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On a class of vector-valued entire Dirichlet series in n variables

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Abstract. We consider a class F of entire Dirichlet series in n variables, whose coefficients belong to a commutative Banach algebra E. With a well defined norm, F is proved to be a Banach algebra with identity. Further results on quasi-invertibility, spectrum and continuous linear functionals are proved for elements belonging to F.

1. Introduction

Let E be a commutative Banach algebra with the identity such that all non-null elements of E are invertible. In this case E is also a field. Let

$$f\left(s_1, s_2, \ldots, s_n\right)=\sum_{m_1=1}^{\infty} \sum_{m_2=1}^{\infty} \cdots \sum_{m_n=1}^{\infty} a_{m_1, m_2, \ldots, m_n} e^{\left(\lambda_{1_{m_1}} s_1+\lambda_{2_{m_2}} s_2+\cdots+\lambda_{n_{m_n}} s_n\right)} \tag{1.1}$$

be an n-tuple Dirichlet series, where $a_{m_1,m_2,\ldots,m_n} \in E$, $s_j=\sigma_j+it_j$, $j=1,2,\ldots,n$, and

$$0<\lambda_{p_1}<\lambda_{p_2}<\cdots<\lambda_{p_k} \rightarrow \infty \text { as } k \rightarrow \infty, \text { for } p=1,2, \ldots, n.$$

To simplify the form of n-tuple Dirichlet series, we use the following notation:

$$\begin{gathered}
s=\left(s_1, s_2, \ldots, s_n\right) \in \mathbb{C}^n, \\
m=\left(m_1, m_2, \ldots, m_n\right) \in \mathbb{C}^n, \\
\lambda_{n_{m_n}}=\left(\lambda_{1_{m_1}}, \lambda_{2_{m_2}}, \ldots, \lambda_{n_{m_n}}\right) \in \mathbb{R}^n.
\end{gathered}$$

We further define

$$\lambda_{n_{m_n}} s=\lambda_{1_{m_1}} s_1+\lambda_{2_{m_2}} s_2+\cdots+\lambda_{n_{m_n}} s_n,$$

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$$\left|\lambda_{n_{m_n}}\right| = \lambda_{1_{m_1}} + \lambda_{2_{m_2}} + \cdots + \lambda_{n_{m_n}},$$
$$|m| = m_1 + m_2 + \cdots + m_n.$$

Thus, the series (1.1) can be written as

$$f(s) = \sum_{m=1}^{\infty} a_m \, e^{\lambda_{n_{m_n}} s}. \tag{1.2}$$

Janusauskas [1] showed that if there exists a tuple $p > \bar{0} = (0, 0, \dots, 0)$ such that

$$\limsup_{|m|\to\infty} \frac{\sum_{k=1}^{\infty} \log m_k}{p\lambda_{n_{m_n}}} = 0, \tag{1.3}$$

then the domain of absolute convergence of (1.2) coincides with its domain of convergence. Sarkar [6] proved that a necessary and sufficient condition for the series (1.2) with $a_m \in \mathbb{C}$ and satisfying (1.3) to be entire, i.e., to converge in the whole complex plane, is

$$\lim_{|m|\to\infty} \frac{\log |a_m|}{\left|\lambda_{n_{m_n}}\right|} = -\infty. \tag{1.4}$$

Liang and Gao [5] investigated the convergence and growth of n-tuple Dirichlet series and thus established an equivalence relation between order and the coefficients. Vaish [7] proved a necessary and sufficient condition so that Goldberg order of multiple Dirichlet series defining an entire function remained unaltered under rearrangements of the coefficients of the series. Kumar and Manocha in [2] generalised the condition of weighted norm for a Dirichlet series in one variable. In the present paper, we extend some results from [2] and also prove some new results for a Dirichlet series in n variables.

2. Basic results

In this section, some basic results are proved which are required to prove our main results.

