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Applications of Chebyshev polynomials on a Sakaguchi type class of analytic functions associated with quasi-subordination

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Abstract. We introduce a class of analytic functions which is defined in terms of a quasi-subordination. Coefficient estimates including the relevant classical Fekete–Szegö inequality of functions belonging to the aforementioned class are derived. Improved results for associated classes involving subordination and majorization are also discussed.

1. Introduction

Let A denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad z \in E, \tag{1.1}$$

that are analytic in the open unit disk $E = \{z : |z| < 1\}$ and normalized by $f(0) = 0 = f'(0) - 1$.

For two analytic functions f and g such that $f(0) = g(0)$, we say that f is subordinate to g in E and write $f(z) \prec g(z)$, $z \in E$, if there exists a Schwarz function w with $w(0) = 0$ and $|w(z)| \leq |z|$, $z \in E$, such that $f(z) = g(w(z))$, $z \in E$. Also, if the function g is univalent in E, then we have the following equivalence (see [8]):

$$f(z) \prec g(z) \Longleftrightarrow f(0) = g(0) \quad \text{and} \quad f(E) \subset g(E). \tag{1.2}$$

Furthermore, f is said to be quasi-subordinate to g in E, and written as $f(z) \prec_q g(z)$, $z \in E$, if there exists an analytic function φ such that

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$|\varphi(z)| \leq 1$ $(z \in E)$, the function f/φ is analytic in E, and

$$\frac{f(z)}{\varphi(z)} \prec g(z), \quad z \in E. \tag{1.3}$$

That is, there exists a Schwartz function w such that $f(z) = \varphi(z)g(w(z))$, $z \in E$. The definition can be found in [12].

Also, one observes that if $\varphi(z) = 1$ $(z \in E)$, then the quasi-subordination $\prec_q$ becomes the well-known subordination $\prec$, and for the Schwartz function $w(z) = z$ $(z \in E)$, the quasi-subordination $\prec_q$ becomes the majorization $\ll$. In this case (see [6]),

$$f(z) \prec_q g(z) \Longrightarrow f(z) = \varphi(z)g(z) \Longrightarrow f(z) \ll g(z), \quad z \in E. \tag{1.4}$$

Recently, Sharma and Raina [14] worked on a Sakaguchi type class of analytic functions associated with quasi-subordination whose geometric condition satisfies

$$\left(f'(z)\left(\frac{(1-b)z}{f(z)-f(bz)}\right)^{\lambda} - 1\right) \prec_q (\phi(z)-1), \quad z \in E, \tag{1.5}$$

where $b \in \mathbb{C}$, $|b| \leq 1$, $\phi \in P$, $\lambda \geq 0$, and powers are considered to have only principal values. Olatunji et al. [10] extended this class of functions to the class $\mathcal{G}_q^{\lambda}(\phi, s, b)$ whose geometric functions satisfy

$$\left(f'(z)\left(\frac{(s-b)z}{f(sz)-f(bz)} - 1\right)^{\lambda} - 1\right) \prec_q (\phi(z)-1), \quad z \in E, \tag{1.6}$$

where $s, b \in \mathbb{C}$, $s \neq b$, $|s| \leq 1$, $|b| \leq 1$, $\lambda \geq 0$, $\phi \in P$ is the modified sigmoid function, and powers are also considered to have only principal values. The interesting results obtained are too numerous to discuss.

Chebyshev polynomials play a major role in numerical analysis. These polynomials are of four kinds but most of the books and articles related to specific orthogonal polynomials of the Chebyshev family contain essentially results on Chebyshev polynomials of the first and second kinds, namely

$$T_n(x) = \cos n\theta \ \text{ and } \ U_n(x) = \frac{\sin(n+1)\theta}{\sin\theta}, \quad x \in [-1, 1],$$

which have numerous applications (see Doha [1] and Mason [7]). Here the subscript n denotes the polynomial degree and $x = \cos\theta$ (see [4] for details).

For the purpose of our results, we need the following lemma and definition.

