

On commutativity of semiprime Banach algebras

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ABSTRACT. In the present paper, we discuss the commutativity of semi-prime rings. Further, using this result, we establish that if \mathfrak{A} is a semiprime Banach algebra, and \mathcal{H}_1 and \mathcal{H}_2 are nonvoid open subsets of \mathfrak{A} which admit a continuous derivation $d : \mathfrak{A} \rightarrow \mathfrak{A}$ such that $d(x^m) \circ d(y^n) \pm x^m \circ y^n = 0$ for all $x \in \mathcal{H}_1$ and $y \in \mathcal{H}_2$, where m, n are no longer fixed but they depend on the pair of elements x and y , then \mathfrak{A} is commutative.

1. Introduction

Throughout this paper, \mathfrak{A} is a Banach algebra with identity, $Z(\mathfrak{A})$ is the center of \mathfrak{A} and \mathfrak{N} is a closed linear subspace of \mathfrak{A} . As usual, $[x, y] = xy - yx$ and $x \circ y = xy + yx$. A linear mapping $d : \mathfrak{A} \rightarrow \mathfrak{A}$ is said to be a derivation on \mathfrak{A} if $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in \mathfrak{A}$. A Banach algebra is a complex normed algebra \mathfrak{A} whose underlying vector space is a Banach space. In fact, any Banach algebra \mathfrak{A} without a unity can be embedded into a unital Banach algebra $\mathfrak{A}_I = \mathfrak{A} \oplus \mathbb{C}$ as an ideal of codimension one. In particular, we may identify \mathfrak{A} with the ideal $\{(x, 0) : x \in \mathfrak{A}\}$ in \mathfrak{A}_I via the isometric isomorphism $x \rightarrow (x, 0)$. Recall that an algebra \mathfrak{A} is said to be prime if, for any $a, b \in \mathfrak{A}$, $a\mathfrak{A}b = (0)$ implies $a = 0$ or $b = 0$, and \mathfrak{A} is semiprime if for any $a \in \mathfrak{A}$, $a\mathfrak{A}a = (0)$ implies $a = 0$.

In mid 1940s, after the development of the general structure theory for rings, a great deal of work has been done to show that under certain types of hypotheses, rings are commutative or almost commutative. A classical result of ring theory established by Jacobson which generalized the theorem of Wedderburn states that every finite division ring is commutative, and any Boolean ring is a commutative ring. This theorem is stated as follows: Any

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ring in which $x^n(x) = x$, $n(x) > 1$ is a positive integer, is necessarily commutative. Inspired by this result, several techniques have been developed to investigate conditions under which a ring becomes commutative, for instance, generalizing Herstein's conditions, using restrictions on polynomials, introducing derivations and generalized derivations on rings, looking special properties for rings, etc. For more details and references, see the review article [18]. One can also achieve this goal for semisimple Banach algebras involving derivations.

Singer and Wermer [23] proved that any bounded linear derivation on a commutative Banach algebra maps the algebra into its radical. On the other hand, Johnson and Sinclair [13] proved that any linear derivation on a semisimple Banach algebra is continuous. Combining these two results, one can obtain that there are no nonzero linear derivations on a commutative semisimple Banach algebra. Firstly, a non-commutative extension of Singer–Wermer theorem has been proved by Yood [24], by showing that if for all pairs $x, y \in \mathfrak{A}$, where \mathfrak{A} is a non-commutative Banach algebra, and the element $[d(x), y] \in \text{rad}(\mathfrak{A})$, then d maps \mathfrak{A} into $\text{rad}(\mathfrak{A})$. Later on, Brešar and Vukman [5], generalized Yood's theorem by considering $[d(x), x] \in \text{rad}(\mathfrak{A})$ (see [8, 9, 19, 20, 21, 22] and the references therein), $\text{rad}(\mathfrak{A})$ is the Jacobson radical of \mathfrak{A} .