Lemma 1. *The following conditions are equivalent.*

(a) $\limsup_{|m|\to\infty} \frac{\log(m_1+m_2+\cdots+m_n}{\lambda_{1n_{m_1}}+\lambda_{2_{m_2}}+\cdots+\lambda_{n_{m_n}}} = D < \infty.$

(b) $\limsup_{m_k\to\infty} \frac{\log m_k}{\lambda_{k_{m_k}}} = D_k < \infty, \quad k \in \mathbb{N} = \{1, 2, \dots\}.$

(c) *There exists* α, $0 < \alpha < \infty$, *such that the series*

$$\sum_{m_1, m_2, \dots, m_n = 1}^{\infty} e^{-\alpha\left(\lambda_{1n_{m_1}}+\lambda_{2_{m_2}}+\cdots+\lambda_{n_{m_n}}\right)} \tag{2.1}$$

converges.

Proof. (a)⇒(b). Let $\epsilon > 0$ be arbitrary. There exists $M_0 = M_0(\epsilon)$ such that

$$\log(m_1 + m_2 + \cdots + m_n) < (D + \epsilon)\left(\lambda_{1_{m_1}} + \lambda_{2_{m_2}} + \cdots + \lambda_{n_{m_n}}\right)$$

for $m_1 + m_2 + \cdots + m_n \geq M_0$. According to the relation

$$\sqrt{m_1 m_2 \ldots m_n} \leq (m_1 + m_2 + \cdots + m_n),$$

we get

$$\frac{\log m_k}{\lambda_{k_{m_k}}} < 2(D + \epsilon)\frac{\lambda_{1n_{m_1}} + \lambda_{2_{m_2}} + \cdots + \lambda_{n_{m_n}}}{\lambda_{k_{m_k}}}, \quad 1 \leq k \leq n, \quad m_k \geq M,$$

where $M = M(\epsilon)$.

(b)⇒(c). For arbitrary numbers $\epsilon_k > 0$, $1 \leq k \leq n$, we may choose $M_k(\epsilon_k)$, $1 \leq k \leq n$, such that

$$\log m_k < (D_k + \epsilon_k)\lambda_{k_{m_k}}, \quad 1 \leq k \leq n, \quad m_k \geq M_k(\epsilon_k).$$

Then the series (2.1) converges for $\alpha > \max_{1 \leq k \leq n}(D_k + \epsilon_k)$.

(c)⇒(a). Let S be the sum of the series (2.1). There exists $M_0 \in \mathbb{N}$ such that

$$\sum_{m_1, m_2, \ldots, m_n = 1}^{m_{1_0}, m_{2_0}, \ldots, m_{n_0}} e^{-\alpha\left(\lambda_{1_{m_1}} + \lambda_{2_{m_2}} + \cdots + \lambda_{n_{m_n}}\right)} < S \text{ for } m_{1_0}, m_{2_0}, \ldots, m_{n_0} \geq M_o.$$

Hence

$$\begin{aligned} &m_{1_0} m_{2_0} \ldots m_{n_0}\, e^{-\alpha\left(\lambda_{1_{m_1}} + \lambda_{2_{m_2}} + \cdots + \lambda_{n_{m_n}}\right)} \\ &\qquad \leq \sum_{m_1, m_2, \ldots, m_n = 1}^{m_{1_0}, m_{2_0}, \ldots, m_{n_0}} e^{-\alpha\left(\lambda_{1_{m_1}} + \lambda_{2_{m_2}} + \cdots + \lambda_{n_{m_n}}\right)} < S < \infty. \end{aligned}$$

Thus

$$\begin{aligned} &\limsup_{m_1 + m_2 \cdots + m_n \to \infty} \frac{\log(m_1 + m_2 + \cdots + m_n)}{\lambda_{1_{m_1}} + \lambda_{2_{m_2}} + \cdots + \lambda_{n_{m_n}}} \\ &\qquad \leq \limsup_{m_1 + m_2 \cdots + m_n \to \infty} \frac{\log(m_1 m_2 \ldots m_n)}{\lambda_{1_{m_1}} + \lambda_{2_{m_2}} + \cdots + \lambda_{n_{m_n}}} \leq \alpha. \end{aligned}$$

□

Theorem 1. *A necessary and sufficient condition for the series* (1.1) *satisfying* (1.3) *to be entire is that*

$$\limsup_{m_1 + m_2 + \cdots + m_n \to \infty} \frac{\log(\|a_{m_1, m_2, \ldots, m_n}\|)}{\lambda_{1_{m_1}} + \lambda_{2_{m_2}} + \cdots + \lambda_{n_{m_n}}} = -\infty. \tag{2.2}$$