Lemma 1.1 (see [5]). *Let the Schwartz function w be given by*

$$w(z) = w_1 z + w_2 z^2 + w_3 z^3 + \dots, \quad z \in E. \tag{1.7}$$

Then

$$|w_1| \leq 1, \quad |w_2 - t w_1^2| \leq 1 + (|t|-1)|w_1|^2 \leq \max\{1, |t|\}, \tag{1.8}$$

where $t \in \mathbb{C}$.

Definition 1.2. We say that $f \in A$ of the form (1.1) belongs to $\mathcal{G}_q^{\lambda}(H, s, b)$ if

$$\left(f'(z) \left(\frac{(s-b)z}{f(sz) - f(bz)} \right)^{\lambda} - 1 \right) \prec_q (H(z,t) - 1), \tag{1.9}$$

and the power is considered to have principal value. Also,

$$H(z,t) = \frac{1}{1 - 2tz + z^2}.$$

By letting $t = \cos\alpha$, $\alpha \in (-\pi/3, \pi/3)$, we obtain

$$\begin{aligned} H(z,t) &= 1 + \sum_{n=1}^{\infty} \frac{\sin(n+1)\alpha}{\sin\alpha} z^n \\ &= 1 + 2\cos\alpha z + \left(3\cos^2\alpha - \sin^2\alpha\right) z^2 + \dots, \quad z \in E. \end{aligned} \tag{1.10}$$

So, we can write

$$H(z,t) = 1 + U_1(t)z + U_2(t)z^2 + \dots, \quad z \in E,\ t \in (-1,1), \tag{1.11}$$

where

$$U_{n-1} = \frac{\sin\left(n \cos^{-1} t\right)}{\sqrt{1-t^2}}, \quad n \in \mathbb{N},$$

are the Chebyshev polynomials of second kind. Furthermore, we know that

$$U_n(t) = 2tU_{n-1}(t) - U_{n-2}(t)$$

and

$$\begin{aligned} &U_1(t) = 2t, \quad U_2(t) = 4t^2 - 1, \quad U_3(t) = 8t^3 - 4t, \\ &U_4(t) = 16t^4 - 12t^2 + 1, \ \dots . \end{aligned} \tag{1.12}$$

The Chebyshev polynomials $T_n(t)$, $t \in [-1,1]$, of first kind have the generating function of the form

$$\sum_{n=0}^{\infty} T_n(t) z^n = \frac{1 - tz}{1 - 2tz + z^2} \quad (z \in E).$$

The following relationships exist between the Chebyshev polynomials of first and second kinds (see [2]):

$$\begin{aligned} \frac{dT_n(t)}{dt} &= nU_{n-1}(t), \\ T_n(t) &= U_n(t) - tU_{n-1}(t), \\ 2T_n(t) &= U_n(t) - U_{n-2}(t). \end{aligned}$$

From Definition 1.2 it follows that $f \in \mathcal{G}_q^{\lambda}(H,s,b)$ if and only if there exists an analytic function φ with $|\varphi(z)| \le z,\ z \in E$, such that

$$\frac{\left(f'(z)\left(\frac{(s-b)z}{f(sz)-f(bz)}\right)^{\lambda}-1\right)}{\varphi(z)} \prec (H(z,t)-1), \quad z \in E. \tag{1.13}$$

If $\varphi(z) \equiv 1,\ z \in \mathbb{U}$, in the subordination condition (1.13), then the class $\mathcal{G}_q^{\lambda}(H,s,b)$ is denoted by $\mathcal{G}^{\lambda}(H,s,b)$, and the functions satisfy the condition

$$f'(z)\left(\frac{(s-b)z}{f(sz)-f(bz)}\right)^{\lambda} \prec H(z,t), \quad z \in E.$$

A sharp bound for the functional $|a_3-\mu a_2^2|$ for univalent functions $f \in E$ of the form (1.1) with real μ $(0 \le \mu \le 1)$ was obtained by the Fekete–Szegö [3] approach (see also [15]). Since then, the problem of finding the sharp bounds for this functional of any compact family of functions $f \in \Gamma$ with any complex μ is generally known as the classical Fekete–Szegö inequality. Authors like [9], [11], and [14] have studies several subclasses of functions making use of the Fekete–Szegö problem. We are motivated by the earlier works [2], [4] and [10].

