

Summands in locally almost square and locally octahedral spaces

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ABSTRACT. We study the question whether properties like local/weak almost squareness and local octahedrality pass down from an absolute sum $X \oplus_F Y$ to the summands X and Y .

1. Introduction

First we fix some notation. Throughout this paper we denote by X and Y real Banach spaces; X^* denotes the dual of X , B_X its closed unit ball, and S_X denotes its unit sphere.

Let us now begin by recalling the following definition (see [8]): X is called *octahedral* (OH) if for every finite-dimensional subspace F of X and every $\varepsilon > 0$ there is some $y \in S_X$ such that

$$\|x + y\| \geq (1 - \varepsilon)(\|x\| + 1), \quad x \in F.$$

The space ℓ^1 is the standard example of an octahedral space. In fact, a Banach space possesses an equivalent octahedral norm if and only if it contains an isomorphic copy of ℓ^1 (see Theorem 2.5 (p. 106) in [7]).

In the paper [11], the following weaker forms of octahedrality were introduced: X is called *weakly octahedral* (WOH) if for every finite-dimensional subspace F of X , every $x^* \in B_{X^*}$, and each $\varepsilon > 0$ there is some $y \in S_X$ such that

$$\|x + y\| \geq (1 - \varepsilon)(|x^*(x)| + 1), \quad x \in F.$$

The space X is called *locally octahedral* (LOH) if for every $x \in X$ and every $\varepsilon > 0$ there exists $y \in S_X$ such that

$$\|sx + y\| \geq (1 - \varepsilon)(|s|\|x\| + 1), \quad s \in \mathbb{R}.$$

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The motivation for these definitions was to give dual characterisations of the so-called diameter-two-properties, see [11].

There are many equivalent formulations of the three octahedrality properties (see for instance [9] and [11]). We will recall only those which we need here (they can be found in [11]): a Banach space X is octahedral if and only if for every $n \in \mathbb{N}$, all $x_1, \dots, x_n \in S_X$, and every $\varepsilon > 0$ there exists an element $y \in S_X$ such that $\|x_i + y\| \geq 2 - \varepsilon$ for all $i = 1, \dots, n$. The space X is locally octahedral if and only if for every $x \in S_X$ and all $\varepsilon > 0$ there exists $y \in S_X$ such $\|x \pm y\| \geq 2 - \varepsilon$. We will use these characterisations later without further mention.

Now we come to the classes of almost square spaces and their relatives. In the paper [1], the following definitions were introduced. A Banach space X is said to be *almost square* (ASQ) if the following holds: for all $n \in \mathbb{N}$ and all $x_1, \dots, x_n \in S_X$ there exists a sequence $(y_k)_{k \in \mathbb{N}}$ in B_X such that $\|y_k\| \rightarrow 1$ and $\|x_i + y_k\| \rightarrow 1$ for all $i = 1, \dots, n$. The space X is called *weakly almost square* (WASQ) if for every $x \in S_X$ there is a weak null sequence $(y_k)_{k \in \mathbb{N}}$ in B_X such that $\|y_k\| \rightarrow 1$ and $\|x \pm y_k\| \rightarrow 1$. The space X is called *locally almost square* (LASQ) if it fulfils the definition of an LASQ space without the additional condition that the sequence $(y_k)_{k \in \mathbb{N}}$ converges weakly to zero.

Obviously, WASQ implies LASQ. It was shown in [1] that ASQ implies WASQ and that the converse of this statement does not hold, while it is not known whether LASQ is strictly weaker than WASQ.

The model example of an ASQ space is c_0 and it was further proved in [1] that every ASQ space contains an isomorphic copy of c_0 and, conversely, every separable Banach space containing an isomorphic copy of c_0 can be equivalently renormed to become ASQ. In [4] it was shown that the same holds true also for nonseparable spaces. Also, it was proved in [1] that if X is ASQ, then X^* is OH.

Next we will recall the necessary basics on absolute sums. A norm F on \mathbb{R}^2 is called *absolute* if $F(a, b) = F(|a|, |b|)$ for all $(a, b) \in \mathbb{R}^2$ and it is called *normalised* if $F(1, 0) = 1 = F(0, 1)$. If F is an absolute, normalised norm on \mathbb{R}^2 , and X and Y are two Banach spaces, then the absolute sum of X and Y with respect to F , denoted by $X \oplus_F Y$, is defined as the direct product $X \times Y$ equipped with the norm $\|(x, y)\|_F = F(\|x\|, \|y\|)$. Then $X \oplus_F Y$ is again a Banach space.

For every $1 \leq p \leq \infty$, the p -norm $\|\cdot\|_p$ on \mathbb{R}^2 is an absolute, normalised norm and the corresponding sum is just the usual p -direct sum of two Banach spaces. We also note the following important facts (see for instance Lemmas 1 and 2 on p. 36 in [5]): if F is an absolute, normalised norm on \mathbb{R}^2 , then we have for all $a, b, c, d \in \mathbb{R}$

$$1) \quad |a| \leq |c| \text{ and } |b| \leq |d| \Rightarrow F(a, b) \leq F(c, d),$$

- 2) $|a| < |c|$ and $|b| < |d| \Rightarrow F(a, b) < F(c, d)$,
- 3) $\|(a, b)\|_\infty \leq F(a, b) \leq \|(a, b)\|_1$.

It follows in particular that $|a|, |b| \leq F(a, b)$ for all $a, b \in \mathbb{R}$.

