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An entire function sharing fixed points with its linear differential polynomial

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Abstract. We study the uniqueness of entire functions, when they share a linear polynomial, in particular, fixed points, with their linear differential polynomials.

1. Definitions and results

Let f be a nonconstant meromorphic function defined in the open complex plane $\mathbb{C}$, and let $a = a(z)$ be a polynomial. Let us denote by $E(a; f)$ and $\overline{E}(a; f)$ the set of zeros of $f - a$, counted with multiplicities, and the set of all distinct zeros of $f - a$, respectively. If $A \subset \mathbb{C}$, then we denote by $n_A(r, a; f)$ the number of zeros of $f - a$, counted with multiplicities, that lie in $\{z : |z| \leq r\} \cap A$. The corresponding integrated counting function is defined by

$$N_A(r, a; f) = \int_0^r \frac{n_A(t, a; f) - n_A(0, a; f)}{t} dt + n_A(0, a; f) \log r.$$

We also denote by $\overline{N}_A(r, a; f)$ the reduced counting functions of those zeros of $f - a$ that lie in $\{z : |z| \leq r\} \cap A$.

Clearly, if $A = \mathbb{C}$, then $N_A(r, a; f) = N(r, a; f)$ and $\overline{N}_A(r, a; f) = \overline{N}(r, a; f)$. The standard definitions and notation of the value distribution theory are available in [1].

The uniqueness of an entire function sharing a nonzero finite value with its first two derivatives was considered by Jank et al. [2] in 1986. The following is their result.

Theorem A (see [2]). *Let f be a nonconstant entire function and let a be a nonzero finite value. If $\overline{E}(a; f) = \overline{E}(a; f^{(1)}) \subset \overline{E}(a; f^{(2)})$, then $f \equiv f^{(1)}$.*

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Considering $f = e^{\omega z} + \omega - 1$ and $a = \omega$, where ω is a $(k-1)$th imaginary root of unity and $k(\geq 3)$ is an integer, Zhong [10] pointed out that in Theorem A one can not replace the second derivative by any higher order derivative. Under this context, Zhong [10] proved the following theorem.

Theorem B (see [10]). *Let f be a nonconstant entire function and let a be a nonzero finite number. If $E(a; f) = E(a; f^{(1)})$ and $\overline{E}(a; f) \subset \overline{E}(a; f^{(n)}) \cap \overline{E}(a; f^{(n+1)})$ for $n \geq 1$, then $f \equiv f^{(n)}$.*

Considering a shared linear polynomial, Lahiri and Ghosh [3] extended Theorem A in the following manner.

Theorem C (see [3]). *Let f be a nonconstant entire function and let $a(z) = \alpha z + \beta$, where $\alpha(\neq 0), \beta$ are constants. If $E(a; f) \subset E(a; f^{(1)}) \subset E(a; f^{(2)})$, then either $f = \lambda e^z$ or $f = \alpha z + \beta + (\alpha z + \beta - 2\alpha)\exp(\frac{\alpha z+\beta-2\alpha}{\alpha})$, where $\lambda(\neq 0)$ is a constant.*

In 1999, Li [7] considered linear differential polynomials and proved the following result.

Theorem D (see [7]). *Let f be a nonconstant entire function and $L = a_1 f^{(1)} + a_2 f^{(2)} + \cdots + a_n f^{(n)}$, where $a_1, a_2, \ldots, a_n(\neq 0)$ are constants and $a(\neq 0)$ is a finite number. If $\overline{E}(a; f) = \overline{E}(a; f^{(1)}) \subset \overline{E}(a; L) \cap \overline{E}(a; L^{(1)})$, then $f \equiv f^{(1)} \equiv L$.*

In this paper, we consider the uniqueness of an entire function that shares a linear polynomial with linear differential polynomials generated by it. For two subsets A and B of $\mathbb{C}$, we denote by $A\Delta B$ the set $(A-B)\cup(B-A)$, which is called the symmetric difference of the sets A and B.

We now state the main result of the paper.

