

Quasi-monomiality and operational identities for Laguerre–Konhauser-type matrix polynomials and their applications

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ABSTRACT. It is shown that an appropriate combination of methods, relevant to matrix polynomials and to operational calculus can be a very useful tool to establish and treat a new class of matrix Laguerre–Konhauser polynomials. We explore the formal properties of the operational identities to derive a number of properties of the new class of Laguerre–Konhauser matrix polynomials and discuss the links with classical polynomials.

1. Introduction

The analogous extension to the matrix framework for the classical case of Humbert, Hermite, Jacobi, Gegenbauer, Laguerre, and Chebyshev polynomials has been carried out in recent years. Properties and applications of different classes for these matrix polynomials are the focus of a number of previous papers (see, for example, [5]–[13], [15]–[21] and references therein). Motivated by the works mentioned above, we aim in this paper to construct a matrix version of the Laguerre–Konhauser polynomials given in [1] and exploit the monomiality principle to discuss various properties of these polynomials.

Throughout this paper, for a matrix $A \in \mathbb{C}^{N \times N}$, its spectrum is denoted by $\sigma(A)$. The two-norm of A is defined by

$$\|A\| = \sup_{x \neq 0} \frac{\|Ax\|_2}{\|x\|_2},$$

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where, for a vector $y \in \mathbb{C}^{N \times N}$, $\|y\|_2 = (y^T y)^{\frac{1}{2}}$ is the Euclidean norm of y . I will denote the identity matrix in $\mathbb{C}^{N \times N}$. We say that a matrix A in $\mathbb{C}^{N \times N}$ is positive stable if $\Re(\lambda) > 0$ for all $\lambda \in \sigma(A)$, where $\sigma(A)$ is the set of the eigenvalues of A . If A_0, A_1, \dots, A_n are elements of $\mathbb{C}^{N \times N}$ and $A_n \neq 0$, then we call

$$P(x) = A_n x^n + A_{n-1} x^{n-1} + A_{n-2} x^{n-2} + \dots + A_1 x + A_0$$

a matrix polynomial of degree n in x . If $A + nI$ is invertible for every integer $n \geq 0$, then

$$(A)_n = A(A+I)(A+2I) \cdots (A+(n-1)I), \quad n \geq 1; \quad (A)_0 = I.$$

Thus we have

$$(A)_n = \Gamma(A+nI)\Gamma^{-1}(A).$$

For any matrix $A \in \mathbb{C}^{N \times N}$, we have the following relation (see [12])

$$(1-x)^{-A} = \sum_{n=0}^{\infty} \frac{(A)_n x^n}{n!}, \quad |x| < 1. \quad (1.1)$$

In [4] Dattoli and Torre introduced Laguerre polynomials of two variables in the form

$$L_n(x, y) = n! \sum_{s=0}^n \frac{(-1)^s x^s y^{n-s}}{(s!)^2 (n-s)!},$$

combined with the principle of monomiality, which provided new means of analysis for the derivation of the solution of large classes of partial differential equations often encountered in physical problems. Also two interesting unifications and generalizations of Laguerre polynomials $L_n(x, y)$ are considered by Dattoli et al. [2] in the forms

$${}_1L_{n,\rho}(x, y) = n! \sum_{s=0}^n \frac{x^{s-\rho} y^{n-s}}{s!(n-s)!\Gamma(\rho+s+1)}$$

and

$$L_n^m(x, y) = (m+n)! \sum_{s=0}^n \frac{(-1)^s x^s y^{n-s}}{s!(n-s)!(m+k)!}.$$

The Konhauser polynomials of second kind are defined by (see [13])

$$Z_n^\beta(x; k) = \frac{\Gamma(kn + \beta + 1)}{n!} \sum_{j=0}^n \binom{n}{j} \frac{x^{kj}}{\Gamma(kj + \beta + 1)},$$

where $\beta \in \mathbb{C}_{-1}^+$, $n \in \mathbb{N}$, and $k \in \mathbb{N}$. Bin-Saad [1] studied the Laguerre-Konhauser polynomials ${}_kL_n^{(\alpha, \beta)}(x, y)$ as follows:

$${}_kL_n^{(\alpha, \beta)}(x, y) = n! \sum_{s=0}^n \sum_{r=0}^{n-s} \frac{(-1)^{s+r} x^{r+\alpha} y^{ks+\beta}}{s!r!(n-s-r)!\Gamma(\alpha+r+1)\Gamma(ks+\beta+1)}, \quad (1.2)$$

where $k = 1, 2, 3, \dots$ and $\alpha, \beta \in \mathbb{R}$. These polynomials can be written in the more elegant forms:

$${}_k L_n^{(\alpha, \beta)}(x, y) = n! \sum_{s=0}^n \frac{(-1)^s x^\alpha y^{ks+\beta} L_{n-s}^\alpha(x)}{s! \Gamma(\alpha + n - s + 1) \Gamma(ks + \beta + 1)},$$

and

$${}_k L_n^{(\alpha, \beta)}(x, y) = n! \sum_{s=0}^n \frac{(-1)^s x^{s+\alpha} y^\beta Z_{n-s}^\beta(x)}{s! \Gamma(\alpha + s + 1) \Gamma(kn - ks + \beta + 1)}.$$

It may be of interest to point out that the series representation (1.2), in particular, yields the following relationships:

$$(-1)^\rho y^{n+\rho} {}_1 L_n^{(\rho, 0)}\left(\frac{-x}{y}\right) = {}_1 L_{n, \rho}(x, y), \quad (1.3)$$

$$\frac{(m+n)!}{n!} y^{n+m} x^{-m} {}_1 L_n^{(m, 0)}\left(\frac{x}{y}, 0\right) = L_n^m(x, y), \quad (1.4)$$

$$\frac{\Gamma(kn + \beta + 1)}{n!} y^{-\beta} {}_k L_n^{(0, \beta)}(0, y) = Z_n^\beta(y; k), \quad (1.5)$$

$$\frac{\Gamma(n + \alpha + 1)}{n!} x^{-\alpha} x^{-n} {}_1 L_n^{(\alpha, 0)}(x, 0) = L_n^{(\alpha)}(x).$$

In this work, we will deal with operational definitions ruled by the operators \hat{D}_x and \hat{D}_x^{-1} , where \hat{D}_x denotes the derivative operator and \hat{D}_x^{-1} defines the inverse of the derivative. The following two formulas are well-known consequences of the derivative operator \hat{D}_x and the integral operator \hat{D}_x^{-1} (see [14]):

$$\hat{D}_x^m x^\lambda = \frac{\Gamma(\lambda + 1)}{\Gamma(\lambda - m + 1)} x^{\lambda - m}, \quad (1.6)$$

$$\hat{D}_x^{-m} x^\lambda = \frac{\Gamma(\lambda + 1)}{\Gamma(\lambda + m + 1)} x^{\lambda + m}, \quad (1.7),$$

where $m \in \mathbb{N} \cup \{0\}$, $\lambda \in \mathbb{C} / \{-1, -2, \dots\}$.

2. The Laguerre–Konhauser-type matrix polynomials

Let us consider the generating relation

$$\begin{aligned} f^{(A, B, C)}(x, y; t) &= \left[1 - t(1 - \hat{D}_y^{-k})\right]^{-C} {}_1 F_1 \left[C; A + I; \frac{-xt}{1 - t(1 - \hat{D}_y^{-k})} \right] \\ &\quad \times \{x^A y^B \Gamma^{-1}(A + I) \Gamma^{-1}(B + I)\}, \end{aligned}$$

where $\{A, B, C\} \subset \mathbb{C}^{N \times N}$, whose eigenvalues μ satisfy $\Re(\mu) > 0$, and

$${}_1 F_1 [a; c; x] = \sum_{n=0}^{\infty} \frac{(a)_n x^n}{n! (c)_n} \quad (2.1)$$

is the confluent hypergeometric series (see [22]). Making use of the series representation (2.1), the binomial relation (1.1), and applying the results (1.6) and (1.7), we can conclude that

$$f^{(A,B,C)}(x, y; t) = \sum_{n=0}^{\infty} \frac{(C)_n}{n!} \left\{ \sum_{r=0}^{\infty} \sum_{s=0}^n \frac{(-1)^r (C+nI)_s x^{A+rI} y^{B+ksI}}{r!s!} \times \Gamma^{-1}(A+(r+1)I) \Gamma^{-1}(B+(ks+1)I) \right\} t^{n+s}.$$