We will also need the following (see [12]): for every $t \in (-1, 1)$ there exists a unique $f(t) \in (0, 1]$ such that $F(t, f(t)) = 1$. We will call the function f the *upper boundary curve* of $B_{(\mathbb{R}^2, F)}$. It is even, concave (hence continuous), decreasing on $[0, 1)$, and increasing on $(-1, 0]$. Thus it can be extended to a concave, continuous, even function on $[-1, 1]$, which will also be denoted by f .

Octahedrality properties in p -direct sums were already studied in [11]. Among others, the following results were proved.

- (i) For $1 < p \leq \infty$, $X \oplus_p Y$ is LOH/WOH if and only if X and Y are LOH/WOH.
- (ii) $X \oplus_\infty Y$ is OH if and only if X and Y are OH.
- (iii) For $1 < p < \infty$, $X \oplus_p Y$ is never OH (provided that X and Y are nontrivial).

In [1] it is proved that the properties LOH and LASQ are stable under arbitrary (even infinite) absolute sums, and that WOH and WASQ are stable under all absolute sums which fulfil a simple density assumption, including in particular all finite absolute sums. Among others, also the following results were obtained in [1] for any two nontrivial Banach spaces X and Y .

- (i) For $1 \leq p < \infty$, $X \oplus_p Y$ is LASQ/WASQ if and only if X and Y are LASQ/WASQ.
- (ii) $X \oplus_\infty Y$ is LASQ/WASQ/ASQ if and only if X or Y is LASQ/WASQ/ASQ.
- (iii) For $1 \leq p < \infty$, $X \oplus_p Y$ is never ASQ (provided that X and Y are nontrivial).

The purpose of this note is to extend these results by showing that (i) and (iii) also hold if we replace $\|\cdot\|_p$ by any absolute, normalised norm $F \neq \|\cdot\|_\infty$. We will also prove some results on summands in LOH spaces, which imply in particular that X and Y are LOH whenever $X \oplus_F Y$ is LOH and F is strictly convex.

Finally, we will also discuss some results on ultrapowers of LOH, LASQ, etc. spaces and the closedness of these classes with respect to the Banach–Mazur distance.

2. Almost square properties in absolute sums

We start with the following lemma, which is surely well known, but since the author was not able to find it explicitly in the literature, a proof is included here for reader’s convenience.

Lemma 2.1. *Let F be an absolute, normalised norm on \mathbb{R}^2 .*

- (a) $F(1, 1) = 1 \Leftrightarrow F = \|\cdot\|_\infty$.
 (b) $F(1, 1) = 2 \Leftrightarrow F = \|\cdot\|_1$.

Proof. (a) Assume that $F(1, 1) = 1$. Let $(a, b) \in \mathbb{R}^2$ be such that $F(a, b) = 1$. Then $|a|, |b| \leq 1$. If both $|a| < 1$ and $|b| < 1$, then we would have $F(a, b) < F(1, 1) = 1$ (by the general monotonicity properties of absolute norms listed in Section 1).

It follows that $|a| = 1$ or $|b| = 1$, hence $\|(a, b)\|_\infty = 1$.

Thus we have $S_{(\mathbb{R}^2, F)} \subseteq S_{(\mathbb{R}^2, \|\cdot\|_\infty)}$, which implies $F = \|\cdot\|_\infty$.

(b) Suppose that $F(1, 1) = 2$, i. e., the midpoint of $(0, 1)$ and $(1, 0)$ lies on the unit sphere of (\mathbb{R}^2, F) . It follows that the whole line segment from $(0, 1)$ to $(1, 0)$ lies on $S_{(\mathbb{R}^2, F)}$, thus $F(t, 1 - t) = 1$ for every $t \in [0, 1]$.

Hence we have for every $(a, b) \neq (0, 0)$

$$1 = F(|a|/(|a| + |b|), 1 - |a|/(|a| + |b|)) = F(|a|/(|a| + |b|), |b|/(|a| + |b|)),$$

i. e., $F(a, b) = \|(a, b)\|_1$. □

Before we can come to the first main result on sums of LASQ (etc.) spaces, we have to prove another auxiliary lemma.

Lemma 2.2. *Let F be an absolute, normalised norm on \mathbb{R}^2 with $F \neq \|\cdot\|_\infty$ and let $\varepsilon > 0$. Then there is a $\delta > 0$ such that the following holds:*

$$a, b \geq 0, F(a, b) = 1 \text{ and } F(a, 1) \leq 1 + \delta \Rightarrow b \geq 1 - \varepsilon.$$

Proof. Denote by f the upper boundary curve of $B_{(\mathbb{R}^2, F)}$. If the claim was false, then we could find sequences $(a_n)_{n \in \mathbb{N}}, (b_n)_{n \in \mathbb{N}}$ in $[0, \infty)$ such that $F(a_n, b_n) = 1$, $F(a_n, 1) \leq 1 + 1/n$ and $b_n < 1 - \varepsilon$ for each $n \in \mathbb{N}$.

Since $a_n, b_n \leq 1$ for every $n \in \mathbb{N}$, we can find subsequences $(a_{n_k}), (b_{n_k})$ such that $a_{n_k} \rightarrow a$ and $b_{n_k} \rightarrow b$ for some $a, b \in [0, 1]$. It follows that $F(a, b) = 1 = F(a, 1)$ and $b \leq 1 - \varepsilon$.

Since $F \neq \|\cdot\|_\infty$, it follows from Lemma 2.1 that $F(1, 1) > 1$ and hence $a < 1$. But then $b = f(a) = 1$, by definition of f . This is a contradiction since $b < 1$. □

Now we can prove the first main result of this paper.