Theorem 1.1. *Let f be a nonconstant entire function and $L = a_2 f^{(2)} + a_3 f^{(3)} + \cdots + a_n f^{(n)}$, where $a_2, a_3, \ldots, a_n(\neq 0)$ are constants and $n(\geq 2)$ is a positive integer. Also, let $a(z) = \alpha z + \beta$, where $\alpha(\neq 0)$, β are constants. Suppose that $A = \overline{E}(a; f)\Delta\overline{E}(a; f^{(1)})$ and $B = \overline{E}(a; f^{(1)})\backslash\{\overline{E}(a; L)\cap\overline{E}(a; L^{(1)})\}$. If the conditions*

(i) $N_A(r, a; f) + N_A(r, a; f^{(1)}) = O\{\log T(r, f)\}$,

(ii) $N_B(r, a; f^{(1)}) = S(r, f)$,

(iii) *each common zero of $f - a$ and $f^{(1)} - a$ has the same multiplicity,*

are satisfied, then $f = L = \lambda e^z$, where $\lambda(\neq 0)$ is a constant.

Putting $A = B = \emptyset$, we obtain the following corollary which improves Theorem B for $n \geq 2$.

Corollary 1.1. *Let f be a nonconstant entire function and $L = a_2 f^{(2)} + a_3 f^{(3)} + \cdots + a_n f^{(n)}$, where $a_2, a_3, \ldots, a_n(\neq 0)$ are constants and $n(\geq 2)$ is an integer. Also let $a(z) = \alpha z + \beta$, where $\alpha(\neq 0), \beta$ are constants. Suppose* *that* $E(a; f) = E(a; f^{(1)})$ *and* $\overline{E}(a; f^{(1)}) \subset \{\overline{E}(a; L) \cap \overline{E}(a; L^{(1)})\}$. *Then* $f = L = \lambda e^z$, *where* $\lambda(\neq 0)$ *is a constant.*

The following examples show that the hypotheses (i) and (ii) of Theorem 1.1 are essential.

Example 1.1. Let $f(z) = e^z$, $L = f^{(2)} + f^{(3)}$ and $a(z) = z$. Then clearly $N_A(r, a; f) + N_A(r, a; f^{(1)}) = O\{\log T(r, f)\}$ and $N_B(r, a; f^{(1)}) = T(r, f) + O(1) \neq S(r, f)$. Also we note that the hypothesis (iii) of Theorem 1.1 holds, but $f \not\equiv L$.

Example 1.2. Let $f(z) = e^z + z^2$, $L = f^{(3)} + f^{(4)}$ and $a(z) = 2z$. Then clearly $N_A(r, a; f) + N_A(r, a; f^{(1)}) = T(r, e^z) + O(1) \neq O\{\log T(r, f)\}$ and $N_B(r, a; f^{(1)}) = S(r, f)$. Since $E(a; f^{(1)}) = \emptyset$, we note that the hypothesis (iii) of Theorem 1.1 holds, but $f \not\equiv L$.

We denote by $N_{(2}(r, a; f)$ the counting function, counted with multiplicities, of the multiple zeros of $f - a$.

A related result concerning the derivatives of an entire function can be found in [4].

2. Lemmas

In this section, we present some lemmas.

Lemma 2.1 (see [9])**.** *Let g be a transcendental entire function and let $\phi(\not\equiv 0)$ be a meromorphic function satisfying $T(r, \phi) = S(r, g)$. Then*

$$T(r, g) \leq C_n\{N(r, 0; g) + \overline{N}(r, 0; g^{(n)} - \phi)\} + S(r, g),$$

where C_n is a constant depending only on $n(\geq 1)$.

Lemma 2.2. *Let f be a transcendental entire function and let $a = a(z)$ be a meromorphic function satisfying $a - a^{(n)} \not\equiv 0$ and $T(r, a) = S(r, f)$. Then*

$$T(r, f) \leq C_n\{N(r, a; f) + \overline{N}(r, a; f^{(n)})\} + S(r, f),$$

where C_n is a constant depending only on $n(\geq 1)$.

Proof. Putting $g = f - a$ and $\phi = a - a^{(n)}$ in Lemma 2.1, we obtain the result. □

Lemma 2.3 (see [5])**.** *Let f be transcendental entire function of finite order and let $a = a(z) = \alpha z + \beta$, where $\alpha(\neq 0), \beta$ are constants. Suppose that $A = \overline{E}(a; f)\Delta\overline{E}(a; f^{(1)})$. If $N_A(r, a; f) + N_A(r, a; f^{(1)}) = O\{\log T(r, f)\}$ and each common zero of $f - a$ and $f^{(1)} - a$ have the same multiplicity, then $m(r, a; f) = m(r, \frac{1}{f-a}) = S(r, f)$.*

To prove the following lemma, we adapt some techniques from [5].