Thus, we find that

$$\begin{aligned} & \left[1 - t(1 - \hat{D}_y^{-k}) \right]^{-C} {}_1F_1 \left[C; A+I; \frac{-xt}{1 - t(1 - \hat{D}_y^{-k})} \right] \\ & \times \{ x^A y^B \Gamma^{-1}(A+I) \Gamma^{-1}(B+I) \} = \sum_{n=0}^{\infty} (C)_{nk} L_n^{(A,B)}(x, y) \frac{x^n}{n!}, \end{aligned} \quad (2.2)$$

where, for $k = 1, 2, 3, \dots$,

$$\begin{aligned} {}_k L_n^{A,B}(x, y) &= n! \left\{ \sum_{s=0}^n \sum_{r=0}^{n-s} \frac{(-1)^{s+r} x^{A+rI} y^{B+ksI}}{s!r!(n-s-r)!} \right. \\ & \left. \times \Gamma^{-1}(A+(r+1)I) \Gamma^{-1}(B+(ks+1)I) \right\}, \end{aligned} \quad (2.3)$$

C is any matrix in $\mathbb{C}^{N \times N}$, and A, B are matrices in $\mathbb{C}^{N \times N}$ whose eigenvalues μ satisfy $\Re(\mu) > 1$. When $C = A + I$, the generating relation (2.2) simplifies at once to the following elegant result:

$$\begin{aligned} & \left[1 - t(1 - \hat{D}_y^{-k}) \right]^{-A-I} \exp \left[\frac{-xt}{1 - t(1 - \hat{D}_y^{-k})} \right] \\ & \times \{ x^A y^B \Gamma^{-1}(A+I) \Gamma^{-1}(B+I) \} = \sum_{n=0}^{\infty} (A+I)_{nk} L_n^{(A,B)}(x, y) \frac{x^n}{n!}. \end{aligned} \quad (2.4)$$

The definition (2.3) suggests us (as particular cases) to define the following matrix versions of the polynomials defined by (1.3), (1.4) and (1.5):

$$\begin{aligned} & (-1)^A y^{A+nI} {}_1L_n^{(A,0)} \left(\frac{-x}{y}, 0 \right) \\ & = n! \sum_{s=0}^n \frac{x^{sI-A} y^{n-s} \Gamma^{-1}(A+(s+1)I)}{s!(n-s)!} = {}_1L_{n,A}(x, y), \\ & \frac{\Gamma(A+nI+I)}{n!} y^{A+nI} x^{-A} {}_1L_n^{(A,0)} \left(\frac{x}{y}, 0 \right) \\ & = \Gamma(A+nI+I) \sum_{s=0}^n \frac{(-1)^s x^s y^{n-s} \Gamma^{-1}(A+kI)}{s!(n-s)!} = L_n^{(A)}(x, y), \end{aligned}$$

$$\begin{aligned} & \frac{\Gamma(B + knI + I)}{n!} y^B {}_1L_n^{(0,B)}(x, 0) \\ &= \frac{\Gamma(B + knI + I)}{n!} \sum_{j=0}^n \binom{n}{j} x^{kj} \Gamma^{-1}(B + (kj + 1)I) = Z_n^B(y; k). \end{aligned} \quad (2.5)$$