Proposition 2.3. *If F is any absolute, normalised norm on \mathbb{R}^2 with $F \neq \|\cdot\|_\infty$, and X and Y are nontrivial Banach spaces, then the following holds:*

- (i) *If $X \oplus_F Y$ is LASQ, then X and Y are LASQ.*
- (ii) *If $X \oplus_F Y$ is WASQ, then X and Y are WASQ.*
- (iii) *$X \oplus_F Y$ is not ASQ.*

Note that the converses of (i) and (ii) also hold by the general results in [1].

Proof. First we will prove statement (ii). So let $Z := X \oplus_F Y$ be WASQ and let $y \in S_Y$. Then there is a weakly null sequence $(z_n = (u_n, v_n))_{n \in \mathbb{N}}$ in B_Z such that $\|z_n \pm (0, y)\|_F \rightarrow 1$ and $\|z_n\|_F \rightarrow 1$. Actually, we may assume that $\|z_n\|_F = F(\|u_n\|, \|v_n\|) = 1$ for every $n \in \mathbb{N}$.

By Lemma 2.2 there exists a sequence $(\delta_n)_{n \in \mathbb{N}}$ in $(0, \infty)$ such that $\delta_n \rightarrow 0$ and for every $n \in \mathbb{N}$ the following holds:

$$a, b \geq 0, F(a, b) = 1 \text{ and } F(a, 1) \leq 1 + \delta_n \Rightarrow b \geq 1 - 2^{-n}.$$

By passing to a subsequence if necessary, we may assume that $F(\|u_n\|, \|y \pm v_n\|) \leq 1 + \delta_n$ for every $n \in \mathbb{N}$. It follows that

$$F(\|u_n\|, 1) \leq \frac{1}{2}(F(\|u_n\|, \|y + v_n\|) + F(\|u_n\|, \|y - v_n\|)) \leq 1 + \delta_n$$

and hence $\|v_n\| \geq 1 - 2^{-n}$ for every n .

Since we also have $\|v_n\| \leq F(\|u_n\|, \|v_n\|) = 1$ for each n , we obtain $\|v_n\| \rightarrow 1$. Also, $(v_n)_{n \in \mathbb{N}}$ is a weakly null sequence in Y , since $(z_n)_{n \in \mathbb{N}}$ is weakly null in Z .

We further have $\|y \pm v_n\| \leq F(\|u_n\|, \|y \pm v_n\|) \leq 1 + \delta_n$ and thus

$$1 + \delta_n \geq \|y + v_n\| \geq 2 - \|y - v_n\| \geq 1 - \delta_n, \quad n \in \mathbb{N},$$

which implies $\|y \pm v_n\| \rightarrow 1$. Thus Y is WASQ.

Since $X \oplus_F Y \cong Y \oplus_{\tilde{F}} X$, where $\tilde{F}(a, b) := F(b, a)$, the same argument also shows that X is LASQ. This completes the proof of (ii) and statement (i) is proved analogously.

Now we will prove (iii). Assume to the contrary that $X \oplus_F Y$ is ASQ. Since $F \neq \|\cdot\|_\infty$, we have $F(1, 1) > 1$ (Lemma 2.1). Choose $\varepsilon > 0$ such that $(1 - \varepsilon)F(1, 1) > 1$.

By Lemma 2.2 (applied to F and \tilde{F}) there exists a $\delta > 0$ such that for all $a, b \geq 0$ with $F(a, b) = 1$ the following holds:

$$\begin{aligned} F(a, 1) \leq 1 + \delta &\Rightarrow b \geq 1 - \varepsilon, \\ F(1, b) \leq 1 + \delta &\Rightarrow a \geq 1 - \varepsilon. \end{aligned}$$

Now let $x \in S_X$ and $y \in S_Y$. Since $X \oplus_F Y$ is ASQ, there exist $u \in X$, $v \in Y$ such that $F(\|u\|, \|v\|) = 1$ and $F(\|x \pm u\|, \|v\|) \leq 1 + \delta$, $F(\|u\|, \|y \pm v\|) \leq 1 + \delta$.

A similar calculation as in the proof of (ii) shows that $F(\|u\|, 1) \leq 1 + \delta$ and $F(1, \|v\|) \leq 1 + \delta$. It follows that $\|u\|, \|v\| \geq 1 - \varepsilon$.

But then $1 = F(\|u\|, \|v\|) \geq (1 - \varepsilon)F(1, 1) > 1$ and with this contradiction the proof is finished. \square

3. Local octahedrality properties in absolute sums

Next we turn our attention to LOH sums. First recall that a Banach space X is *strictly convex* (SC) if $x, y \in S_X$ and $\|x + y\| = 2$ imply $x = y$. The

p -norms are strictly convex for $1 < p < \infty$. We will call a point $x \in S_X$ an SC-point of X if $\|x+y\| < 2$ for every $y \in S_X$ with $y \neq x$. Thus X is strictly convex if and only if every point of S_X is an SC-point.

Given an absolute, normalised norm F on \mathbb{R}^2 , set

$$r_F := \inf\{a \in [0, 1] : \exists b \geq 0 F(a, b) = 1 \text{ and } F(a+1, b) = 2\}.$$

The following lemma is intuitively clear, but we include a proof for the sake of completeness.

Lemma 3.1. *Let F be an absolute, normalised norm on \mathbb{R}^2 with upper boundary curve f . Then*

- (i) $r_F = 1 \Leftrightarrow (1, 0)$ is an SC-point of (\mathbb{R}^2, F) or $f(1) > 0$;
- (ii) $r_F = 0 \Leftrightarrow F = \|\cdot\|_1$.