A mapping $f : R \rightarrow R$ is said to be strong commutativity preserving (*scp*) on R if $[f(x), f(y)] = [x, y]$ for all $x, y \in R$. Over the past decades a lot of work related to commutativity preserving mappings has been done by various authors (see [3, 10] and the references therein). Inspired by these works, Bell and Daif [4] obtained commutativity of prime and semiprime rings admitting derivations and endomorphisms which are *scp* on R or on certain subsets on R . In fact, it was shown that if a semiprime ring R admits a derivation d which is *scp* on a nonzero ideal I of R , i.e., $[d(x), d(y)] = [x, y]$ for all $x, y \in I$, then I is central. In 2002, Ashraf and Rehman [1] obtained the same conclusion if the commutator is replaced by an anticommutator which stated that if a prime ring R admits a derivation d such that $d(x) \circ d(y) = x \circ y$ for all $x, y \in R$, then R is commutative. In [12], Herstein proved that a ring R is commutative if it has no nonzero nilpotent ideal and there is a fixed integer $n > 1$ such that $(xy)^n = x^n y^n$ for all $x, y \in R$. As an application, Yood [25] proved these results in the case of a Banach algebra. Motivated by the above results, in the present paper we prove that if \mathfrak{A} is a semiprime Banach algebra, $\mathcal{H}_1, \mathcal{H}_2$ are nonvoid open subset of \mathfrak{A} , and \mathfrak{A} admits a continuous derivation $d : \mathfrak{A} \rightarrow \mathfrak{A}$ such that $d(x^m) \circ d(y^m) \pm (x^m \circ y^m) = 0$ for all $x \in \mathcal{H}_1$ and $y \in \mathcal{H}_2$, where m, n are no longer fixed but they depend on the pair of elements x and y , then \mathfrak{A} is commutative.

2. The results on semiprime rings

In all that follows, R is a semiprime ring and U is the maximal right ring of quotients. The center of U , denoted by C , is called the extended centroid of R . For an explanation of the maximal right ring of quotients, we refer the reader to [2]. We shall use the fact that any derivation of a semiprime ring R can be uniquely extended to a derivation of its Utumi quotient ring U (maximal right ring of quotients), and so any derivation of R can be defined on the whole U [2, Proposition 2.5.1]. Moreover, if R is semiprime, then so is its Utumi quotient ring. The extended centroid C of a semiprime ring R coincides with the center of its Utumi quotient ring [7, p. 38]. Also, if B is the set of all idempotents in C , one may assume that R is a B -algebra which is orthogonally complete. For any maximal ideal P of B , PR forms a minimal prime ideal of R , which is invariant under any nonzero derivation of R [7, p. 42]. We use the theory of differential identities, and the fact that any semiprime ring R and its maximal right ring of quotients satisfy the same differential identities (for the explanation of differential identities we refer the reader to [2, 7, 14, 16]).

We begin with the following lemma.

Lemma 2.1. *Let R be a prime ring with characteristic different from 2, let U be the maximal right ring of quotients, and let I be a nonzero ideal of R . Assume that d is an inner derivation of U , in the sense that there exists a noncentral element $b \in U$ such that $d(x) = [b, x]$ for all $x \in R$. If I satisfies*

$$[b, x^m][b, y^n] + [b, y^n][b, x^m] \pm (x^m y^n + y^n x^m) = 0, \quad (2.1)$$

where m, n are fixed positive integers, then R is commutative.

Proof. Assume that R is not commutative, otherwise we have nothing to prove. By given hypothesis and Theorem 2 [6], U satisfies (2.1). Moreover, since U remains prime by the primeness of R , replace U by R and suppose that $b \in R$. By [11], since R is a centrally closed prime C -algebra, $RC = R$. Using Martindale's theorem [17], we see that RC (and so R) is a primitive ring. As R is a primitive, there exists a vector space \mathcal{V} and a division ring \mathcal{D} such that R is a dense ring of \mathcal{D} -linear transformations over \mathcal{V} . Suppose that $\dim_{\mathcal{D}} \mathcal{V} \geq 2$, otherwise we are done.

Our aim is to show that for any $v \in \mathcal{V}$, v and bv are linearly \mathcal{D} -dependent. If $v = 0$, then $\{v, bv\}$ is \mathcal{D} -dependent. So we may assume that $bv \neq 0$. Suppose that v and bv are linearly \mathcal{D} -independent for some $v \in \mathcal{V}$. For this we consider the following two cases.

If $b^2v \notin \text{Span}_{\mathcal{D}}\{v, bv\}$, then the set $\{v, bv, b^2v\}$ is linearly \mathcal{D} -independent. By the density of R , there exist $x, y \in R$ such that

$$\begin{aligned} xv = v, \quad xbv = 0, \quad xb^2v = v, \\ yv = 0, \quad ybv = bv, \quad yb^2v = b^2v. \end{aligned}$$

We can easily see that

$$0 = ([b, x^m][b, y^n] + [b, y^n][b, x^m] \pm (x^m y^n + y^n x^m))v = v \neq 0,$$

a contradiction.