Also, it may be of interest to point out that the series representation (2.3) yields the relationship

$$\frac{\Gamma(A + nI + I)}{n!} (\lambda x)^{-A} {}_1L_n^{(A,0)}(x\lambda, 0) = L_n^{(A,\lambda)}(x), \quad (2.6)$$

where $L_n^{(A,\lambda)}(x)$ is the known Laguerre matrix polynomial of one variable (see [11]). Further, in view of relations (2.5) and (2.6) we may write the series representation (2.4) in the more elegant forms

$$\begin{aligned} {}_kL_n^{(A,B)}(x, y) &= n! \left\{ \sum_{s=0}^n \frac{(-1)^s x^{A+sI} y^B Z_n^B(y; k)}{s!} \right. \\ &\quad \left. \times \Gamma^{-1}(A + (s + 1)I) \Gamma^{-1}(B + (kn - ks + 1)I) \right\} \end{aligned}$$

and

$$\begin{aligned} {}_kL_n^{(A,B)}(x\lambda, y) &= n! \left\{ \sum_{s=0}^n \frac{(-1)^s x^A y^{B+ksI} L_n^{(A,\lambda)}(x, y)}{s!} \right. \\ &\quad \left. \times \Gamma^{-1}(A + (n - s + 1)I) \Gamma^{-1}(B + (ks + 1)I) \right\}. \end{aligned}$$

Next, in view of the definition of Kampé de Fériet's double hypergeometric series (see [22], p. 27(28))

$$\begin{aligned} & F_{l;m;n}^{p;q;k} \left[\begin{matrix} (a_p) : & (b_q); & (c_k); & \\ (d_l) : & (e_m); & (f_n); & \end{matrix} \right. x, y \left. \right] \\ &= \sum_{r,s=0}^{\infty} \frac{\prod_{j=1}^p (a_j)_{r+s} \prod_{j=1}^q (b_j)_r \prod_{j=1}^k (c_j)_s x^r y^s}{\prod_{j=1}^l (d_j)_{r+s} \prod_{j=1}^m (e_j)_r \prod_{j=1}^n (f_j)_s r! s!}, \end{aligned}$$

we can easily establish the following series representation for the polynomials ${}_kL_n^{(A,B)}(x, y)$:

$$\begin{aligned} & {}_kL_n^{(A,B)}(x, y) = \{x^A y^B \Gamma^{-1}(A + I) \Gamma^{-1}(B + I)\} \\ & \quad \times F_{0;1;k}^{1;0;0} \left[\begin{matrix} -nI : & -; & -; \\ - : & A + I; & \Delta(k; B + I); \end{matrix} \right. x, \left. \left(\frac{y}{k}\right)^k \right], \end{aligned} \quad (2.7)$$

where, and in what follows, $\Delta(k; B)$ abbreviates the array of k parameters $\frac{B}{k}, \frac{B+I}{k}, \dots, \frac{B+kI-I}{k}$, $k \geq 1$, and dashed line “-” means that the number of parameters is zero. Another interesting operational formula for ${}_kL_n^{(A,B)}(x, y)$

can be derived, as a consequence of the explicit representation (2.7). Indeed, since

$$r!\hat{D}_x^{-r} = x^r,$$

formula (2.7) yields

$$\begin{aligned} {}_kL_n^{(A,B)}(x, y) &= \{x^A y^B \Gamma^{-1}(A+I) \Gamma^{-1}(B+I)\} \\ &\times {}_0F_{0:1;k} \left[\begin{array}{ccc} -nI : & I; & \Delta(k; I); \\ - : & A+I; & \Delta(k; B+I); \end{array} \quad \begin{array}{c} \hat{D}_x^{-1}, \hat{D}_y^k \\ 0 \end{array} \right]. \end{aligned}$$