Lemma 2.4. *Let f be a transcendental entire function and $a(z) = \alpha z + \beta(\not\equiv 0)$. Suppose that*

$$L = a_2 f^{(2)} + a_3 f^{(3)} + \cdots + a_n f^{(n)} \ \ \text{and} \ \ h = \frac{(a - a^{(1)})L - a(f^{(1)} - a^{(1)})}{f - a},$$

where $a_2, a_3, \ldots, a_n(\neq 0)$ are constants. Further, suppose that

$$A = \overline{E}(a; f)\backslash\overline{E}(a; f^{(1)}) \ \ \text{and} \ \ B = \overline{E}(a; f^{(1)})\backslash\{\overline{E}(a; L) \cap \overline{E}(a; L^{(1)})\}.$$

If the conditions

(i) $N_A(r, a; f) + N_B(r, a; f^{(1)}) = S(r, f)$,

(ii) *each common zero of $f - a$ and $f^{(1)} - a$ has the same multiplicity,*

(iii) *h is transcendental entire or meromorphic,*

hold, then $m(r, a; f^{(1)}) = m\left(r, \frac{1}{f^{(1)}-a}\right) = S(r, f)$.

Proof. Since $a - a^{(1)} = (f^{(1)} - a^{(1)}) - (f^{(1)} - a)$, we have that if z_0 is a common zero of $f - a$ and $f^{(1)} - a$ with multiplicity $q(\geq 2)$, then z_0 is a zero of $a - a^{(1)}$ with multiplicity $q - 1$. So

$$N_{(2}(r, a; f) \leq 2N(r, 0; a - a^{(1)}) + N_A(r, a; f) = S(r, f).$$

Hence, by the hypothesis, we see that

$$\begin{aligned} N(r, h) &\leq N_A(r, a; f) + N_B\left(r, a; f^{(1)}\right) + N_{(2}(r, a; f) + S(r, f) \\ &= S(r, f). \end{aligned}$$

Since $m(r, h) = S(r, f)$, we have $T(r, h) = S(r, f)$.

Now, by a simple calculation we get

$$f = a + \frac{1}{h}\left\{\left(a - a^{(1)}\right)(L - a) - a\left(f^{(1)} - a\right)\right\}.$$

Differentiating, we obtain

$$\begin{aligned} f^{(1)} = a^{(1)} &+ \left(\frac{1}{h}\right)^{(1)}\left\{\left(a - a^{(1)}\right)(L - a) - a\left(f^{(1)} - a\right)\right\} \\ &+ \left(\frac{1}{h}\right)\left\{a^{(1)}(L - a) + \left(a - a^{(1)}\right)\left(L^{(1)} - a^{(1)}\right) - a^{(1)}\left(f^{(1)} - a\right)\right. \\ &\left. - a\left(f^{(2)} - a^{(1)}\right)\right\}. \end{aligned}$$

This implies

$$\begin{aligned} \frac{1}{f^{(1)} - a} = \frac{\xi}{\zeta} - \frac{1}{\zeta}\left(\frac{a - a^{(1)}}{h}\right)^{(1)} \frac{L - a_2 a^{(1)}}{f^{(1)} - a} - \frac{a - a^{(1)}}{h\zeta}\,\frac{L^{(1)}}{f^{(1)} - a} \\ + \frac{a}{h\zeta}\,\frac{f^{(2)} - a^{(1)}}{f^{(1)} - a}, \end{aligned} \tag{2.1}$$

where

$$\xi = 1 + \left(\frac{a}{h}\right)^{(1)} \quad \text{and} \quad \zeta = a^{(1)} - a - \left(\frac{a(a - a^{(1)})}{h}\right)^{(1)} + \left(\frac{a - a^{(1)}}{h}\right)^{(1)} a_2 a^{(1)}.$$

We now verify that $\xi \not\equiv 0$ and $\zeta \not\equiv 0$. If $\xi \equiv 0$, then $1 + (a/h)^{(1)} \equiv 0$. Integrating, we get $h = a/(c - z)$, where c is a constant. This implies a contradiction as h is transcendental.