In this work, we focus on coefficient estimates including the Fekete–Szegö inequality of the classes $\mathcal{G}_q^{\lambda}(H,s,b)$, $\mathcal{G}^{\lambda}(H,s,b)$ and the one involving majorization. Our results are new in this direction and they also give birth to many corollaries.

2. Main results

Theorem 2.1. *Let $f \in A$ of the form* (1.1) *belong to $\mathcal{G}_q^{\lambda}(H,s,b)$. Then*

$$|a_2| \le \frac{2t}{|2-\lambda(s+b)|} \tag{2.1}$$

and, for some $\mu \in \mathbb{C}$,

$$|a_3-\mu a_2^2| \le \frac{2t}{|3-\lambda(s^2+sb+b^2)|}\max\left\{1, \left|2t\left(\frac{c\left[3-\lambda(s^2+sb+b^2)\right]}{(2-\lambda(s+b))^2} - \frac{\lambda\left[1+\frac{2-(s+b)}{2-\lambda(s+b)}\right](s+b)}{2(2-\lambda(s+b))}\right)-\frac{4t^2-1}{2t}\right|\right\}. \tag{2.2}$$

Proof. Let $f \in \mathcal{G}_q^{\lambda}(H,s,b)$, then from (1.13) we have

$$f'(z)\left(\frac{(s-b)z}{f(sz)-f(bz)}\right)^{\lambda}-1 = \varphi(z)[H(z,t)-1], \quad z \in E, \tag{2.3}$$

where the function $H(z,t)$ is given by (1.11), the analytic function $\varphi \in E$ is of the form

$$\varphi(z) = d_0 + d_1 z + d_2 z^2 + \dots, \tag{2.4}$$

and Schwartz function w is determined by (1.7). Therefore, the right hand side of (2.3) gives

$$\begin{aligned}
&\varphi(z)[H(w(z),t) - 1] \\
&\quad = \left(d_0 + d_1 z + d_2 z^2 + \dots\right) \left[U_1(t) w_1 z + \left(U_2(t) w_1^2 + U_1(t) w_2\right) z^2 + \dots\right] \\
&\quad = U_1(t) w_1 d_0 z + \left[\left(U_2(t) w_1^2 + U_1(t) w_2\right) d_0 + U_1(t) w_1 d_1\right] z^2 + \dots .
\end{aligned}$$

Using the series expansion of $f(z)$ from (1.1) and the expansion (2.3), we get

$$[2 - \lambda(s+b)] a_2 = U_1(t) w_1 d_0, \tag{2.5}$$

$$\begin{aligned}
&\left[3 - \lambda(s^2 + sb + b^2)\right] a_3 - \lambda \left[2 - \frac{1+\lambda}{2}(s+b)\right](s+b) a_2^2 \\
&\quad = \left(U_2(t) w_1^2 + U_1(t) w_2\right) d_0 + U_1(t) w_1 d_1.
\end{aligned} \tag{2.6}$$

Now, (2.5) gives

$$a_2 = \frac{U_1(t) w_1 d_0}{2 - \lambda(s+b)}, \tag{2.7}$$

which in view of (2.6) yields

$$\begin{aligned}
\left[3 - \lambda\left(s^2 + sb + b^2\right)\right] a_3 &= \left(U_2(t) w_1^2 + U_1(t) w_2\right) d_0 + U_1(t) w_1 d_1 \\
&\quad + \frac{\lambda[4 - (1+\lambda)(s+b)](s+b) U_1^2(t) w_1^2 d_0^2}{2[2 - \lambda(s+b)]^2}.
\end{aligned} \tag{2.8}$$