On the other hand, if we employ the result

$$\hat{D}_y^{r+s} y^n = \frac{n! y^{n-r-s}}{(n-s-r)!},$$

then we find that

$$\begin{aligned} y^{A+nI} L_n^{A,B}(x/y, y) &= \{y^B \Gamma^{-1}(A+I) \Gamma^{-1}(B+I)\} \\ &\times {}_0F_1 \left[-; \Delta(k; B+I); - \left(\frac{y}{k}\right)^k y \hat{D}_y \right] \exp(-\hat{D}_x^{-1} \hat{D}_y) \{x^A y^n\}, \end{aligned}$$

or equivalently,

$$\begin{aligned} y^{A+nI} L_n^{A,B}(x/y, y) &= \{x^A y^B \Gamma^{-1}(A+I) \Gamma^{-1}(B+I)\} \\ &\times {}_0F_1 \left[-; \Delta(k; B+I); - \left(\frac{y}{k}\right)^k y \hat{D}_y \right] {}_1F_1 \left[I; A+I; -\hat{D}_x \hat{D}_y^{-1} \right] \{y^n\}. \end{aligned}$$

Further, according to the inverse operator (1.7), we can rewrite ${}_kL_n^{(A,B)}(x, y)$ in the series representation

$$\begin{aligned} {}_kL_n^{(A,B)}(x, y) &= \left\{ \sum_{s=0}^n \sum_{r=0}^{n-s} \frac{(-n)_{s+r} \hat{D}_x^{-r} \hat{D}_y^{-ks}}{s! r!} \right. \\ &\left. \times [x^A y^B \Gamma^{-1}(A+I) \Gamma^{-1}(B+I)] \right\}. \end{aligned} \quad (2.8)$$

This yields the Rodrigue-type formula

$${}_kL_n^{(A,B)}(x, y) = \left(1 - \hat{D}_x^{-1} - \hat{D}_y^{-k}\right)^n \{x^A y^B \Gamma^{-1}(A+I) \Gamma^{-1}(B+I)\}. \quad (2.9)$$

Directly from (2.8), by exploiting the same procedure leading to equation (2.8), we can establish the following operational connecting relationships of ${}_kL_n^{(A,B)}(x, y)$ with the Laguerre matrix polynomials $L_n^{(A,\lambda)}(x)$ and Konhauser matrix polynomials $Z_n^B(y; k)$:

$$\begin{aligned} {}_kL_n^{(A,B)}(x, y) &= n! y^B \Gamma^{-1}(knI + B + I) (1 - \hat{D}_x^{-1})^n \\ &\times Z_n^B(y/(1 - \hat{D}_x^{-1}); k) \{x^A \Gamma^{-1}(A+I)\} \end{aligned}$$

and

$$\begin{aligned} {}_kL_n^{(A,B)}(x\lambda, y) &= n!x^A\Gamma^{-1}(A+nI+I)(1-\hat{D}_y^{-k})^n \\ &\quad \times L_n^{(A,\lambda)}(y/(1-\hat{D}_y^{-k}))\{y^B\Gamma^{-1}(B+I)\}. \end{aligned}$$

3. Quasi-monomiality and operational identities

The Laguerre–Konhauser polynomials ${}_kL_n^{(A,B)}(x, y)$ are quasi-monomials under the action of the operators (see [3])

$$\begin{aligned} \hat{M} &= 1 - \hat{D}_x^{-1} - \hat{D}_y^{-k}, \\ \hat{P}_1 &= -x^A \hat{D}_x x^{-A+I} \hat{D}_x, \end{aligned} \tag{3.1}$$

$$\hat{P}_2 = -\frac{1}{k} y^{B-kI+I} \hat{D}_y y^{-B+kI} \hat{D}_y^k. \tag{3.2}$$

According to the quasi-monomiality properties we have

$$\begin{aligned} \hat{M} {}_kL_n^{(A,B)}(x, y) &= {}_kL_{n+1}^{(A,B)}(x, y), \\ \hat{P}_1 {}_kL_n^{(A,B)}(x, y) &= n {}_kL_{n-1}^{(A,B)}(x, y), \\ \hat{P}_2 {}_kL_n^{(A,B)}(x, y) &= n {}_kL_{n-1}^{(A,B)}(x, y), \\ \frac{1}{2}(\hat{P}_1 + \hat{P}_2) {}_kL_n^{(A,B)}(x, y) &= n {}_kL_{n-1}^{(A,B)}(x, y), \end{aligned}$$