Proof. Suppose that (1.1) defines an entire function. Then it converges absolutely for all $(s_1, s_2, \ldots, s_n)$. Thus, taking the points with coordinates $(q, q, \ldots, q)$, where $q > 0$, it follows that

$$\sum_{m_1=1}^{\infty} \sum_{m_2=1}^{\infty} \cdots \sum_{m_n=1}^{\infty} \|a_{m_1,m_2,\ldots,m_n}\| \, e^{\left(\lambda_{1m_1}+\lambda_{2m_2}+\cdots+\lambda_{nm_n}\right)q} < \infty.$$

Therefore,

$$\|a_{m_1,m_2,\ldots,m_n}\| \, e^{\left(\lambda_{1m_1}+\lambda_{2m_2}+\ldots+\lambda_{nm_n}\right)q} \le T(q, q, \ldots, q) < 1$$

for sufficiently large values of $m_1 + m_2 + \cdots + m_n$. Hence

$$\limsup_{m_1+m_2+\ldots+m_n \to \infty} \frac{\log \|a_{m_1,m_2,\ldots,m_n}\|}{\lambda_{1m_1} + \lambda_{2m_2} + \cdots + \lambda_{nm_n}} < -q.$$

Since $q > 0$ is arbitrary, the "necessity" part of the theorem is proved.

Conversely, let (2.2) be satisfied. It is sufficient to prove that (1.1) converges for all $(s_1, s_2, \ldots, s_n)$. Let $\sigma > 0$ be such that $\Re s_1 < \sigma, \Re s_2 < \sigma, \ldots, \Re s_n < \sigma$. By Lemma 1 we can find $\alpha > 0$ such that the series (2.1) converges. Now we fix $N_0(\alpha)$ such that

$$\frac{\log \|a_{m_1,m_2,\ldots,m_n}\|}{\lambda_{1m_1} + \lambda_{2m_2} + \cdots + \lambda_{nm_n}} < -\sigma - \alpha \text{ for } m_1, m_2, \ldots, m_n \ge N_0(\alpha).$$

Thus

$$\|a_{m_1,m_2,\ldots,m_n}\| \, e^{\left(\lambda_{1m_1}+\lambda_{2m_2}+\cdots+\lambda_{nm_n}\right)\sigma} < e^{-\alpha\left(\lambda_{1m_1}+\lambda_{2m_2}+\cdots+\lambda_{nm_n}\right)}.$$

Since the series (2.1) converges, the series (1.1) converges absolutely for all $(s_1, s_2, \ldots, s_n)$. $\square$

From the above notations one gets

$$\lim_{|m| \to \infty} \frac{\log \|a_m\|}{\left|\lambda_{n m_n}\right|} = -\infty. \tag{2.3}$$

We denote by F the set of the series (1.2) satisfying (1.3) and for which the sequence of numbers

$$\left|\lambda_{n m_n}\right|^{c_1\left|\lambda_{n m_n}\right|} (|m|!)^{c_2} \|a_m\|$$

is bounded, where $c_1, c_2 \ge 0$ and c_1, c_2 are simultaneously not zero. Thus there exists a number G such that

$$\left|\lambda_{n m_n}\right|^{c_1\left|\lambda_{n m_n}\right|} (|m|!)^{c_2} \|a_m\| < G,$$

$$c_1 \log\left|\lambda_{n m_n}\right| + c_2 \frac{\log(|m|!)}{\left|\lambda_{n m_n}\right|} + \frac{\log \|a_m\|}{\left|\lambda_{n m_n}\right|} < \frac{\log G}{\left|\lambda_{n m_n}\right|},$$

$$\frac{\log \|a_m\|}{\left|\lambda_{n m_n}\right|} < -\left\{c_1 \log\left|\lambda_{n m_n}\right| + c_2 \frac{\log(|m|!)}{\left|\lambda_{n m_n}\right|} + \frac{\log G}{\left|\lambda_{n m_n}\right|}\right\}.$$

This implies (2.3). Hence by Theorem 1 every element of F represents an entire function.

