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Inequalities of Hermite–Hadamard type for *HH*-convex functions

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Abstract. Some inequalities of Hermite–Hadamard type for HH-convex functions defined on positive intervals are given. Applications for special means are also provided.

1. Introduction

Let $I \subset \mathbb{R} \backslash \{0\}$. Following [1] (see also [6]) we say that a function $f : I \to \mathbb{R}$ is *HA-convex* if

$$f\left(\frac{xy}{tx + (1-t)\,y}\right) \le (1-t)\,f(x) + tf(y) \tag{1.1}$$

for all $x, y \in I$ and $t \in [0,1]$. If the inequality in (1.1) is reversed, then f is said to be *HA-concave*.

If $I \subset (0,\infty)$ and f is convex and nondecreasing, then f is HA-convex, and if f is HA-convex and nonincreasing, then f is convex.

If $[a,b] \subset I \subset (0,\infty)$ and if we consider the function $g : [1/b, 1/a] \to \mathbb{R}$, defined by $g(t) = f(1/t)$, then we can state the following fact.

Lemma 1 (see [1]). *The function f is HA-convex (concave) on $[a,b]$ if and only if g is convex (concave) in the usual sense on $[1/b, 1/a]$.*

Therefore, as examples of HA-convex functions we can take $f(t) = g(1/t)$, where g is any convex function on $[1/b, 1/a]$.

In the recent paper [5] we obtained the following characterization result as well.

Lemma 2. *Let $[a,b] \subset (0,\infty)$ and let $f, h : [a,b] \to \mathbb{R}$ be so that $h(t) = tf(t)$ for $t \in [a,b]$. Then f is HA-convex (concave) on the interval $[a,b]$ if and only if h is convex (concave) on $[a,b]$.*

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Following [1], we say that a function $f : I \to (0, \infty)$ with $I \subset \mathbb{R} \setminus \{0\}$ is *HH-convex* if

$$f\left(\frac{xy}{tx + (1-t)\,y}\right) \le \frac{f(x)\,f(y)}{(1-t)\,f(y) + tf(x)} \tag{1.2}$$

for all $x, y \in I$ and $t \in [0,1]$. If the inequality in (1.2) is reversed, then f is said to be *HH-concave.*

We observe that the inequality (1.2) is equivalent to

$$(1-t)\,\frac{1}{f(x)} + t\frac{1}{f(y)} \le \frac{1}{f\left(\frac{xy}{tx+(1-t)y}\right)} \tag{1.3}$$

for all $x, y \in I$ and $t \in [0,1]$. Therefore we have the following fact.

Lemma 3. *A function $f : I \to (0, \infty)$ is HH-convex (concave) on I if and only if $g : I \to (0, \infty)$, $g(x) = \frac{1}{f(x)}$, is HA-concave (convex) on I.*

Taking into account the above lemmas, we can state the following result.

Proposition 1. *Let $f : [a, b] \to (0, \infty)$, where $[a, b] \subset (0, \infty)$. Define the related functions*

$$P_f : [1/b, 1/a] \to (0, \infty)\,, \quad P_f(x) = \frac{1}{f\left(\frac{1}{x}\right)},$$

and

$$Q_f : [a, b] \to (0, \infty)\,, \quad Q_f(x) = \frac{x}{f(x)}.$$

The following statements are equivalent:

(i) *the function f is HH-convex (concave) on $[a, b]$;*
(ii) *the function P_f is concave (convex) on $[1/b, 1/a]$;*
(iii) *the function Q_f is concave (convex) on $[a, b]$.*

For a convex function $h : [c, d] \to \mathbb{R}$, the following inequality is well known in the literature as the *Hermite–Hadamard inequality*:

$$h\left(\frac{c+d}{2}\right) \le \frac{1}{d-c}\int_c^d h(t)\,dt \le \frac{h(c) + h(d)}{2}. \tag{1.4}$$

For related results and references, see e.g. [4].

Motivated by the above results, we establish in this paper some inequalities of Hermite–Hadamard type for *HH*-convex functions defined on positive intervals. Applications for special means are also provided.

2. The results

We have the following result that can be obtained by the use of the regular Hermite–Hadamard inequality (1.4).