which can be combined to prove that ${}_kL_n^{(A,B)}(x, y)$ satisfy the following differential equations:

$$\begin{aligned} \left[(x\hat{D}_x + A - 1)\hat{D}_x \hat{D}_y^k + x\hat{D}_x^2 + (I - A)\hat{D}_x - (A + nI)\hat{D}_y^k \right] {}_kL_n^{(A,B)}(x, y) &= 0, \\ \left[y(1 - x\hat{D}_x)\hat{D}_y^k + (kI - B)(1 - \hat{D}_x)\hat{D}_y^k + y\hat{D}_x \hat{D}_y^k \right. \\ \left. - (knI + B)\hat{D}_x \right] {}_kL_n^{(A,B)}(x, y) &= 0. \end{aligned}$$

From the lowering operators \hat{P}_1 and \hat{P}_2 , we can define operators playing the role of the inverse operators \hat{P}_1^{-1} and \hat{P}_2^{-1} (see [4], equation (15)). Thus we get

$$\begin{aligned} \hat{P}_1^{-1} &= -x^A \hat{D}_x^{-1} x^{-A-I} \hat{D}_x^{-1}, \\ \hat{P}_2^{-1} &= -ky^B \hat{D}_y^{-1} y^{-B-I} \hat{D}_y^{-k}, \end{aligned}$$

and they satisfy

$$\hat{P}_1^{-1} {}_kL_n^{(A,B)}(x, y) = \frac{{}_kL_{n+1}^{(A,B)}(x, y)}{(n+1)} = \hat{P}_2^{-1} {}_kL_n^{(A,B)}(x, y).$$

Clearly, we have

$$\hat{P}_1 \hat{P}_1^{-1} {}_kL_n^{(A,B)}(x, y) = {}_kL_n^{(A,B)}(x, y).$$

Regarding the Lie bracket $[\cdot, \cdot]$, defined by $[A, B] = AB - BA$, we are led to

$$[\hat{P}_1, \hat{M}]_k L_n^{(A,B)}(x, y) = {}_k L_n^{(A,B)}(x, y), \quad [\hat{P}_2, \hat{M}]_k L_n^{(A,B)}(x, y) = {}_k L_n^{(A,B)}(x, y),$$

From relations (3.1) and (3.2), it is also proved that the set of polynomials ${}_k L_n^{(A,B)}(x, y)$ satisfy the partial differential equation

$$\left[x^A \frac{\partial}{\partial x} x^{-A+I} \frac{\partial}{\partial x} - \frac{1}{k} y^{B-kI+I} \frac{\partial}{\partial y} y^{-B+kI} \frac{\partial^k}{\partial y^k} \right] {}_k L_n^{(A,B)}(x, y) = 0.$$

4. Some applications

By starting from identity (2.9), multiplying both sides of (2.9) by $t_n/n!$ and then taking the sum, we obtain

$$\begin{aligned} \sum_{n=0}^{\infty} {}_k L_n^{(A,B)}(x, y) \frac{t^n}{n!} &= \exp\left(t(1 - \hat{D}_x^{-1} - \hat{D}_y^{-k})\right) \\ &\times \{x^A y^B \Gamma^{-1}(A+I) \Gamma^{-1}(B+I)\}. \end{aligned} \quad (4.1)$$

Now, according to the facts that

$$\exp\left(t \hat{D}_x^{-1}\right) \{x^A \Gamma^{-1}(A+I)\} = \{x^A \Gamma^{-1}(A+I)\} {}_0F_1[-; A+I; -tx]$$

and

$$\begin{aligned} \exp\left(t \hat{D}_y^{-k}\right) \{y^B \Gamma^{-1}(B+I)\} &= \{y^B \Gamma^{-1}(B+I)\} \\ &\times {}_0F_k\left[-; \Delta(k; B+I); -\left(\frac{y}{k}\right)^k t\right], \end{aligned}$$

the generating function (4.1) further yields the interesting generating relation

$$\begin{aligned} \sum_{n=0}^{\infty} {}_k L_n^{(A,B)}(x, y) \frac{t^n}{n!} &= \{x^A y^B \Gamma^{-1}(A+I) \Gamma^{-1}(B+I)\} {}_0F_1[-; A+I; -tx] \\ &\times {}_0F_k\left[-; \Delta(k; B+I); -\left(\frac{y}{k}\right)^k t\right]. \end{aligned}$$