If $\zeta \equiv 0$, then

$$a^{(1)} - a - \left(\frac{a(a - a^{(1)})}{h}\right)^{(1)} + \left(\frac{a - a^{(1)}}{h}\right)^{(1)} a_2 a^{(1)} \equiv 0,$$

and so

$$(\alpha - \beta)z - \frac{\alpha z^2}{2} + \alpha_2 = \frac{a(a - \alpha)}{h} - \frac{a_2 \alpha (a - \alpha)}{h},$$

where α_2 is a constant. Therefore,

$$h = \frac{(\alpha z + \beta - \alpha)(\alpha z + \beta - a_2 \alpha)}{-\frac{\alpha z^2}{2} + (\alpha - \beta)z + \alpha_2},$$

which is a contradiction as h is transcendental.

Since clearly $T(r, \xi) + T(r, \zeta) = S(r, f)$, from (2.1) we get

$$m(r, a; f^{(1)}) = m\left(r, \frac{1}{f^{(1)} - a}\right) = S(r, f).$$

This proves the lemma. □

Lemma 2.5 (see [6], p. 58). *Each solution of the differential equation*

$$a_n f^{(n)} + a_{n-1} f^{(n-1)} + \cdots + a_0 f = 0,$$

where $a_0(\not\equiv 0), a_1, \cdots, a_n(\not\equiv 0)$ *are polynomials, is an entire function of finite order.*

Lemma 2.6 (see [1], p. 47). *Let* f *be a nonconstant meromorphic function and let* a_1, a_2, a_3 *be three distinct meromorphic functions satisfying* $T(r, a_\nu) = S(r, f)$ *for* $\nu = 1, 2, 3$. *Then*

$$T(r, f) \le \overline{N}(r, 0; f - a_1) + \overline{N}(r, 0; f - a_2) + \overline{N}(r, 0; f - a_3) + S(r, f).$$

Lemma 2.7 (see [8], p. 92). *Let* $f_1, f_2, \ldots, f_n$ *be meromorphic functions which are nonconstant except possibly for* f_n, *where* $n \ge 3$. *If* $f_n \not\equiv 0$, $\sum_{j=1}^{n} f_j \equiv 1$, *and*

$$\sum_{j=1}^{n} N(r, 0; f_j) + (n - 1) \sum_{j=1}^{n} \overline{N}(r, \infty; f_j) < \{\mu + o(1)\} T(r, f_k)$$

for $k = 1, 2, \ldots, n - 1$ *and for some* $\mu (0 < \mu < 1)$, *then* $f_n \equiv 1$.

3. Proof of Theorem 1.1

Proof. First, we see that f can not be a polynomial. We suppose that f is a polynomial. Then $T(r,f)=O(\log r)$ and $N_A(r,a;f)+N_A(r,a;f^{(1)})=O(\log T(r,f))=S(r,f)$ imply $A=\emptyset$. Also $N_B(r,a;f^{(1)})=S(r,f)$ implies $B=\emptyset$. Therefore,

$$E(a;f)=E\left(a;f^{(1)}\right) \text{ and } \overline{E}\left(a;f^{(1)}\right)\subset\overline{E}(a,L)\cap\overline{E}\left(a;L^{(1)}\right).$$

Let the degree of f be greater than 1. Then $deg(f-a)>deg(f^{(1)}-a)$. Since each common zero of $f-a$ and $f^{(1)}-a$ has the same multiplicity, this contradicts the fact that $E(a;f)=E\left(a;f^{(1)}\right)$.

Next, let $f=A_1z+B_1$, where $A_1(\neq 0), B_1$ are constants. Then $f^{(1)}=A_1$ and $L\equiv L^{(1)}\equiv 0$. Now, $(A_1-\beta)/\alpha$ is the only zero of $f^{(1)}-a$, and $-\beta/\alpha$ is the only zero of $L-a$. Consequently, $\overline{E}\left(a;f^{(1)}\right)\subset\overline{E}(a,L)$ implies that $(A_1-\beta)/\alpha=-\beta/\alpha$ and so $A_1=0$, a contradiction. Therefore f is a transcendental entire function.