Therefore,

$$\begin{aligned}
a_3 &= \frac{U_1(t)}{3 - \lambda\left(s^2 + sb + b^2\right)} \left\{ w_1 d_1 + d_0 \left[w_2 \right.\right. \\
&\quad \left.\left. + \left(\frac{d_0 \lambda \left(1 + \frac{2-(s+b)}{2-\lambda(s+b)}\right)(s+b) U_1(t)}{2(2 - \lambda(s+b))} + \frac{U_2(t)}{U_1(t)} \right) w_1^2 \right] \right\}.
\end{aligned} \tag{2.9}$$

For some $\mu \in \mathbb{C}$, from (2.7) and (2.9) we obtain that

$$\begin{aligned}
a_3 - \mu a_2^2 = \frac{U_1(t)}{3 - \lambda\left(s^2 + sb + b^2\right)} &\left[d_1 w_1 + d_0 \left(w_2 + \frac{U_2(t)}{U_1(t)} w_1^2 \right) \right. \\
&- U_1(t) w_1^2 d_0^2 \left(\frac{c\left(3 - \lambda\left(s^2 + sb + b^2\right)\right)}{(2 - \lambda(s+b))^2} \right. \\
&\left.\left. - \frac{\lambda\left[1 + \frac{2-(s+b)}{2-\lambda(s+b)}\right](s+b)}{2(2 - \lambda(s+b))} \right) \right].
\end{aligned} \tag{2.10}$$

Since φ, given by (2.4), is analytic and bounded in E, using [8, p. 172], we have

$$d_0 \le 1 \quad \text{and} \quad d_1 = (1-d_0^2)y, \quad |y| \le 1. \tag{2.11}$$

Putting the value of d_1 from (2.11) into (2.10), we get

$$\begin{aligned} a_3 - \mu a_2^2 = {} & \frac{U_1(t)}{3-\lambda\left(s^2+sb+b^2\right)} \left\{ y w_1 + \left(w_2 + \frac{U_2(t)}{U_1(t)} w_1^2 \right) d_0 \right. \\ & - \left[U_1(t) \left(\frac{c\left(3-\lambda\left(s^2+sb+b^2\right)\right)}{(2-\lambda(s+b))^2} \right. \right. \\ & \left. \left. \left. - \frac{\lambda\left[1+\frac{2-(s+b)}{2-\lambda(s+b)}\right](s+b)}{2(2-\lambda(s+b))} \right) w_1^2 + y w_1 \right] d_0^2 \right\}. \end{aligned} \tag{2.12}$$

If $d_0 = 0$ in (2.12), then we obtain

$$\left|a_3 - a_2^2\right| \le \frac{2t}{\left|3-\lambda\left(s^2+sb+b^2\right)\right|}. \tag{2.13}$$

But if $d_0 \ne 0$, then let

$$\begin{aligned} F(d_0) := y w_1 + \left(w_2 + \frac{U_2(t)}{U_1(t)} w_1^2 \right) d_0 - \left[U_1(t) \left(\frac{c\left(3-\lambda\left(s^2+sb+b^2\right)\right)}{(2-\lambda(s+b))^2} \right. \right. \\ \left. \left. - \frac{\lambda\left[1+\frac{2-(s+b)}{2-\lambda(s+b)}\right](s+b)}{2(2-\lambda(s+b))} \right) w_1^2 + y w_1 \right] d_0^2, \end{aligned}$$

which is a polynomial in d_0 and hence analytic in $|d_0| \le 1$. Maximum of $|F(d_0)|$ is attained at $d_0 = e^{i\theta}$ $(0 \le \theta < 2\pi)$. We find that

$$\max_{0\le\theta<2\pi} \left| F\left(e^{i\theta}\right) \right| = |F(1)| \tag{2.14}$$

and

$$\begin{aligned} \left|a_3 - \mu a_2^2\right| \le {} & \frac{2t}{3-\lambda\left(s^2+sb+b^2\right)} \left| w_2 \right. \\ & - \left[2t \left(\frac{c\left(3-\lambda\left(s^2+sb+b^2\right)\right)}{(2-\lambda(s+b))^2} \right. \right. \\ & \left. \left. \left. - \frac{\lambda\left[1+\frac{2-(s+b)}{2-\lambda(s+b)}\right](s+b)}{2(2-\lambda(s+b))} \right) - \frac{4t^2-1}{2t} \right] w_1^2 \right|, \end{aligned} \tag{2.15}$$

which, using [5], shows that (2.2) holds. The inequality (2.2) together with (2.13) establishes the result. □