Proof. (i) If $r_F < 1$, then there must be $a \in [0, 1)$ and $b > 0$ such that $F(a, b) = 1$ and $F(a+1, b) = 2$. Hence $(1, 0)$ is not an SC-point of (\mathbb{R}^2, F) .

Moreover, the whole line segment from $(1, 0)$ to (a, b) belongs to the unit sphere of (\mathbb{R}^2, F) , which implies that $f(a+t(1-a)) = (1-t)b$ for $t \in [0, 1)$. Hence $f(s) = \frac{1-s}{1-a}b$ for $s \in [a, 1)$. Thus $f(1) = 0$. This shows “ \Leftarrow ”.

Now assume that $r_F = 1$ and $(1, 0)$ is not an SC-point of (\mathbb{R}^2, F) . Then we can find $(a, b) \in \mathbb{R}^2$ such that $F(a, b) = 1$ and $F(1+a, b) = 2$ but $(a, b) \neq (1, 0)$. Without loss of generality we may assume that $a, b > 0$. Since $r_F = 1$, it follows that $a = 1$, hence $F(1, b) = 1$.

If $f(1) = 0$, there would be $s \in [0, 1)$ such that $f(s) < b$. But then we obtain a contradiction since $1 = F(s, f(s)) < F(1, b) = 1$. Thus we must have $f(1) > 0$.

(ii) Suppose that $r_F = 0$. This easily implies $F(1, 1) = 2$ and thus we have $F = \|\cdot\|_1$ by Lemma 2.1. The converse is clear. \square

We need two more auxiliary lemmas.

Lemma 3.2. *Let F be an absolute, normalised norm on \mathbb{R}^2 . If $a, b \geq 0$ are such that $F(a, b) = 1$ and $0 \leq c \leq 1+a$ is such that $F(c, b) = 2$, then $c = 1+a$ and $a \geq r_F$.*

Proof. First note that under the above assumptions we have $2 = F(c, b) \leq F(1+a, b) \leq F(1, 0) + F(a, b) = 2$. Hence, by definition of r_F , we must have $a \geq r_F$.

Now we denote again by f the upper boundary curve of F and distinguish two cases.

Case 1: $r_F = 1$. Then $a = 1$ and $1 < c \leq 2$ (if $c \leq 1$, we would obtain $2 = F(c, b) \leq F(1, b) = 1$). Thus $a' := c - 1 \in (0, 1]$ and $F(1+a', b) = 2$ as well as $1 = F(1, b) \geq F(a', b) \geq F(c, b) - F(1, 0) = 1$.

Since $r_F = 1$, it follows that $a' = 1$, i. e. $c = 2$.

Case 2: $r_F < 1$. Then we have $r_F \leq a < c \leq 1+a$. Put $d := c - a \in (0, 1]$ and let $w > 0$ be such that $F(r_F, w) = 1$ and $F(1+r_F, w) = 2$.

Then the line segment from $(1, 0)$ to (r_F, w) lies completely in $S_{(\mathbb{R}^2, F)}$ and we obtain

$$f(s) = h(s - 1) \quad \forall s \in [r_F, 1], \text{ where } h := \frac{w}{r_F - 1}.$$

We also put $g(s) := (d+1)f(s/(d+1))$ for $s \in [0, d+1]$. Then $F(s, g(s)) = d + 1$ for every such s .

It is easy to see that $c/(d+1) \in [a, 1] \subseteq [r_F, 1]$ and thus we have $g(c) = (d+1)f(c/(d+1)) = h(c - (d+1)) = h(a - 1)$.

If $a < 1$, we have $f(a) = b$ and since $a \geq r_F$ it follows that $f(a) = h(a - 1)$, thus $g(c) = b$.

For $a = 1$ we must have $b = 0 = g(c)$ (otherwise there is $s \in [0, 1)$ such that $f(s) < b$ and hence $1 = F(s, f(s)) < F(1, b) = F(a, b) = 1$).

Thus we always have $g(c) = b$, which implies that $2 = F(c, b) = F(c, g(c)) = d + 1$. Hence $d = 1$ and $c = 1 + a$. \square

Lemma 3.3. *Let F be an absolute, normalised norm on \mathbb{R}^2 and let $\varepsilon > 0$. Then there exists $\delta > 0$ such that the following holds: whenever $a, b \geq 0$ with $F(a, b) = 1$ and $0 \leq c \leq 1 + a$ with $F(c, b) \geq 2 - \delta$, then $c \geq 1 + r_F - \varepsilon$.*

Proof. This follows from Lemma 3.2 and a standard compactness argument. \square

Now we can prove the main result on local octahedrality in absolute sums.

Proposition 3.4. *Let F be an absolute, normalised norm on \mathbb{R}^2 and let X and Y be nontrivial Banach spaces such that $X \oplus_F Y$ is LOH. Then for every $x \in S_X$ and every $\varepsilon > 0$ there is a $z \in S_X$ such that $\|x \pm z\| \geq 2r_F - \varepsilon$.*

Proof. Let $x \in S_X$ and $\varepsilon > 0$. Choose $\delta > 0$ according to Lemma 3.3 for the parameter $\varepsilon/2$. Since $X \oplus_F Y$ is LOH, we can find $(u, v) \in S_{X \oplus_F Y}$ such that $F(\|x \pm u\|, \|v\|) \geq 2 - \delta$.