Theorem 1. *Let $f : [a,b] \to (0,\infty)$ be an HH-convex (concave) function on $[a,b] \subset (0,\infty)$. Then we have*

$$f\left(\frac{2ab}{a+b}\right) \geq (\leq) \frac{ab}{b-a}\int_a^b \frac{1}{t^2 f(t)}dt \geq (\leq) \frac{f(b)+f(a)}{2} \tag{2.1}$$

and

$$\frac{\frac{a+b}{2}}{f\left(\frac{a+b}{2}\right)} \geq (\leq) \frac{1}{b-a}\int_a^b \frac{t}{f(t)}dt \geq (\leq) \frac{af(b)+bf(a)}{2f(a)f(b)}. \tag{2.2}$$

Proof. Since f is *HH*-convex (concave) on $[a,b]$, by Proposition 1 we have that P_f is concave (convex) on $[1/b, 1/a]$. By Hermite–Hadamard inequality (1.4) for P_f we have

$$f\left(\frac{1}{\frac{1/a+1/b}{2}}\right) \geq (\leq) \frac{1}{1/a-1/b}\int_{1/b}^{1/a} \frac{1}{f(1/s)}ds \geq (\leq) \frac{f\left(\frac{1}{1/b}\right)+f\left(\frac{1}{1/a}\right)}{2},$$

which is equivalent to

$$f\left(\frac{2ab}{a+b}\right) \geq (\leq) \frac{ab}{b-a}\int_{1/b}^{1/a} \frac{1}{f(1/s)}ds \geq (\leq) \frac{f(b)+f(a)}{2}. \tag{2.3}$$

If we make the change of variable $1/s = t$, then $s = 1/t$ and $ds = -dt/t^2$ and from (2.3) we get (2.1).

Since f is *HH*-convex (concave) on $[a,b]$, by Proposition 1 we also have that Q_f is concave (convex) on $[a,b]$. By Hermite–Hadamard inequality (1.4) for Q_f we have

$$\frac{\frac{a+b}{2}}{f\left(\frac{a+b}{2}\right)} \geq (\leq) \frac{1}{b-a}\int_a^b \frac{t}{f(t)}dt \geq (\leq) \frac{\frac{a}{f(a)}+\frac{b}{f(b)}}{2},$$

which is equivalent to (2.2). □

We use the following result obtained by the author in [2] and [3].

Lemma 4. *Let $h : [\alpha,\beta] \to \mathbb{R}$ be a convex (concave) function on $[\alpha,\beta]$. Then we have the inequalities*

$$0 \leq (\geq) \frac{h(\alpha)+h(\beta)}{2} - \frac{1}{\beta-\alpha}\int_\alpha^\beta h(t)\,dt \leq (\geq) \frac{1}{8}\left[h'_-(\beta)-h'_+(\alpha)\right](\beta-\alpha) \tag{2.4}$$

and

$$0 \le (\ge) \frac{1}{\beta - \alpha} \int_\alpha^\beta h(t)\,dt - h\left(\frac{\alpha+\beta}{2}\right) \le (\ge) \frac{1}{8}\left[h'_-(\beta) - h'_+(\alpha)\right](\beta - \alpha). \tag{2.5}$$

The constant $1/8$ *is best possible in* (2.4) *and* (2.5).

We have the following reverse inequalities.

Theorem 2. *Let* $f : [a,b] \to (0,\infty)$ *be an HH-convex (concave) function on* $[a,b] \subset (0,\infty)$*. Then we have*

$$\begin{aligned} 0 &\ge (\le) \frac{f(b)+f(a)}{2} - \frac{ab}{b-a}\int_a^b \frac{1}{t^2 f(t)}dt \\ &\ge (\le) \frac{1}{8ab}\left(\frac{a^2}{f^2(a)}f'_+(a) - \frac{b^2}{f^2(b)}f'_-(b)\right)(b-a), \end{aligned} \tag{2.6}$$

$$\begin{aligned} 0 &\ge (\le) \frac{ab}{b-a}\int_a^b \frac{1}{t^2 f(t)}dt - f\left(\frac{2ab}{a+b}\right) \ge (\le) \\ &\ge (\le) \frac{1}{8ab}\left(\frac{a^2}{f^2(a)}f'_+(a) - \frac{b^2}{f^2(b)}f'_-(b)\right)(b-a), \end{aligned} \tag{2.7}$$