If $b^2v \in \text{Span}_{\mathcal{D}}\{v, bv\}$, then $b^2v = v\alpha + bv\beta$ for some $\alpha, \beta \in \mathcal{D}$. In view of the density of R , there exist $x, y \in R$ such that

$$\begin{aligned} xv = v, \quad xbv = 0, \\ yv = 0, \quad ybv = bv. \end{aligned}$$

Hence we get

$$0 = ([b, x^m][b, y^n] + [b, y^n][b, x^m] \pm (x^m y^n + y^n x^m))v = 2v\alpha \neq 0,$$

a contradiction. So v and bv are linearly \mathcal{D} -dependent for all $v \in \mathcal{V}$, and a standard argument shows that $b \in C$, a contradiction. This finishes the proof. \square

Proposition 2.1. *Let R be a prime ring with characteristic different from 2, let I be a nonzero ideal of R , and let m, n be two fixed positive integers. If R admits a derivation d such that*

$$d(x^m)d(y^n) + d(y^n)d(x^m) \pm (x^m y^n + y^n x^m) = 0 \quad (2.2)$$

for all $x, y \in I$, then R is commutative.

Proof. If $d = 0$, then $x^m y^n + y^n x^m = 0$ for all $x, y \in I$. By Chuang [6, Theorem 1], this generalized polynomial identity is also satisfied by U and hence R as well. Note that this is a polynomial identity and thus there exists a field \mathbb{F} such that $R \subseteq M_k(\mathbb{F})$, the ring of $k \times k$ matrices over a field \mathbb{F} , where $k > 1$. Moreover, R and $M_k(\mathbb{F})$ satisfy the same polynomial identity [15, Lemma 1], i.e., $x^m y^n + y^n x^m = 0$ for all $x, y \in M_k(\mathbb{F})$. But by choosing $x = e_{11}$, $y = e_{11} + e_{22}$, we get $0 = x^m y^n + y^n x^m = 2e_{11} \neq 0$, a contradiction.

Now we assume d is a nonzero derivation satisfying (2.2). This condition is a differential identity satisfied by I . In the light of Kharchenko's theory [14], either $d(x) = [b, x]$ is the inner derivation induced by an element $b \in U$ or I satisfies the generalized polynomial identity

$$\begin{aligned} \sum_{i=0}^{m-1} x^i z x^{m-1-i} \sum_{j=0}^{n-1} y^j w y^{n-1-j} + \sum_{j=0}^{n-1} y^j w y^{n-1-j} \sum_{i=0}^{m-1} x^i z x^{m-1-i} \\ \pm (x^m y^n + y^n x^m) = 0 \end{aligned}$$

for all $x, y \in I$. In the latter case, set $w = 0$ to obtain the identity $x^m y^n + y^n x^m = 0$ for all $x, y \in I$. Then, as above, we get a contradiction.

Next if $d(x) = [b, x]$, then (2.1) holds for any $x, y \in I$. In view of Lemma 2.1, we get the required result. \square

We immediately get the following corollary from the above proposition.

Corollary 2.1. *Let R be a prime ring with characteristic different from 2, and let m, n be two fixed positive integers. If R admits a derivation d such that (2.2) holds for all $x, y \in R$, then R is commutative.*

Now we prove our result for semiprime rings.

Theorem 2.1. *Let R be a semiprime ring with characteristic different from 2, and let m, n be two fixed positive integers. If R admits a nonzero derivation d such that*

$$d(x^m) \circ d(y^n) \pm (x^m \circ y^n) = 0 \quad (2.3)$$

for all $x, y \in R$, then R is commutative.

Proof. By [7, p. 38], $Z(U) = C$, the extended centroid of R , and by [2, Proposition 2.5.1], d can be uniquely extended on U , the maximal right ring of quotients of R . In view of Lee [16], R and U satisfy the same differential identities, therefore (2.3) is satisfied for all $x, y \in U$. Let B be the complete Boolean algebra of idempotents in C , and let M be any maximal ideal of B . Then U is a B -algebra which is orthogonally complete (see Chuang [7, p. 42]), and by [2, Proposition 2.5.1]), MU is a prime ideal of U , which is d -invariant. Define $\bar{U} = U/MU$ and denote by \bar{d} the derivation induced by d on \bar{U} , i.e., $\bar{d}(\bar{u}) = \overline{d(u)}$ for all $u \in U$. Then \bar{d} in \bar{U} has the same property as d on U . It is obvious that \bar{U} is prime. Therefore, by Corollary 2.1, \bar{U} is commutative. This implies that, for any maximal ideal M of B , $[U, U] \subseteq MU$ and hence $[U, U] \subseteq \bigcap_M MU = 0$, where MU runs over all prime ideals of U . In particular, $[R, R] = 0$ and so R is commutative. This completes the proof. \square

3. Application on semiprime Banach algebras

Now we apply our purely ring-theoretic result on semiprime Banach algebras to obtain the commutativity of Banach algebras. Let us introduce some well known and elementary definitions for the sake of completeness. Here \mathfrak{A} will denote a real or complex Banach algebra with center $Z(\mathfrak{A})$ and \mathfrak{N} a closed linear subspace of \mathfrak{A} . The Jacobson radical of an algebra is the intersection of all primitive ideals of \mathfrak{A} , and is denoted by $rad(\mathfrak{A})$. If the Jacobson radical reduces to the zero element, then \mathfrak{A} is called semisimple.