A similar procedure yields

$$\begin{aligned} \sum_{n=0}^{\infty} {}_k L_n^{(A,B)}(x, y) t^n &= \{x^A y^B \Gamma^{-1}(A+I) \Gamma^{-1}(B+I)\} \\ &\times \frac{1}{(1-t)} {}_1F_1\left[I; A+I; \frac{-tx}{(1-t)}\right] {}_1F_k\left[I; \Delta(k; B+I); -\left(\frac{y}{k}\right)^k \frac{t}{(1-t)}\right]. \end{aligned}$$

On account of the identity

$$\begin{aligned} \left(1 - t(1 - \hat{D}_y^{-k})\right)^{-C} \{y^B \Gamma^{-1}(B+I)\} &= \{y^B \Gamma^{-1}(B+I)\} (1-t)^{-C} \\ &\times {}_0F_k\left[-; \Delta(k; B+I); -\left(\frac{y}{k}\right)^k t\right], \end{aligned}$$

the generating relation (2.4) can be written in the more compact form

$$\begin{aligned} & \{x^A y^B (1-t)^{-A-I} \Gamma^{-1}(A+I) \Gamma^{-1}(B+I)\} \\ & \times F_{0:1;k}^{1:0;0} \left[\begin{array}{c} A+I : \quad -; \quad -; \\ - : \quad A+I; \quad \Delta(k; B+I); \end{array} \quad \frac{-tx}{(1-t)}, -\frac{t}{(1-t)} \left(\frac{y}{k}\right)^k \right] \\ & = \sum_{n=0}^{\infty} (A+I)_{nk} L_n^{(A,B)}(x, y) \frac{x^n}{n!}. \end{aligned}$$

From relation (2.9), we can state that

$$\begin{aligned} & (1-\hat{M})^n \{x^A y^B \Gamma^{-1}(A+I) \Gamma^{-1}(B+I)\} \\ & = \sum_{s=0}^n (-1)^s \binom{n}{s}_k L_n^{(A,B)}(x, y) \\ & = \{x^{A+nI} y^B \Gamma^{-1}(A+nI+I) \Gamma^{-1}(B+I)\} \\ & \quad \times {}_2F_k \left[-nI, -A-nI; \Delta(k; B+I); -\left(\frac{y}{k}\right)^k / x \right]. \end{aligned} \tag{4.2}$$

The Jacobi matrix polynomials $P_n^{(A,B)}(x)$, $n \in \mathbb{N}$, have been given in [7], for parameter matrices A and B whose eigenvalues μ satisfy $\Re(\mu) > 0$, as follows:

$$\begin{aligned} P_n^{(A,B)}(x) & = \Gamma^{-1}(B+I) \Gamma(B+(n+1)I) \frac{(-1)^n}{n!} \\ & \quad \times {}_2F_1 \left[A+B+nI, -nI; B+I; \frac{x+1}{2} \right]. \end{aligned} \tag{4.3}$$

Now, in view of representation (4.2), equation (4.3) provides us the following connection between Laguerre matrix polynomials ${}_k L_n^{(A,B)}(x, y)$ and Jacobi matrix polynomials $P_n^{A,B}(x, y)$ in the form:

$$\begin{aligned} & \sum_{s=0}^n (-1)^s \binom{n}{s}_1 L_n^{(-A-B,B)} \left(x, -\frac{x(x+1)}{2} \right) \\ & = (-1)^{-1} n! x^{A+nI} y^B \Gamma^{-1}(B+(n+1)I) \Gamma^{-1}(-A-B+(n+1)I) P_n^{A,B}(x, y). \end{aligned}$$

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References

- [1] Maged G. Bin-Saad, *Associated Laguerre-Konhauser polynomials, quasi-monomiality and operational identities*, J. Math. Anal. Appl. **324**(2) (2006), 438–448.
- [2] G. Dattoli, S. Lorenzutta, A. M. Mancho, and A. Torre, *Generalized polynomials and associated operational identities*, J. Comput. Appl. Math. **108** (1999), 209–218.