Now

$$\begin{aligned}N_{(2}\left(r,a;f^{(1)}\right)&\leq N_A\left(r,a;f^{(1)}\right)+N_B\left(r,a;f^{(1)}\right)\\&\quad+N_{(2}\left(r,a;f^{(1)}|f=a\right)+S(r,f)\\&=N_{(2}\left(r,a;f^{(1)}|f=a\right)+S(r,f),\end{aligned}\tag{3.1}$$

where $N_{(2}\left(r,a;f^{(1)}|f=a\right)$ denotes the counting function (counted with multiplicities) of those multiple zeros of $f^{(1)}-a$, which are also zeros of $f-a$.

We note that a common zero of $f-a$ and $f^{(1)}-a$ of multiplicity $q(\geq 2)$ is a zero of $a-a^{(1)}=\left(f^{(1)}-a^{(1)}\right)-\left(f^{(1)}-a\right)$ with multiplicity $q-1(\geq 1)$. Therefore,

$$N_{(2}\left(r,a;f^{(1)}|f=a\right)\leq 2N\left(r,0;a-a^{(1)}\right)=S(r,f).$$

So, from (3.1) we get

$$N_{(2}\left(r,a;f^{(1)}\right)=S(r,f).\tag{3.2}$$

First, we suppose that $L^{(1)}\not\equiv f^{(1)}$. Then, using (3.2), we get by the hypothesis that

$$\begin{aligned}N\left(r,a;f^{(1)}\right)&\leq N_B\left(r,a;f^{(1)}\right)+N\left(r,\frac{a}{a-\alpha};\frac{L^{(1)}}{f^{(1)}-\alpha}\right)+S(r,f)\\&\leq T\left(r,\frac{L^{(1)}}{f^{(1)}-\alpha}\right)+S(r,f)=N\left(r,\frac{L^{(1)}}{f^{(1)}-\alpha}\right)+S(r,f)\\&\leq N\left(r,\alpha;f^{(1)}\right)+S(r,f).\end{aligned}\tag{3.3}$$

Again,

$$\begin{aligned} m(r,a;f) &\le m\left(r, \frac{f^{(1)}-\alpha}{f-a}\,\frac{1}{f^{(1)}-\alpha}\right) \le m\left(r,\alpha;f^{(1)}\right)+S(r,f) \\ &= T\left(r,f^{(1)}\right) - N\left(r,\alpha;f^{(1)}\right)+S(r,f) \\ &= m\left(r,f^{(1)}\right) - N\left(r,\alpha;f^{(1)}\right)+S(r,f) \\ &\le m(r,f) - N\left(r,\alpha;f^{(1)}\right)+S(r,f) \\ &= T(r,f) - N\left(r,\alpha;f^{(1)}\right)+S(r,f), \end{aligned}$$

and so

$$N\left(r,\alpha;f^{(1)}\right) \le N(r,a;f)+S(r,f).$$

Thus from (3.3) we get

$$N\left(r,a;f^{(1)}\right) \le N(r,a;f)+S(r,f). \tag{3.4}$$

Again,

$$\begin{aligned} N(r,a;f) &\le N_A(r,a;f) + N\left(r,a;f^{(1)} \mid f=a\right) \\ &\le N\left(r,a;f^{(1)}\right)+S(r,f). \end{aligned} \tag{3.5}$$

Therefore, from (3.4) and (3.5), we deduce that

$$N\left(r,a;f^{(1)}\right) = N(r,a;f)+S(r,f). \tag{3.6}$$

Let h, defined as in Lemma 2.4, be transcendental. Then

$$\begin{aligned} T(r,f) = m(r,f) &\le m\left(r, \frac{1}{h}\left\{\left(a-a^{(1)}\right)L - af^{(1)}\right\}\right)+S(r,f) \\ &\le m\left(r,f^{(1)}\right) + m\left(r, \left(a-a^{(1)}\right)\frac{L}{f^{(1)}} - a\right)+S(r,f) \\ &\le m\left(r,f^{(1)}\right)+S(r,f) = T\left(r,f^{(1)}\right)+S(r,f) \\ &= m\left(r,f^{(1)}\right)+S(r,f) \le m(r,f)+S(r,f) = T(r,f)+S(r,f). \end{aligned}$$

Therefore,

$$T\left(r,f^{(1)}\right) = T(r,f)+S(r,f). \tag{3.7}$$

Again, by Lemma 2.4 we get $m\left(r,a;f^{(1)}\right) = S(r,f)$. Then, from (3.6) and (3.7), we have that

$$m(r,a;f) + m\left(r,a;f^{(1)}\right) = S(r,f). \tag{3.8}$$

Next we suppose that h is rational. Then by Lemma 2.5 we see that f is of finite order. So, by the hypothesis and Lemma 2.3, we get the equality $m(r,a;f)=S(r,f)$.