We shall use the following readily established fact. Let $p(t) = \sum_{r=0}^n b_r t^r$ be a polynomial in the real variable t with coefficients in \mathfrak{A} . If $p(t) \in \mathfrak{N}$ for all t is an infinite subset of the reals, then every b_r lies in \mathfrak{N} .

Theorem 3.1. *Let \mathfrak{A} be a semiprime Banach algebra, and let \mathcal{H}_1 and \mathcal{H}_2 be nonvoid open subsets of \mathfrak{A} . If \mathfrak{A} admits a continuous derivation $d : \mathfrak{A} \rightarrow \mathfrak{A}$ such that (2.3) holds for all $x \in \mathcal{H}_1$ and $y \in \mathcal{H}_2$, where m, n are no longer fixed but they depend on the pair of elements x and y , then \mathfrak{A} is commutative.*

Proof. Fix $x \in \mathcal{H}_1$ and define

$$\mathcal{U}_{m,n} = \{y \in \mathfrak{A} \mid d(x^m) \circ d(y^n) + (x^m \circ y^n) \neq 0, \quad d(x^m) \circ d(y^n) + (x^m \circ y^n) \neq 0\}.$$

We claim that each $\mathcal{U}_{m,n}$ is open in \mathfrak{A} , that is, we have to show that $\mathcal{U}_{m,n}^c$, the complement of $\mathcal{U}_{m,n}$ is closed. For this, we consider a sequence $(w_k) \in \mathcal{U}_{m,n}^c$ such that $w_k \rightarrow w$ as $k \rightarrow \infty$, and we need to show that $w \in \mathcal{U}_{m,n}$. Since $w_k \in \mathcal{U}_{m,n}^c$, we have

$$d(x^m) \circ d(w_k^n) + (x^m \circ w_k^n) = 0.$$

Taking the limit in k and using the continuity of d , we obtain

$$d(x^m) \circ d(\lim_{k \rightarrow \infty} w_k^n) + x^m \circ \lim_{k \rightarrow \infty} w_k^n = 0.$$

Since $w_k \rightarrow w$ as $k \rightarrow \infty$, we conclude that

$$d(x^m) \circ d(w^n) + (x^m \circ w^n) = 0.$$

Similarly, one can see that

$$d(x^m) \circ d(w^n) - (x^m \circ w^n) = 0.$$

Hence $w \in \mathcal{U}_{m,n}^c$, i.e., each $\mathcal{U}_{m,n}$ is open. By the Baire category theorem, if every $\mathcal{U}_{m,n}$ is dense, then their intersection is also dense, which contradicts the existence of \mathcal{H}_1 and \mathcal{H}_2 . Thus there are positive integers r, s such that $\mathcal{U}_{r,s}$ is not dense in \mathfrak{A} . Therefore, there exists a nonvoid open subset $\mathcal{H}_3 \in \mathcal{U}_{r,s}^c$ such that, for every $y \in \mathcal{H}_3$, either

$$d(x^r) \circ d(y^s) + x^r \circ y^s = 0,$$

or

$$d(x^r) \circ d(y^s) - x^r \circ y^s = 0.$$

Let $z_0 \in \mathcal{H}_3$ and $w \in \mathfrak{A}$, then $z_0 + tw \in \mathcal{H}_3$ for all sufficiently small real t . Therefore, for each t , either $d(x^r) \circ d((z_0 + tw)^s) + x^r \circ (z_0 + tw)^s = 0$ or $d(x^r) \circ d((z_0 + tw)^s) - x^r \circ (z_0 + tw)^s = 0$. Then at least one of the above must hold for infinitely many t . Suppose that