- [3] G. Dattoli and A. Torre, *Operational methods and two variable Laguerre polynomials*, Atti Accad. Sci. Torino Cl. Sci. Fis. Mat. Natur. **132** (1998), 1–7.
- [4] G. Dattoli and A. Torre, *Exponential operators, quasi-monomials and generalized polynomials*, Radiation Physics and Chemistry **57** (2000), 21–26.
- [5] G. Dattoli, A. Torre, and M. Carpanese, *Operational rules and arbitrary order Hermite generating functions*, J. Math. Anal. Appl. **227** (1998), 98–111.
- [6] E. Defez and L. Jódar, *Chebyshev matrix polynomials and second order matrix differential equations*, Util. Math. **61** (2002), 107–123.
- [7] E. Defez and L. Jódar, *Jacobi matrix differential equation, polynomial solution and their properties*, Comput. Math. Appl. **48** (2004), 789–803.
- [8] A. Horadam, *Polynomials associated with Gegenbauer polynomials*, Fibonacci Quart. **23** (1985), 295–399.
- [9] A. Horadan and S. Pethe, *Gegenbauer polynomials revisited*, Fibonacci Quart. **19** (1981), 393–398.
- [10] L. Jódar and R. Company, *Hermite matrix polynomials and second order matrix differential equations*, J. Approx. Theory Appl. **12**(2) (1996), 20–30.
- [11] L. Jódar, R. Company, and E. Navarro, *Laguerre matrix polynomials and systems of second order differential equations*, Appl. Numer. Math. **15**(1) (1994), 53–64.
- [12] L. Jódar and J. C. Cortés, *On the hypergeometric matrix function*, J. Comput. Appl. Math. **99**(1-2) (1998), 205–217.
- [13] J. D. E. Konhauser, *Biorthogonal polynomials suggested by the Laguerre polynomials*, Pacific J. Math. **21** (1967), 303–314.
- [14] K. S. Miller and B. Ross, *An Introduction to Fractional Calculus and Fractional Differential Equations*, J. Wiley and Sons, New York, 1993.
- [15] M. A. Pathana, Maged G. Bin-Saad, and Fadhl Al-Sarahi, *On matrix polynomials associated with Humbert polynomials*, J. Korea Soc. Math. Educ. Ser. B Pure Appl. Math. **3** (2014), 207–218.
- [16] A. Shehata, *Connections between Legendre with Hermite and Laguerre matrix polynomials*, Gazi Univ. J. Sci. **28**(2) (2015), 221–230.
- [17] A. Shehata, *On modified Laguerre matrix polynomials*, J. Natur. Sci. Math. **8**(2) (2015), 153–166.
- [18] A. Shehata, *Some relations on Laguerre matrix polynomials*, Malays. J. Math. Sci. **9**(3) (2015), 443–462.
- [19] A. Shehata, *Some relations on Humbert matrix polynomials*, Math. Bohem. **141**(4) (2016), 407–429.
- [20] A. Shehata and Bayram Çekim, *Some relations on Hermite-Hermite matrix polynomials*, Politehn. Univ. Bucharest Sci. Bull. Ser. A Appl. Math. Phys. **78**(1) (2016), 181–194.
- [21] A. Shehata and Subuhi Khan, *On Bessel-Maitland matrix function*, Mathematica **57**(1-2) (2015), 90–103.
- [22] H. M. Srivastava and R. W. Karlsson, *Multiple Gaussian Hypergeometric Series*, Ellis Horwood Limited, Chichester, New York, 1985.

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