Define binary operations in F as

$$\begin{aligned} f(s)+g(s) &= \sum_{m=1}^{\infty} (a_m+b_m)\, e^{\lambda_{n_{m_n}} s}, \\ \gamma f(s) &= \sum_{m=1}^{\infty} (\gamma a_m)\, e^{\lambda_{n_{m_n}} s}, \quad \gamma \in E, \\ f(s)\cdot g(s) &= \sum_{m=1}^{\infty} \left|\lambda_{n_{m_n}}\right|^{c_1\left|\lambda_{n_{m_n}}\right|} (|m|!)^{c_2}\, a_m b_m\, e^{\lambda_{n_{m_n}} s}. \end{aligned}$$

The norm in F is defined by

$$\|f\| = \sup_{|m|\geq 1} \left|\lambda_{n_{m_n}}\right|^{c_1\left|\lambda_{n_{m_n}}\right|} (|m|!)^{c_2} \|a_m\|. \tag{2.4}$$

It is not difficult to see that F forms a linear space over the field E.

3. Main results

Theorem 2. *F is a commutative Banach algebra with identity over the field E.*

Proof. In order to prove this theorem we need to show that F is complete under the norm defined by (2.4). Let $\{f_{t_1}\}$ be a Cauchy sequence in F, where

$$f_{t_1}(s) = \sum_{m=1}^{\infty} a_m^{(t_1)}\, e^{\lambda_{n_{m_n}} s}.$$

Then, for given $\epsilon > 0$, we can find $t \geq 1$ such that $\|f_{t_1} - f_{t_2}\| < \epsilon$ for $t_1, t_2 \geq t$, i.e.,

$$\sup_{|m|\geq 1} \left|\lambda_{n_{m_n}}\right|^{c_1\left|\lambda_{n_{m_n}}\right|} (|m|!)^{c_2} \left\|a_m^{(t_1)} - a_m^{(t_2)}\right\| < \epsilon, \quad t_1, t_2 \geq t.$$

This shows that $\{a_m^{(t_1)}\}$ forms a Cauchy sequence in a Banach space E for all values of $|m| \geq 1$. Hence

$$\lim_{t_1\to\infty} a_m^{(t_1)} = a_m, \quad |m| \geq 1.$$

Letting $t_2 \to \infty$, we get

$$\sup_{|m|\geq 1} \left|\lambda_{n_{m_n}}\right|^{c_1\left|\lambda_{n_{m_n}}\right|} (|m|!)^{c_2} \left\|a_m^{(t_1)} - a_m\right\| < \epsilon, \quad t_1 \geq t.$$

Thus $f_{t_1} \to f$ as $t_1 \to \infty$. Also

$$\sup_{|m|\geq 1} \left|\lambda_{n_{m_n}}\right|^{c_1\left|\lambda_{n_{m_n}}\right|} (|m|!)^{c_2} \|a_m\| \leq \sup_{|m|\geq 1} \left|\lambda_{n_{m_n}}\right|^{c_1\left|\lambda_{n_{m_n}}\right|} (|m|!)^{c_2} \left\|a_m^{(t_1)} - a_m\right\|$$

$$+ \sup_{|m|\geq 1} \left|\lambda_{n_{m_n}}\right|^{c_1\left|\lambda_{n_{m_n}}\right|} (|m|!)^{c_2} \left\|a_m^{(t_1)}\right\| < \infty.$$

The identity element in F is

$$e(s) = \sum_{m=1}^{\infty} e \left|\lambda_{n_{m_n}}\right|^{-c_1\left|\lambda_{n_{m_n}}\right|} (|m|!)^{-c_2} e^{\lambda_{n_{m_n}} s},$$

where e is the identity element of E. Now, if $f, g \in F$, then

$$\begin{aligned}\|f \cdot g\| &= \sup_{|m|\geq 1} \left|\lambda_{n_{m_n}}\right|^{c_1\left|\lambda_{n_{m_n}}\right|} (|m|!)^{c_2} \left\{\left\| \left|\lambda_{n_{m_n}}\right|^{c_1\left|\lambda_{n_{m_n}}\right|} (|m|!)^{c_2} a_m\, b_m\right\|\right\} \\ &\leq \|f\|\,\|g\|.\end{aligned}$$

This proves the theorem. □

Theorem 3. *The function* (1.2) *is invertible in F if and only if the sequence*

$$\left\{\left\|d_m \left|\lambda_{n_{m_n}}\right|^{-c_1\left|\lambda_{n_{m_n}}\right|} (|m|!)^{-c_2}\right\|\right\} \tag{3.1}$$

is bounded, where d_m is the inverse of a_m.