Because of $\|x \pm u\| \leq 1 + \|u\|$ and $F(\|u\|, \|v\|) = 1$ this implies $\|x \pm u\| \geq 1 + r_F - \varepsilon/2$. It follows that $\|u\| \geq \|x + u\| - 1 \geq r_F - \varepsilon/2$. Now put $z := u/\|u\|$. Then

$$\|x \pm z\| \geq \|x \pm u\| - \|u - u/\|u\|\| \geq 1 + r_F - \frac{\varepsilon}{2} - (1 - \|u\|) \geq 2r_F - \varepsilon.$$

\square

It follows in particular from Proposition 3.4 that X is LOH if $X \oplus_F Y$ is LOH and $r_F = 1$ (which, by Lemma 3.1, is equivalent to the fact that $f(1) > 0$ or $(1, 0)$ is an SC-point of (\mathbb{R}^2, F)).

More generally, for any Banach space X we may define

$$s(X) := \sup\{s \in [0, 2] : \forall x \in S_X \forall \varepsilon > 0 \exists y \in S_X \|x \pm y\| \geq s - \varepsilon\}.$$

Then X is LOH if and only if $s(X) = 2$ and Proposition 3.4 reads: *if $X \oplus_F Y$ is LOH, then $s(X) \geq 2r_F$.*

Note that $s(\mathbb{R}) = 0$, while it easily follows from Riesz's Lemma that $s(X) \geq 1$ whenever $\dim(X) \geq 2$. The following statements are also easy to verify: $s(\ell^\infty) = s(c_0) = 1 = s(\ell^\infty(n))$ for $n \geq 2$, where ℓ^∞ is the space of bounded sequences, c_0 is the space of null sequences (both equipped with the supremum norm), and $\ell^\infty(n)$ is the space \mathbb{R}^n equipped with the maximum norm. It is also easy to prove that $s(H) = \sqrt{2}$ for any Hilbert space H with $\dim(H) \geq 2$.

Putting everything together, we obtain the following corollary (for (b) note that $r_F > 0$ if $F \neq \|\cdot\|_1$ (Lemma 3.1)).

Corollary 3.5. *Let F be an absolute, normalised norm on \mathbb{R}^2 , and let X and Y be nontrivial Banach spaces. Then the following holds.*

- (a) *If $r_F = 1$ and $X \oplus_F Y$ is LOH, then so is X . In particular, this holds if F is strictly convex or $F = \|\cdot\|_\infty$.*
- (b) *If $F \neq \|\cdot\|_1$, then $\mathbb{R} \oplus_F Y$ is not LOH.*
- (c) *If $r_F > 1/2$, then $\ell^\infty \oplus_F Y$, $c_0 \oplus_F Y$ and $\ell^\infty(n) \oplus_F Y$ for $n \in \mathbb{N}$ are not LOH.*
- (d) *If $r_F > 1/\sqrt{2}$ and H is a Hilbert space, then $H \oplus_F Y$ is not LOH.*

Of course, it is also possible to prove results analogous to Proposition 3.4 and Corollary 3.5 for the second summand by modifying the definition of r_F accordingly (i. e., using instead $r_{\tilde{F}}$, where $\tilde{F}(a, b) := F(b, a)$).

The author does not know whether there are any analogous results for WOH spaces, but let us remark that the above proof-techniques could also be used to show that $X \oplus_F Y$ is not octahedral if X and Y are nontrivial Banach spaces, $F \neq \|\cdot\|_\infty$ and $r_F = 1 = r_{\tilde{F}}$. However, this result already follows from the more general results on octahedrality in absolute sums that were proved in the recent paper [10]. The authors of this paper investigated the stability of average roughness (which is a generalisation of octahedrality) with respect to absolute sums. They introduced the notion of *positive octahedrality* for an absolute, normalised norm F on \mathbb{R}^2 , meaning that there exist $c, d \geq 0$ with $F(c, d) = 1$ and $F(c+1, d) = F(c, d+1) = 2$. They proved that $X \oplus_F Y$ is octahedral whenever X and Y are octahedral and F is positively octahedral, and, conversely, if $X \oplus_F Y$ is octahedral for some nontrivial Banach spaces X and Y , then F has to be positively octahedral.

Let us note one more corollary concerning the so called local-diameter-two-property. For $x^* \in S_{X^*}$ and $\varepsilon > 0$, the *slice* of B_X induced by x^* and ε is the set $S(x^*, \varepsilon) := \{z \in B_X, x^*(z) > 1 - \varepsilon\}$. Following the terminology of [2], a Banach space X is said to have the *local diameter-two-property* (LD2P) if every slice of B_X has diameter 2. Then the following result holds (recall that a norm is smooth if it is Gâteaux-differentiable at each nonzero point).

Corollary 3.6. *If F is a smooth, absolute, normalised norm on \mathbb{R}^2 , and X and Y are nontrivial Banach spaces such that $X \oplus_F Y$ has the LD2P, then X has the LD2P.*

Proof. It is well known that a finite-dimensional Banach space is smooth if and only if its dual is strictly convex. If we put

$$F^*(c, d) := \sup\{|ac| + |bd| : (a, b) \in B_{(\mathbb{R}^2, F)}\},$$

then F^* is an absolute, normalised norm on \mathbb{R}^2 and $(X \oplus_F Y)^* \cong X^* \oplus_{F^*} Y^*$ (this is a standard fact from the theory of absolute sums, which is easy to prove). The claim now follows from Corollary 3.5 and the duality between LOH and LD2P ([11], see also Section 5). \square

4. Ultrapowers of OH/LOH and ASQ/LASQ spaces

Next we will consider ultrapowers of OH/LOH and ASQ/LASQ spaces. First we recall the necessary definitions. Given a free ultrafilter \mathcal{U} on \mathbb{N} and a bounded sequence $(a_n)_{n \in \mathbb{N}}$ of real numbers, there exists (by a compactness argument) a unique number $a \in \mathbb{R}$ such that for every $\varepsilon > 0$ we have $\{n \in \mathbb{N} : |a_n - a| < \varepsilon\} \in \mathcal{U}$. It is called the *limit of $(a_n)_{n \in \mathbb{N}}$ along \mathcal{U}* and will be denoted by $\lim_{n, \mathcal{U}} a_n$.

