Tr}(A))^2. \tag{12}$$

Applying inequality (12) to the left side of inequality (11), we have

$$\operatorname{Tr}\left(\left(\frac{A+B}{2}\right)^2\right) \leq \int_0^1 (\operatorname{Tr}(tA+(1-t)B))^2 dt \leq \frac{(\operatorname{Tr}(A))^2 + (\operatorname{Tr}(B))^2}{2}.$$

Let M_n^+ stand for all positive matrices in M_n, and let $\lambda(A)$ denote the set of all eigenvalues of a matrix A. If A is a positive semi-definite matrix, then all of its eigenvalues are non-negative.

As mentioned in Example 3.2, $\Phi(A) = \frac{\operatorname{Tr}(A)}{n}$ is a unital positive linear invertible preserving map. By Corollary 2.9, we have the following result.

Theorem 3.3. *Let $f : I \subseteq (0,\infty) \to \mathbb{R}$ be an operator convex function for matrices in M_n^+ with $\lambda(A), \lambda(B) \subseteq I$, and let $\Phi(A) = \frac{\operatorname{Tr}(A)}{n}$ be a map from M_n to $\mathbb{R}^+$. Then*

$$f\left(\operatorname{Tr}\left(\frac{A+B}{2n}\right)\right) \leq \int_0^1 f\left(\operatorname{Tr}\left(\frac{tA+(1-t)B}{2n}\right)\right) dt \leq \operatorname{Tr}\left(\frac{f(A)+f(B)}{2n}\right). \tag{13}$$

Moreover, by inequalities (13), for the operator convex function $f(t) = t^r$ we obtain that

$$\left(\operatorname{Tr}\left(\frac{A+B}{2n}\right)\right)^r \leq \int_0^1 \left(\operatorname{Tr}\left(\frac{tA+(1-t)B}{2n}\right)\right)^r dt \leq \operatorname{Tr}\left(\frac{A^r+B^r}{2n}\right).$$

So, we have

$$(\operatorname{Tr}(A+B))^r \leq \int_0^1 (\operatorname{Tr}(tA+(1-t)B)^r dt \leq (2n)^{r-1} \operatorname{Tr}(A^r+B^r)$$

for $-1 \leq r \leq 0$ or $1 \leq r \leq 2$.

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