Theorem 2.2. *Let $f \in A$ of the form* (1.1) *belong to $\mathcal{G}^{\lambda}(H,s,b)$. Then the inequalities* (2.1) *and* (2.2) *hold.*

Proof. Similar as in the proof of Theorem 2.1, if $\varphi(z) = 1$, then (2.4) evidently implies that $d_0 = 1$ and $d_n = 0$. Hence, in view of (2.7), (2.10), and [5], we obtain the desired result. □

Theorem 2.3. *Let $s \neq b$, $s, b \in \mathbb{C}$. If a function $f \in A$ of the form* (1.1) *satisfies*

$$\left[f'(z)\left(\frac{(s-b)z}{f(sz)-f(bz)}\right)^{\lambda} - 1\right] \ll [H(z,t)-1], \quad z \in E,$$

then the inequalities (2.1) *and* (2.2) *hold.*

Proof. Following the proof of Theorem 2.1, letting $w(z) = z$ in (1.7), so that $w_1 = 1$ and $w_n = 0$, $n = 2, 3, 4, \ldots$, in view of (2.7) and (2.10) we get (2.1) and

$$\begin{aligned} a_3 - \mu a_2^2 \leq \frac{U_1(t)}{3-\lambda(s^2+sb+b^2)} \Bigg[d_1 + \frac{U_2(t)}{U_1(t)} d_0 \\ -U_1(t) \left(\frac{c\left(3-\lambda(s^2+sb+b^2)\right)}{(2-\lambda(s+b))^2} \right. \\ \left. - \frac{\lambda \left[1 + \frac{2-(s+b)}{2-\lambda(s+b)}\right](s+b)}{2[2-\lambda(s+b)]} \right) d_0^2 \Bigg]. \end{aligned} \tag{2.16}$$

Putting the value of d_1 from (2.11) into (2.16), we get

$$\begin{aligned} a_3 - \mu a_2^2 \leq \frac{U_1(t)}{3-\lambda(s^2+sb+b^2)} \Bigg[y + \frac{U_2(t)}{U_1(t)} d_0 \\ - \left(U_1(t) \left(\frac{c\left(3-\lambda(s^2+sb+b^2)\right)}{(2-\lambda(s+b))^2} \right. \right. \\ \left. \left. - \frac{\lambda \left[1 + \frac{2-(s+b)}{2-\lambda(s+b)}\right](s+b)}{2[2-\lambda(s+b)]} \right) + y \right) d_0^2 \Bigg]. \end{aligned} \tag{2.17}$$

If $d_0 = 0$ in (2.17), then we get

$$\left|a_3 - \mu a_2^2\right| \leq \frac{2t}{|3-\lambda(s^2+sb+b^2)|}.$$

If $d \neq 0$, then let

$$\begin{aligned} G(d_0) := y + \frac{U_2(t)}{U_1(t)} d_0 - \Bigg[U_1(t) \left(\frac{c\left(3-\lambda(s^2+sb+b^2)\right)}{(2-\lambda(s+b))^2} \right. \\ \left. - \frac{\lambda \left[1 + \frac{2-(s+b)}{2-\lambda(s+b)}\right](s+b)}{2[2-\lambda(s+b)]} \right) + y \Bigg] d_0^2, \end{aligned}$$

which being a polynomial in d_0 is analytic in $|d_0| \leq 1$, and the maximum of $|G(d_0)|$ is attained at $d_0 = e^{i\theta}$ $(0 \leq \theta < 2\pi)$. We thus find that

$$\max_{0\leq\theta<2\pi} \left|G\left(e^{i\theta}\right)\right| = |G(1)|,$$

and, consequently, (2.2) holds. □

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