For a Banach space X , denote by $\ell^\infty(X)$ the space of all bounded sequences in X and set $\mathcal{N}_{\mathcal{U}} := \{(x_n) \in \ell^\infty(X) : \lim_{n, \mathcal{U}} \|x_n\| = 0\}$. The *ultrapower* $X^{\mathcal{U}}$ of X with respect to \mathcal{U} is the quotient space $\ell^\infty(X)/\mathcal{N}_{\mathcal{U}}$ equipped with the (well-defined) norm $\|[(x_n)]\|_{\mathcal{U}} := \lim_{n, \mathcal{U}} \|x_n\|$. Then $X^{\mathcal{U}}$ is again a Banach space (for more information on ultraproducts see for example [13]).

We have the following observations concerning octahedrality in ultrapowers.

Proposition 4.1. *Let X be a Banach space and \mathcal{U} a free ultrafilter on \mathbb{N} . Then the following assertions are equivalent.*

- (i) X is octahedral.
- (ii) For all $z_1, \dots, z_n \in S_{X^{\mathcal{U}}}$ there exists an element $z \in S_{X^{\mathcal{U}}}$ such that $\|z_i + z\|_{\mathcal{U}} = 2$ for every $i \in \{1, \dots, n\}$.
- (iii) $X^{\mathcal{U}}$ is octahedral.

Likewise, the following statements are equivalent.

- (i) X is locally octahedral.
- (ii) For every $z \in S_{X^{\mathcal{U}}}$ there is some $\tilde{z} \in S_{X^{\mathcal{U}}}$ such that $\|z \pm \tilde{z}\|_{\mathcal{U}} = 2$.
- (iii) $X^{\mathcal{U}}$ is locally octahedral.

Proof. We will only prove the statement for octahedral spaces. The proof for local octahedrality is completely analogous.

So let us first assume that X is OH and let $z_1, \dots, z_n \in S_{X^{\mathcal{U}}}$. Let $(x_{i,k})_{k \in \mathbb{N}}$ be a representative of z_i . We may assume that $x_{i,k} \neq 0$ for all

$k \in \mathbb{N}$ and all $i = 1, \dots, n$. (Since $\lim_{k, \mathcal{U}} \|x_{i,k}\| = 1$, we have, for example, $A_i := \{k \in \mathbb{N} : \|x_{i,k}\| > 1/2\} \in \mathcal{U}$. So if we replace $(x_{i,k})_{k \in \mathbb{N}}$ by the sequence $(\tilde{x}_{i,k})_{k \in \mathbb{N}}$ which agrees with $(x_{i,k})_{k \in \mathbb{N}}$ on A_i and takes some fixed value $x \in S_X$ on $\mathbb{N} \setminus A_i$, then we obtain a representative of z_i whose entries are all nonzero.)

Since X is octahedral, we can find, for each $k \in \mathbb{N}$, an element $x_k \in S_X$ such that

$$\left\| \frac{x_{i,k}}{\|x_{i,k}\|} + x_k \right\| \geq 2 - 2^{-k}, \quad i = 1, \dots, n.$$

Let $z := [(x_k)_{k \in \mathbb{N}}] \in S_{X^{\mathcal{U}}}$. For each $k \in \mathbb{N}$ and every $i \in \{1, \dots, n\}$ we have

$$\|x_{i,k} + x_k\| \geq \left\| \frac{x_{i,k}}{\|x_{i,k}\|} + x_k \right\| - \left\| \frac{x_{i,k}}{\|x_{i,k}\|} - x_{i,k} \right\| \geq 2 - 2^{-k} - |1 - \|x_{i,k}\||.$$

Since $\lim_{k, \mathcal{U}} \|x_{i,k}\| = \|z_i\|_{\mathcal{U}} = 1$, it follows that $\|z_i + z\|_{\mathcal{U}} = \lim_{k, \mathcal{U}} \|x_{i,k} + x_k\| = 2$ for all $i \in \{1, \dots, n\}$. This proves (i) \Rightarrow (ii).

(ii) \Rightarrow (iii) is clear.

(iii) \Rightarrow (i). Let $x_1, \dots, x_n \in S_X$ and $\varepsilon > 0$. We consider X as a subspace of $X^{\mathcal{U}}$ (via the canonical embedding). Since $X^{\mathcal{U}}$ is octahedral, there exists $y = [(y_k)_{k \in \mathbb{N}}] \in S_{X^{\mathcal{U}}}$ such that

$$\lim_{k, \mathcal{U}} \|x_i + y_k\| = \|x_i + y\|_{\mathcal{U}} \geq 2 - \varepsilon, \quad i = 1, \dots, n.$$

It follows that $B_i := \{k \in \mathbb{N} : \|x_i + y_k\| \geq 2 - 2\varepsilon\} \in \mathcal{U}$ for all $i = 1, \dots, n$. Since $\lim_{k, \mathcal{U}} \|y_k\| = \|y\|_{\mathcal{U}} = 1$ we also have $A := \{k \in \mathbb{N} : |\|y_k\| - 1| \leq \varepsilon\} \in \mathcal{U}$. Hence $M := A \cap B_1 \cap \dots \cap B_n \in \mathcal{U}$ and in particular, $M \neq \emptyset$.