Since

$$T\left(r,f^{(1)}\right)=m\left(r,f^{(1)}\right)\leq m(r,f)+S(r,f)=T(r,f)+S(r,f),$$

from (3.6) we get

$$m\left(r,a;f^{(1)}\right)\leq m(r,a;f)+N(r,a;f)-N\left(r,a;f^{(1)}\right)+S(r,f)=S(r,f).$$

Hence in this case also we obtain (3.8).

We now put

$$\phi=\frac{f^{(1)}-a}{f-a}\ \text{ and }\ \psi=\frac{L-a}{f^{(1)}-a}.$$

Then by (3.8) we get $m(r,\phi)+m(r,\psi)=S(r,f)$. Also, from the hypothesis we have

$$N(r,\phi)\leq N_A(r,a;f)+N_B\left(r,a;f^{(1)}\right)+N_{(2}(r,a;f)+S(r,f)=S(r,f),$$

because

$$N_{(2}(r,a;f)\leq N_A(r,a;f)+2N\left(r,0;a-a^{(1)}\right)+S(r,f)=S(r,f).$$

Again, by (3.2) and the hypothesis, we get

$$N(r,\psi)\leq N_A\left(r,a;f^{(1)}\right)+N_B\left(r,a;f^{(1)}\right)+N_{(2}\left(r,a;f^{(1)}\right)+S(r,f)=S(r,f).$$

Therefore,

$$T(r,\phi)+T(r,\psi)=S(r,f). \tag{3.9}$$

Let z_1 be a simple zero of $f-a$ such that $z_1\notin A\cup B$ and $a(z_1)-a^{(1)}(z_1)\neq 0$. Then $f(z_1)=f^{(1)}(z_1)=L(z_1)=L^{(1)}(z_1)=a(z_1)$. Now, by Taylor's expansion in some neighbourhood of z_1, we get

$$\begin{aligned}f(z)-a(z)&=(f-a)(z_1)+(f-a)^{(1)}(z_1)(z-z_1)+O(z-z_1)^2\\&=\left(a(z_1)-a^{(1)}(z_1)\right)(z-z_1)+O(z-z_1)^2,\end{aligned}$$

$$\begin{aligned}f^{(1)}(z)-a(z)&=\left(f^{(1)}-a\right)(z_1)+\left(f^{(1)}-a\right)^{(1)}(z_1)(z-z_1)+O(z-z_1)^2\\&=\left\{f^{(2)}(z_1)-a^{(1)}(z_1)\right\}(z-z_1)+O(z-z_1)^2\end{aligned}$$

and

$$\begin{aligned}L(z)-a(z)&=(L-a)(z_1)+(L-a)^{(1)}(z_1)(z-z_1)+O(z-z_1)^2\\&=\left(a(z_1)-a^{(1)}(z_1)\right)(z-z_1)+O(z-z_1)^2.\end{aligned}$$

Therefore, in a neighbourhood of z_1, we obtain

$$\begin{aligned}\phi(z) &= \frac{\left\{f^{(2)}(z_1) - a^{(1)}(z_1)\right\}(z - z_1) + O(z - z_1)^2}{\left(a(z_1) - a^{(1)}(z_1)\right)(z - z_1) + O(z - z_1)^2} \\ &= \frac{f^{(2)}(z_1) - \alpha + O(z - z_1)}{a(z_1) - \alpha + O(z - z_1)} = \frac{f^{(2)}(z_1) - \alpha}{a(z_1) - \alpha} + O(z - z_1)\end{aligned} \tag{3.10}$$

and

$$\begin{aligned}\psi(z) &= \frac{\left(a(z_1) - a^{(1)}(z_1)\right)(z - z_1) + O(z - z_1)^2}{\left(f^{(2)}(z_1) - a^{(1)}(z_1)\right)(z - z_1) + O(z - z_1)^2} \\ &= \frac{a(z_1) - \alpha + O(z - z_1)}{f^{(2)}(z_1) - \alpha + O(z - z_1)} = \frac{a(z_1) - \alpha}{f^{(2)}(z_1) - \alpha} + O(z - z_1).\end{aligned} \tag{3.11}$$

We put $M = \psi - 1/\phi$. Then from (3.9) we get $T(r, M) = S(r, f)$. Also, in some neighbourhood of z_1, we have, by (3.10) and (3.11), that $M(z) = O(z - z_1)$.