$$\begin{aligned} 0 &\ge (\le) \frac{af(b)+bf(a)}{2f(a)f(b)} - \frac{1}{b-a}\int_a^b \frac{t}{f(t)}dt \\ &\ge (\le) \frac{1}{8}\left(\frac{f(b)-bf'_-(b)}{f^2(b)} - \frac{f(a)-af'_+(a)}{f^2(a)}\right)(b-a) \end{aligned} \tag{2.8}$$

and

$$\begin{aligned} 0 &\ge (\le) \frac{1}{b-a}\int_a^b \frac{t}{f(t)}dt - \frac{\frac{a+b}{2}}{f\left(\frac{a+b}{2}\right)} \\ &\ge (\le) \frac{1}{8}\left(\frac{f(b)-bf'_-(b)}{f^2(b)} - \frac{f(a)-af'_+(a)}{f^2(a)}\right)(b-a). \end{aligned} \tag{2.9}$$

Proof. The first part in all inequalities (2.6)–(2.9) follows from Theorem 1.

Now, if we take the derivative of $P_f(x)$, then we have

$$\begin{aligned} P'_f(x) &= \left(\frac{1}{f\left(\frac{1}{x}\right)}\right)' = \left(f^{-1}\left(\frac{1}{x}\right)\right)' = -f^{-2}\left(\frac{1}{x}\right)\left(f\left(\frac{1}{x}\right)\right)' \\ &= -f^{-2}\left(\frac{1}{x}\right)f'\left(\frac{1}{x}\right)\left(-\frac{1}{x^2}\right) = f^{-2}\left(\frac{1}{x}\right)f'\left(\frac{1}{x}\right)\left(\frac{1}{x^2}\right). \end{aligned}$$

Therefore we have

$$P'_{+f}\left(\frac{1}{b}\right) = b^2 f^{-2}(b)\, f'_-(b) = \frac{b^2}{f^2(b)}f'_-(b)$$

and

$$P'_{-f}\left(\frac{1}{a}\right) = a^2 f^{-2}(a)\, f'_+(a) = \frac{a^2}{f^2(a)} f'_+(a)$$

and by the right hand side inequalities in Lemma 4 we get the corresponding inequalities in (2.6) and (2.7).

If we take the derivative of Q_f, we have

$$Q'_f(x) = \left(\frac{x}{f(x)}\right)' = \frac{f(x) - x f'(x)}{f^2(x)}.$$

Therefore

$$Q'_{+f}(a) = \frac{f(a) - a f'_+(a)}{f^2(a)} \quad \text{and} \quad Q'_{-f}(b) = \frac{f(b) - b f'_-(b)}{f^2(b)},$$

and by the right hand side inequalities in Lemma 4 we get the corresponding inequalities in (2.8) and (2.9). □

Theorem 3. *Let $f : [a,b] \to (0,\infty)$ be an HH-convex (concave) function on $[a,b] \subset (0,\infty)$. Then we have*

$$\frac{ab}{b-a}\int_a^b \frac{f(t)}{t^2}dt \leq (\geq) \frac{G^2(f(a), f(b))}{L(f(a), f(b))}. \tag{2.10}$$

Proof. By the definition of *HH*-convex (concave) function, we have by integrating on $[0,1]$ over λ, that

$$\int_0^1 f\left(\frac{ab}{(1-\lambda)b + \lambda a}\right) d\lambda \leq (\geq) \int_0^1 \frac{f(a) f(b)}{(1-\lambda) f(b) + \lambda f(a)} d\lambda. \tag{2.11}$$

Consider the change of variable $\frac{ab}{(1-\lambda)b+\lambda a} = t$. Then $(1-\lambda)b + \lambda a = \frac{ab}{t}$ and $(b-a)\, d\lambda = \frac{ab}{t^2}dt$. Using this change of variable, we have

$$\int_0^1 f\left(\frac{ab}{(1-\lambda)b + \lambda a}\right) d\lambda = \frac{ab}{b-a}\int_a^b \frac{f(t)}{t^2}dt.$$

If $f(b) = f(a)$, then

$$\int_0^1 \frac{f(a) f(b)}{(1-\lambda) f(b) + \lambda f(a)} d\lambda = f(a).$$

If $f(b) \neq f(a)$, then by the change of variable $(1-\lambda) f(b) + \lambda f(a) = s$, we have

$$\begin{aligned}\int_0^1 \frac{f(a) f(b)}{(1-\lambda) f(b) + \lambda f(a)} d\lambda &= \frac{f(a) f(b)}{f(a) - f(b)} \int_{f(b)}^{f(a)} \frac{ds}{s} \\ &= \frac{f(a) f(b)}{L(f(a), f(b))} = \frac{G^2(f(a), f(b))}{L(f(a), f(b))}.\end{aligned}$$

By making use of (2.11) we deduce the desired result (2.10). □

We also have the following theorem.