$$d(x^r) \circ d((z_0 + tw)^s) - x^r \circ (z_0 + tw)^s = 0 \tag{3.1}$$

holds for these t . We have

$$(z_0 + tw)^s = \mathcal{G}_{s,0}(z_0, w) + \mathcal{G}_{s-1,1}(z_0, w)t + \mathcal{G}_{s-2,2}(z_0, w)t^2 + \cdots \\ + \mathcal{G}_{1,s-1}(z_0, w)t^{s-1} + \mathcal{G}_{0,s}(z_0, w)t^s,$$

where $\mathcal{G}_{i,j}(z_0, w)$ denotes the sum of all terms in which z_0 appears exactly i times and w appears exactly j times such that $i + j = s$ (i and j are positive integers). Now

$$d(x^r) \circ d((z_0 + tw)^s) - x^r \circ (z_0 + tw)^s$$

can be written as

$$d(x^r) \circ d(\mathcal{G}_{s,0}(z_0, w)) - x^r \circ \mathcal{G}_{s,0}(z_0, w) \\ + (d(x^r) \circ d(\mathcal{G}_{s-1,1}(z_0, w)) - x^r \circ \mathcal{G}_{s-1,1}(z_0, w))t \\ + (d(x^r) \circ d(\mathcal{G}_{s-2,2}(z_0, w)) - x^r \circ \mathcal{G}_{s-2,2}(z_0, w))t^2 \\ \vdots \\ + (d(x^r) \circ d(\mathcal{G}_{1,s-1}(z_0, w)) - x^r \circ \mathcal{G}_{1,s-1}(z_0, w))t^{s-1} \\ + (d(x^r) \circ d(\mathcal{G}_{0,s}(z_0, w)) - x^r \circ \mathcal{G}_{0,s}(z_0, w))t^s.$$

The above expression is a polynomial in t and the coefficient of t^s in this polynomial is $d(x^r) \circ d(w^s) - x^r \circ w^s$. Therefore, we obtain

$$d(x^r) \circ d(w^s) - x^r \circ w^s = 0.$$

Similarly, if (3.1) holds for these t , then in the similar fashion we conclude that

$$d(x^r) \circ d(w^s) + x^r \circ w^s = 0.$$

Thus, for given $x \in \mathcal{H}_1$, there are positive integers r, s such that for each $w \in \mathfrak{A}$ either $d(x^r) \circ d(w^s) - x^r \circ w^s = 0$ or $d(x^r) \circ d(w^s) + x^r \circ w^s = 0$. Let

$$\mathcal{F}_1 = \{w \in \mathfrak{A} \mid d(x^r) \circ d(w^s) - x^r \circ w^s = 0\}$$

and

$$\mathcal{F}_2 = \{w \in \mathfrak{A} \mid d(x^r) \circ d(w^s) + x^r \circ w^s = 0\}.$$

Then \mathfrak{A} is the union of \mathcal{F}_1 and \mathcal{F}_2 , and each \mathcal{F}_k , $k = 1, 2$, is closed (as we have shown earlier). Thus, by the Baire category theorem, at least one of \mathcal{F}_1 and \mathcal{F}_2 must contain a nonvoid open subset of \mathfrak{A} .

Suppose \mathcal{F}_1 contains a nonvoid open subset \mathcal{H}_4 of \mathfrak{A} . Let $u_0 \in \mathcal{H}_4$ and $v \in \mathfrak{A}$. Then $u_0 + tv \in \mathcal{H}_4$ for sufficiently small t . For these t , we have

$$d(x^r) \circ d((u_0 + tv)^s) - x^r \circ (u_0 + tv)^s = 0.$$

This can be written as a polynomial in t (as earlier) in which the coefficient of t^s is $d(x^r) \circ d(v^s) - x^r \circ v^s = 0$. Therefore, we have $d(x^r) \circ d(v^s) - x^r \circ v^s = 0$ for all $v \in \mathfrak{A}$. Likewise, if \mathcal{F}_2 contains a nonvoid open subset then $d(x^r) \circ d(v^s) + x^r \circ v^s = 0$ for all $v \in \mathfrak{A}$.

Consequently, given $x \in \mathcal{H}_1$, there are positive integers r and s so that either $d(x^r) \circ d(v^s) - x^r \circ v^s = 0$ or $d(x^r) \circ d(v^s) + x^r \circ v^s = 0$ for all $v \in \mathfrak{A}$. Now, we reverse the roles of \mathcal{H}_1 and \mathcal{H}_2 in the above setting. Proceeding in the same way, we find that either $d(x^r) \circ d(v^s) - x^r \circ v^s = 0$ or $d(x^r) \circ d(v^s) + x^r \circ v^s = 0$ for all $x, v \in \mathfrak{A}$. Then by Theorem 2.1, \mathfrak{A} is commutative. This completes the proof. \square

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