Proof. Let $f \in F$ be invertible and let $g(s) = \sum_{m=1}^{\infty} b_m\, e^{\lambda_{n_{m_n}} s}$ be its inverse. Then $f(s) \cdot g(s) = e(s)$. Therefore,

$$\left|\lambda_{n_{m_n}}\right|^{c_1\left|\lambda_{n_{m_n}}\right|} (|m|!)^{c_2} a_m\, b_m = e \left|\lambda_{n_{m_n}}\right|^{-c_1\left|\lambda_{n_{m_n}}\right|} (|m|!)^{-c_2},$$

which implies that

$$\left|\lambda_{n_{m_n}}\right|^{c_1\left|\lambda_{n_{m_n}}\right|} (|m|!)^{c_2} \|b_m\| = \left\|e \left\{\left|\lambda_{n_{m_n}}\right|^{c_1\left|\lambda_{n_{m_n}}\right|} (|m|!)^{c_2} a_m\right\}^{-1}\right\|$$

or, equivalently,

$$\left|\lambda_{n_{m_n}}\right|^{c_1\left|\lambda_{n_{m_n}}\right|} (|m|!)^{c_2} \|b_m\| = \left\|d_m \left|\lambda_{n_{m_n}}\right|^{-c_1\left|\lambda_{n_{m_n}}\right|} (|m|!)^{-c_2}\right\|.$$

Thus (3.1) is a bounded sequence since $g \in F$.

Conversely, suppose that the sequence (3.1) is bounded. Define $g(s)$ such that

$$g(s) = \sum_{m=1}^{\infty} e \left|\lambda_{n_{m_n}}\right|^{-2c_1\left|\lambda_{n_{m_n}}\right|} (|m|!)^{-2c_2}\, a_m^{-1}\, e^{\lambda_{n_{m_n}} s}.$$

Then

$$f(s) \cdot g(s) = \sum_{m=1}^{\infty} e \left|\lambda_{n_{m_n}}\right|^{-c_1\left|\lambda_{n_{m_n}}\right|} (|m|!)^{-c_2}\, e^{\lambda_{n_{m_n}} s} = e(s).$$

The proof is complete. □

Definition 1. A function $g \in F$ is said to be quasi-inverse of $f \in F$ if $f(s) * g(s) = 0$, where

$$f(s) * g(s) = f(s) + g(s) + f(s) \cdot g(s).$$

Theorem 4. *An element* (1.2) *is quasi-invertible in* F *if and only if*

$$\inf_{|m|\geq 1} \left\{ \left\| 1 + \left|\lambda_{n_{m_n}}\right|^{c_1\left|\lambda_{n_{m_n}}\right|} (|m|!)^{c_2} a_m \right\| \right\} > 0. \tag{3.2}$$

The quasi-inverse of f *is the function* $g(s) = \sum_{m=1}^{\infty} b_m\, e^{\lambda_{n_{m_n}} s}$, *where*

$$b_m = \frac{-a_m}{1 + \left|\lambda_{n_{m_n}}\right|^{c_1\left|\lambda_{n_{m_n}}\right|} (|m|!)^{c_2} a_m}. \tag{3.3}$$

Proof. Let $f \in F$ be quasi-invertible, then there exists $g \in F$ such that $f(s) * g(s) = 0$. This implies

$$a_m + b_m + \left|\lambda_{n_{m_n}}\right|^{c_1\left|\lambda_{n_{m_n}}\right|} (|m|!)^{c_2} a_m b_m = 0 \ \text{ for } \ |m| \geq 1.$$

Suppose that (3.2) does not hold, that is,

$$\inf_{|m|\geq 1} \left\{ \left\| 1 + \left|\lambda_{n_{m_n}}\right|^{c_1\left|\lambda_{n_{m_n}}\right|} (|m|!)^{c_2} a_m \right\| \right\} = 0. \tag{3.4}$$