If $M \not\equiv 0$, then

$$\begin{aligned}\overline{N}(r, a; f) &\le N_A(r, a; f) + N_B\left(r, a; f^{(1)}\right) + N_{(2}(r, a; f) \\ &\quad + N(r, 0; a - a^{(1)}) + N(r, 0; M) \\ &= S(r, f),\end{aligned}$$

and so, by (3.6) and Lemma 2.2, we have $T(r, f) = S(r, f)$, a contradiction. Thus $M \equiv 0$ and so

$$L \equiv f. \tag{3.12}$$

Differentiating (3.12) we get $L^{(1)} \equiv f^{(1)}$, which contradicts our hypothesis that $L^{(1)} \not\equiv f^{(1)}$. Therefore, indeed we have $L^{(1)} \equiv f^{(1)}$.

Next we suppose that $L^{(1)} \not\equiv L$. Then, by the hypothesis and (3.2), we get

$$\begin{aligned}N\left(r, a; f^{(1)}\right) &\le N_B\left(r, a; f^{(1)}\right) + N\left(r, 1; \frac{L^{(1)}}{L}\right) + S(r, f) \\ &\le T\left(r, \frac{L^{(1)}}{L}\right) + S(r, f) = N\left(r, \frac{L^{(1)}}{L}\right) + S(r, f) \\ &= \overline{N}(r, 0; L) + S(r, f).\end{aligned} \tag{3.13}$$

Again,

$$\begin{aligned}m(r, a; f) &= m\left(r, \frac{L}{f - a}\frac{1}{L}\right) \le m(r, 0; L) + S(r, f) \\ &= T(r, L) - N(r, 0; L) + S(r, f) = m(r, L) - N(r, 0; L) + S(r, f)\end{aligned}$$

$$\begin{aligned}&\le m\left(r,\frac{L}{f}\right)+m(r,f)-N(r,0;L)+S(r,f)\\&=m(r,f)-N(r,0;L)+S(r,f)=T(r,f)-N(r,0;L)+S(r,f)\end{aligned}$$

and so

$$N(r,0;L)\le N(r,a;f)+S(r,f).$$

Now, by (3.13) we get

$$N\left(r,a;f^{(1)}\right)\le N(r,a;f)+S(r,f). \tag{3.14}$$

Also,

$$\begin{aligned}N(r,a;f)&\le N_A(r,a;f)+N\left(r,a;f^{(1)}\mid f=a\right)\\&\le N\left(r,a;f^{(1)}\right)+S(r,f).\end{aligned}\tag{3.15}$$

From (3.14) and (3.15) we get (3.6).

Now, using Lemmas 2.3–2.5 and (3.6), we similarly obtain (3.8). Further, using ϕ and ψ and proceeding likewise, we get (3.12).

Solving $L-f\equiv 0$, we find that

$$f=c_1e^{\alpha_1 z}+c_2e^{\alpha_2 z}+\cdots+c_ke^{\alpha_k z}, \tag{3.16}$$

where $\alpha_1,\alpha_2,\dots,\alpha_k$ are the roots of $\sum_{j=2}^{n}a_jz^j=1$ and $c_1,c_2,\dots,c_k$ are constants or polynomials, not all identically zero, and $k(\le n)$ is an integer. Differentiating (3.16), we get

$$f^{(1)}=\left(c_1^{(1)}+c_1\alpha_1\right)e^{\alpha_1 z}+\left(c_2^{(1)}+c_2\alpha_2\right)e^{\alpha_2 z}+\cdots+\left(c_k^{(1)}+c_k\alpha_k\right)e^{\alpha_k z}. \tag{3.17}$$