Now let $k_0 \in M$. Then we have, for each $i \in \{1, \dots, n\}$,

$$\left\| x_i + \frac{y_{k_0}}{\|y_{k_0}\|} \right\| \geq \|x_i + y_{k_0}\| - \left\| y_{k_0} - \frac{y_{k_0}}{\|y_{k_0}\|} \right\| \geq 2 - 2\varepsilon - |\|y_{k_0}\| - 1| \geq 2 - 3\varepsilon,$$

thus X is octahedral. \square

For weakly octahedral spaces, the situation seems to be more complicated. Let us first introduce one more notation: if $s = (x_n^*)_{n \in \mathbb{N}}$ is a sequence in S_{X^*} , then we may define a norm-one functional φ_s on $X^{\mathcal{U}}$ by $\varphi_s([(x_n)]) := \lim_{n, \mathcal{U}} x_n^*(x_n)$.

Using the characterisation for WOH spaces from Proposition 2.2 in [11], one can easily prove the following: if X is WOH, then for all $z_1, \dots, z_n \in S_{X^{\mathcal{U}}}$ and every sequence $s = (x_n^*)_{n \in \mathbb{N}}$ in S_{X^*} there exists $z \in S_{X^{\mathcal{U}}}$ such that

$$\|z_i + tz\|_{\mathcal{U}} \geq |\varphi_s(z_i)| + t \quad \forall i \in \{1, \dots, n\}, \quad t > 0.$$

However, it is not clear whether the converse of this statement also holds, nor whether is it equivalent to the weak octahedrality of $X^{\mathcal{U}}$.

Similarly to Proposition 4.1 one can also prove the following result for ASQ/LASQ spaces (we skip the details).

Proposition 4.2. *Let X be a Banach space and \mathcal{U} a free ultrafilter on \mathbb{N} . Then the following assertions are equivalent.*

- (i) X is ASQ.
- (ii) For all $z_1, \dots, z_n \in S_{X\mathcal{U}}$ there exists an element $z \in S_{X\mathcal{U}}$ such that $\|z_i + z\|_{\mathcal{U}} = 1$ for every $i \in \{1, \dots, n\}$.
- (iii) $X^{\mathcal{U}}$ is ASQ.

Likewise, the following statements are equivalent.

- (i) X is LASQ.
- (ii) For every $z \in S_{X\mathcal{U}}$ there is some $\tilde{z} \in S_{X\mathcal{U}}$ such that $\|z \pm \tilde{z}\|_{\mathcal{U}} = 1$.
- (iii) $X^{\mathcal{U}}$ is LASQ.

For WASQ spaces, the situation is again a bit more involved. First we note the following equivalent characterisation of WASQ spaces with separable dual.

Lemma 4.3. *Let X be a Banach space. If X is WASQ, then the following holds: for every $x \in S_X$, every $\varepsilon > 0$, and all $x_1^*, \dots, x_n^* \in S_{X^*}$ there exists a $y \in S_X$ such that $\|x \pm y\| \leq 1 + \varepsilon$ and $x_i^*(y) \leq \varepsilon$ for every $i = 1, \dots, n$.*

If X^ is separable, then the converse of this statement also holds.*

Proof. Assume first that X is WASQ and let $x \in S_X$, $\varepsilon > 0$, and $x_1^*, \dots, x_n^* \in S_{X^*}$. Since X is WASQ, there exists a weakly null sequence $(y_k)_{k \in \mathbb{N}}$ in B_X such that $\|y_k\| \rightarrow 1$ and $\|x \pm y_k\| \rightarrow 1$. Replacing y_k by $y_k/\|y_k\|$ if necessary, we can assume that $\|y_k\| = 1$ for all k .

Since $(y_k)_{k \in \mathbb{N}}$ is weakly convergent to zero and $\|x \pm y_k\| \rightarrow 1$, we can take k large enough so that $x_i^*(y_k) \leq \varepsilon$ for all $i = 1, \dots, n$ and $\|x \pm y_k\| \leq 1 + \varepsilon$.

Now assume that X fulfils the condition of the Lemma and that X^* is separable. We choose a sequence $(x_i^*)_{i \in \mathbb{N}}$ which is dense in S_{X^*} .

If $x \in S_X$, then, using the condition of the Lemma, we can find a sequence $(y_k)_{k \in \mathbb{N}}$ in S_X such that the following holds: $\|x \pm y_k\| \leq 1 + 1/k$ for all $k \in \mathbb{N}$ and $x_i^*(y_k) \leq 1/k$, $-x_i^*(y_k) \leq 1/k$ for all $i = 1, \dots, k$ and all $k \in \mathbb{N}$.

Then we have $\lim_{k \rightarrow \infty} x_i^*(y_k) = 0$ for every $i \in \mathbb{N}$. Since $(x_i^*)_{i \in \mathbb{N}}$ is dense in S_{X^*} , this easily implies that $(y_k)_{k \in \mathbb{N}}$ is a weakly null sequence.

We also have

$$1 + 1/k \geq \|x \pm y_k\| \geq 2 - \|x \mp y_k\| \geq 1 - 1/k$$

for every $k \in \mathbb{N}$, hence $\|x \pm y_k\| \rightarrow 1$. Thus X is WASQ. □

Using this lemma, it is easy to show the next result (the details are omitted).