There exists a subsequence $\{m_t\}$ of a sequence of indices $\{m\}$ such that

$$\left|\lambda_{n_{m_{tn}}}\right|^{c_1\left|\lambda_{n_{m_{tn}}}\right|} (|m_t|!)^{c_2} \left\|a_{m_t}\right\| = -1 \ \text{ as } \ |t| \to \infty. \tag{3.5}$$

Now (3.3) implies that

$$\left|\lambda_{n_{m_{tn}}}\right|^{c_1\left|\lambda_{n_{m_{tn}}}\right|} (|m_t|!)^{c_2} \left\|b_{m_t}\right\| \geq \frac{\left|\lambda_{n_{m_{tn}}}\right|^{c_1\left|\lambda_{n_{m_{tn}}}\right|} (|m_t|!)^{c_2} \left\|a_{m_t}\right\|}{\left\| 1 + \left|\lambda_{n_{m_{tn}}}\right|^{c_1\left|\lambda_{n_{m_{tn}}}\right|} |m_t!|^{c_2}\, a_{m_t} \right\|}.$$

Using (3.4) and (3.5), we see that

$$\|g_t(s)\| \ \to \ \infty \ \text{ as } \ |t| \ \to \ \infty,$$

which is a contradiction.

Conversely, if (3.2) holds, then the function g belongs to F and

$$\begin{aligned} f(s) * g(s) &= \sum_{m=1}^{\infty} \{a_m + b_m + |\lambda_{n_{m_n}}|^{c_1|\lambda_{n_{m_n}}|} (|m|!)^{c_2} a_m b_m\} e^{\lambda_{n_{m_n}} s} \\ &= 0. \end{aligned}$$

Hence f is quasi-invertible. This completes the proof of the theorem. □

Definition 2. Let $f \in F$. The set

$$\sigma(f) = \{l \in \mathbb{C} : f - le \text{ is not invertible}\}$$

is called the spectrum of f.

Theorem 5. *The spectrum $\sigma(f)$, where $f \in F$, is precisely of the form*

$$\sigma(f) = cl \left\{ \left|\lambda_{n_{m_n}}\right|^{c_1 \left|\lambda_{n_{m_n}}\right|} (|m|!)^{c_2} \, a_m \; : \; |m| \, \geq \, 1 \right\}.$$

Proof. By Theorem 3, $f(s) = \sum_{m=1}^{\infty} a_m \, e^{\lambda_{n_{m_n}} s}$ is invertible in F if and only if the sequence (3.1) is a bounded. Thus, the function $\{f(s) \, - \alpha \, e(s)\}$ is not invertible if and only if

$$\left\{ \left\| (a_m - \alpha \, e \, \left|\lambda_{n_{m_n}}\right|^{-c_1 \left|\lambda_{n_{m_n}}\right|} (|m|!)^{-c_2})^{-1} \cdot \left(\left|\lambda_{n_{m_n}}\right|^{-c_1 \left|\lambda_{n_{m_n}}\right|} (|m|!)^{-c_2} \right) \right\| \right\}$$

is not bounded. This is possible only if there exists a subsequence $\{m_r\}$ of the sequence of indicies $\{m\}$ such that

$$\left\| \left|\lambda_{n_{m_{rn}}}\right|^{c_1 \left|\lambda_{n_{m_{rn}}}\right|} (|m_r|!)^{c_2} \, a_{m_r} - \alpha \right\| \to 0 \text{ as } |r| \to \infty.$$

Thus $\alpha \in cl \left\{ \left|\lambda_{n_{m_n}}\right|^{c_1 \left|\lambda_{n_{m_n}}\right|} (|m|!)^{c_2} \, a_m \; : \; |m| \, \geq \, 1 \right\}$. The proof is complete. ☐