From (3.16), (3.17), and $\phi=\left(f^{(1)}-a\right)/(f-a)$, we get

$$\begin{aligned}&\left(\phi c_1-c_1^{(1)}-c_1\alpha_1\right)e^{\alpha_1 z}+\left(\phi c_2-c_2^{(1)}-c_2\alpha_2\right)e^{\alpha_2 z}+\dots\\&\qquad+\left(\phi c_k-c_k^{(1)}-c_k\alpha_k\right)e^{\alpha_k z}\equiv a(\phi-1).\end{aligned}$$

We suppose that $\phi\not\equiv 1$. Then, from the above, we have

$$\sum_{j=1}^{k}\frac{\phi c_j-c_j^{(1)}-c_j\alpha_j}{a(\phi-1)}\,e^{\alpha_j z}\equiv 1. \tag{3.18}$$

We note that $T(r,f)=O(T(r,e^{\alpha_j z}))$ for $j=1,2,\dots,k$.

If the left hand side of (3.18) contains more than two terms, then by Lemma 2.7 we get

$$\frac{\phi c_j-c_j^{(1)}-c_j\alpha_j}{a(\phi-1)}e^{\alpha_j z}\equiv 1 \tag{3.19}$$

for one value of $j\in\{1,2,\dots,k\}$. From (3.19) we see that $T(r,e^{\alpha_j z})=S(r,f)=S(r,e^{\alpha_j z})$, a contradiction.

We now suppose that the left hand side of (3.18) contains only two terms, say,

$$\frac{\phi c_j - c_j^{(1)} - c_j\alpha_j}{a(\phi-1)}\, e^{\alpha_j z} + \frac{\phi c_l - c_l^{(1)} - c_l\alpha_l}{a(\phi-1)}\, e^{\alpha_l z} \equiv 1.$$

By Lemma 2.6 we get

$$\begin{aligned} T\left(r, e^{\alpha_j z}\right) &\le \overline{N}\left(r, 0; e^{\alpha_j z}\right) + \overline{N}\left(r, \infty; e^{\alpha_j z}\right) \\ &\quad + \overline{N}\left(r, \frac{a(\phi-1)}{\phi c_j - c_j^{(1)} - c_j\alpha_j}; e^{\alpha_j z}\right) + S\left(r, e^{\alpha_j z}\right) \\ &= \overline{N}\left(r, 0; e^{\alpha_l z}\right) + S\left(r, e^{\alpha_j z}\right) = S\left(r, e^{\alpha_j z}\right), \end{aligned}$$

a contradiction.

Finally, we suppose that the left hand side of (3.18) contains only one term, say,

$$\frac{\phi c_j - c_j^{(1)} - c_j\alpha_j}{a(\phi-1)}\, e^{\alpha_j z} \equiv 1.$$

Then $T(r, e^{\alpha_j z}) = S(r, f) = S(r, e^{\alpha_j z})$, a contradiction.

Therefore, $\phi \equiv 1$ and so $f^{(1)} \equiv f$. Hence, by (3.12) we get $L \equiv L^{(1)}$, a contradiction to the supposition. Thus, indeed, we have $L \equiv L^{(1)}$.

Now $L \equiv L^{(1)} \equiv f^{(1)}$ implies $L = L^{(1)} = f^{(1)} = \lambda e^z$, where $\lambda(\neq 0)$ is a constant. Therefore $f = \lambda e^z + K$, where K is a constant. By Lemma 2.6 we get

$$\begin{aligned} T\left(r, \lambda e^z\right) &\le \overline{N}\left(r, 0; \lambda e^z\right) + \overline{N}\left(r, \infty; \lambda e^z\right) + \overline{N}\left(r, a-K; \lambda e^z\right) + S\left(r, \lambda e^z\right) \\ &= \overline{N}(r, a; f) + S\left(r, \lambda e^z\right), \end{aligned}$$

which implies $\overline{N}(r, a; f) \neq S(r, f)$. Again, since

$$N_A(r, a; f) + N_B\left(r, a; f^{(1)}\right) = S(r, f),$$

we get

$$\overline{E}(a; f) \cap \overline{E}\left(a; f^{(1)}\right) \neq \emptyset.$$

But this implies $K = 0$ and so $f = L = \lambda e^z$. The proof is complete. □

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