Proposition 4.4. *Let X be a Banach space and \mathcal{U} a free ultrafilter on \mathbb{N} . If X is WASQ, then for every $z \in S_{X\mathcal{U}}$ and all double-sequences $(x_{ik}^*)_{i,k \in \mathbb{N}}$ in S_{X^*} there is some $\tilde{z} \in S_{X\mathcal{U}}$ satisfying $\|z \pm \tilde{z}\|_{\mathcal{U}} = 1$ and $\varphi_{s_i}(\tilde{z}) = 0$ for every $i \in \mathbb{N}$, where $s_i := (x_{ik}^*)_{k \in \mathbb{N}}$.*

If X^* is separable, then the converse also holds.

Again, it is not clear whether the property in Proposition 4.4 is equivalent to the weak almost squareness of $X^{\mathcal{U}}$.

5. The classes of OH/WOH/LOH/ASQ/LASQ spaces are closed with respect to the Banach–Mazur distance

Finally, we would like to show that the classes of OH/WOH/LOH/ASQ and LASQ spaces are closed with respect to the Banach–Mazur distance. Recall that this distance between two isomorphic Banach spaces X and Y is defined by

$$d(X, Y) := \inf\{\|T\|\|T^{-1}\| \mid T : X \rightarrow Y \text{ is an isomorphism}\}.$$

Proposition 5.1. *Let X be a Banach space such that for every $\delta > 0$ there is some OH/WOH/LOH/ASQ/LASQ space Y isomorphic to X with $d(X, Y) < 1 + \delta$. Then X is also OH/WOH/LOH/ASQ/LASQ.*

Proof. The proofs are all similar, so we will only show the most complicated case of WOH spaces. Let $x_1, \dots, x_n \in S_X$, $x^* \in B_{X^*}$ and $0 < \varepsilon < 1/2$. Choose $0 < \delta < \varepsilon^2$ such that $\sqrt{\delta}/(1 - 2\sqrt{\delta}) \leq \varepsilon$.

By assumption there is a WOH space Y isomorphic to X such that $d(X, Y) < 1 + \delta$. Hence we can find an isomorphism $T : X \rightarrow Y$ such that $\|T\| = 1$ and $\|T^{-1}\| < 1 + \delta$.

Put $y_i := Tx_i \in B_Y \setminus \{0\}$ for $i = 1, \dots, n$ and $y^* := (T^*)^{-1}x^* = x^* \circ T^{-1} \in (1 + \delta)B_{Y^*}$.

Since Y is WOH, there exists, by Proposition 2.2 in [11], an element $y \in S_Y$ such that

$$\|y_i + ty\| \geq (1 - \delta)(|y^*(y_i)|/(1 + \delta) + t) \quad \forall i \in \{1, \dots, n\}, \quad t > 0. \quad (5.1)$$

Let $z := T^{-1}y/\|T^{-1}y\| \in S_X$. Then we have for each $i \in \{1, \dots, n\}$ and every $t > 0$

$$\|x_i + tz\| \geq \|x_i + tT^{-1}y\| - t\|T^{-1}y\| - 1 \geq \|y_i + ty\| - t\delta.$$

Combining this with (5.1) and observing that $y^*(y_i) = x^*(x_i)$ we obtain

$$\|x_i + tz\| \geq \|(1 + \delta)y_i + ty\| - \delta - t\delta \geq (1 - \delta)(|x^*(x_i)| + t) - (1 + t)\delta$$

for every $i \in \{1, \dots, n\}$ and every $t > 0$.

Now if $t \geq \varepsilon$, then by the choice of δ we obtain $(\sqrt{\delta} - 2\delta)t \geq \delta$, which implies $(1 - 2\delta)t - \delta \geq (1 - \sqrt{\delta})t$.

Thus for every $t \geq \varepsilon$ and every $i \in \{1, \dots, n\}$ we have

$$\|x_i + tz\| \geq (1 - \delta)|x^*(x_i)| + (1 - \sqrt{\delta})t \geq (1 - \varepsilon)(|x^*(x_i)| + t).$$

By Proposition 2.2 in [11] this implies that X is WOH. \square

Concerning WASQ spaces, using Lemma 4.3 one can show the following result (again an easy proof is omitted).

Proposition 5.2. *Let X be a Banach space such that for every $\delta > 0$ there is some WASQ space Y isomorphic to X with $d(X, Y) < 1 + \delta$. If X^* is separable, then X is also WASQ.*

Finally, we will also consider diameter-two-properties. We have already recalled the definition of the local-diameter-two-property (LD2P) in Section 3. Furthermore, the following terminology was introduced in [2]: a Banach space X has the *diameter-two-property* (D2P) if every nonempty, relatively weakly open subset of B_X has diameter 2, and X has the *strong diameter-two-property* (SD2P) if every convex combination of slices of B_X has diameter 2.

The following results were proved in [11]:

- (a) X has the LD2P $\iff X^*$ is LOH,
- (b) X has the D2P $\iff X^*$ is WOH,
- (c) X has the SD2P $\iff X^*$ is OH.

The result (c) was also proved independently in [3], but it was essentially pointed out already in [6] and [8].

Proposition 5.1 and the duality between diameter-two- and octahedrality properties, together with the fact that $d(X^*, Y^*) \leq d(X, Y)$ holds for all Banach spaces X and Y , immediately yield the following corollary.

Corollary 5.3. *Let X be a Banach space such that for every $\delta > 0$ there is a Banach space Y isomorphic to X which has the SD2P/D2P/LD2P and satisfies $d(X, Y) < 1 + \delta$. Then X also has the SD2P/D2P/LD2P.*

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