Theorem 6. *Every continuous linear functional $\theta : F \to E$ is of the form*

$$\theta(f) \; = \sum_{m=1}^{\infty} a_m \; p_m \, |\lambda_{n_{m_n}}|^{c_1 |\lambda_{n_{m_n}}|} \, (|m|!)^{c_2},$$

where f is defined by (1.2) *and $\{p_m\}$ is a bounded sequence in E.*

Proof. Assume that $\theta : F \to E$ is a continuous linear functional. Since θ is continuous,

$$\theta(f) = \theta \left(\lim_{M \to \infty} f^{(M)} \right),$$

where

$$f^{(M)}(s) = \sum_{m=1}^{M} a_m \, e^{\lambda_{n_{m_n}} s}.$$

Define a sequence $\{f_m\} \subseteq F$ as

$$f_m(s) = e \left|\lambda_{n_{m_n}}\right|^{-c_1 \left|\lambda_{n_{m_n}}\right|} (|m|!)^{-c_2} e^{\lambda_{n_{m_n}} s}.$$

Therefore, using also the linearity of θ, we have

$$\theta(f) = \theta \left(\lim_{M \to \infty} \sum_{m=1}^{M} a_m \, \left|\lambda_{n_{m_n}}\right|^{c_1 \left|\lambda_{n_{m_n}}\right|} (|m|!)^{c_2} f_m \right)$$

$$= \lim_{M\to\infty} \sum_{m=1}^{M} a_m \left|\lambda_{n_{m_n}}\right|^{c_1\left|\lambda_{n_{m_n}}\right|} (|m|!)^{c_2}\, \theta(f_m).$$

Denoting $\theta\,(f_m) = p_m$, we may write

$$\theta(f) = \lim_{M\to\infty} \sum_{m=1}^{M} a_m\, p_m \left|\lambda_{n_{m_n}}\right|^{c_1\left|\lambda_{n_{m_n}}\right|} (|m|!)^{c_2}.$$

We now show that $\{p_m\}$ is a bounded sequence in E. Indeed,

$$\|p_m\| = \|\theta(f_m)\| \;\le\; \tau\,\|f_m\|$$

and $\|f_m\| = 1$ imply

$$\|p_m\| \;\le \tau.$$

Thus $\{p_m\}$ is a bounded sequence in E. This proves the theorem. ☐

Theorem 7. *Let $f(s) = \sum_{m=1}^{\infty} a_m\, e^{\lambda_{n_{m_n}} s} \in F$, where $a_m \neq 0$ for every $|m| \ge 1$, and let $K \in \mathbb{C}^n$ be a set having at least one finite limit point. Define*

$$f_\tau(s) = \sum_{m=1}^{\infty} a_m \left|\lambda_{n_{m_n}}\right|^{-c_1\left|\lambda_{n_{m_n}}\right|} (|m|!)^{-c_2}\, e^{\lambda_{n_{m_n}}(s+\tau)}. \tag{3.6}$$

Then the set $A_f = \{f_\tau : \tau \in K\}$ is a total set with respect to the family of continuous linear functionals $\phi : F \to E$.

Proof. Note that, for all $\tau \in \mathbb{C}^n$,

$$\|f_\tau\| = \sum_{m=1}^{\infty} \|a_m\|\; e^{\Re\left(\lambda_{n_{m_n}}\tau\right)}.$$

Since (1.2) is an entire Dirichlet series which converges absolutely in the whole complex plane, the series on the right hand side of the above equality is convergent for every $\tau \in K$. Hence $f_\tau(s) \in F$ for every $\tau \in K$.

Let ϕ be a continuous linear functional such that $\phi(A_f) \equiv 0$, that is $\phi(f_\tau) = 0$ for all $\tau \in K$. Then, by Theorem 6,

$$h(s) = \sum_{m=1}^{\infty} a_m\, p_m\, e^{\lambda_{n_{m_n}} s} = 0 \;\text{ for all }\; \tau \in K.$$

Since $\{p_m\}$ is a bounded sequence in E and $f \in F$, the function h also belongs to F. But

$$h(\tau) = \sum_{m=1}^{\infty} a_m\, p_m\, e^{\lambda_{n_{m_n}} \tau} = 0 \;\text{ for all }\; \tau \in K.$$

Since K has a finite limit point, we have that $h \equiv 0$. This however implies that

$$a_m\, p_m = 0 \;\text{ for all }\; |m| \ge 1,$$

The assumption $a_m \neq 0$ implies that $p_m = 0$ for all $|m| \geq 1$. Thus $\phi = 0$ and the proof is complete. $\square$

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