Theorem 4. *Let $f:[a,b]\to(0,\infty)$ be an HH-convex (concave) function on $[a,b]\subset(0,\infty)$. Then we have*

$$f\left(\frac{2ab}{a+b}\right)\leq(\geq)\frac{\int_a^b\frac{1}{t^2}f(t)\,f\left(\frac{abt}{(a+b)t-ab}\right)dt}{\int_a^b\frac{f(t)}{t^2}dt}. \tag{2.12}$$

Proof. From the definition of an *HH*-convex (concave) function we have

$$f\left(\frac{2xy}{x+y}\right)\leq(\geq)\frac{2f(x)\,f(y)}{f(x)+f(y)} \tag{2.13}$$

for any $x,y\in[a,b]$.

If we take

$$x=\frac{ab}{(1-\lambda)\,b+\lambda a},\ y=\frac{ab}{(1-\lambda)\,a+\lambda b}\in[a,b],$$

then

$$\begin{aligned}\frac{2xy}{x+y}&=\frac{2\frac{ab}{(1-\lambda)b+\lambda a}\cdot\frac{ab}{(1-\lambda)a+\lambda b}}{\frac{ab}{(1-\lambda)b+\lambda a}+\frac{ab}{(1-\lambda)a+\lambda b}}=\frac{2\frac{1}{(1-\lambda)b+\lambda a}\cdot\frac{ab}{(1-\lambda)a+\lambda b}}{\frac{1}{(1-\lambda)b+\lambda a}+\frac{1}{(1-\lambda)a+\lambda b}}\\&=\frac{2\frac{1}{(1-\lambda)b+\lambda a}\cdot\frac{ab}{(1-\lambda)a+\lambda b}}{\frac{(1-\lambda)a+\lambda b+(1-\lambda)b+\lambda a}{((1-\lambda)b+\lambda a)((1-\lambda)a+\lambda b)}}=\frac{2ab}{a+b},\end{aligned}$$

and by (2.13) we get

$$f\left(\frac{2ab}{a+b}\right)\leq(\geq)\frac{2f\left(\frac{ab}{(1-\lambda)b+\lambda a}\right)f\left(\frac{ab}{(1-\lambda)a+\lambda b}\right)}{f\left(\frac{ab}{(1-\lambda)b+\lambda a}\right)+f\left(\frac{ab}{(1-\lambda)a+\lambda b}\right)},$$

which is equivalent to

$$\begin{aligned}&f\left(\frac{2ab}{a+b}\right)\left[f\left(\frac{ab}{(1-\lambda)\,b+\lambda a}\right)+f\left(\frac{ab}{(1-\lambda)\,a+\lambda b}\right)\right]\\&\qquad\leq(\geq)\,2f\left(\frac{ab}{(1-\lambda)\,b+\lambda a}\right)f\left(\frac{ab}{(1-\lambda)\,a+\lambda b}\right),\end{aligned} \tag{2.14}$$

for any $\lambda\in[0,1]$.

If we integrate the inequality over λ on $[0,1]$ we get

$$\begin{aligned}&f\left(\frac{2ab}{a+b}\right)\left[\int_0^1 f\left(\frac{ab}{(1-\lambda)\,b+\lambda a}\right)d\lambda+\int_0^1 f\left(\frac{ab}{(1-\lambda)\,a+\lambda b}\right)d\lambda\right]\\&\qquad\leq(\geq)\,2\int_0^1 f\left(\frac{ab}{(1-\lambda)\,b+\lambda a}\right)f\left(\frac{ab}{(1-\lambda)\,a+\lambda b}\right)d\lambda.\end{aligned} \tag{2.15}$$

Now, we observe that

$$\int_0^1 f\left(\frac{ab}{(1-\lambda)a+\lambda b}\right)d\lambda = \int_0^1 f\left(\frac{ab}{(1-\lambda)b+\lambda a}\right)d\lambda = \frac{ab}{b-a}\int_a^b \frac{f(t)}{t^2}dt \tag{2.16}$$

and

$$\begin{aligned}&\int_0^1 f\left(\frac{ab}{(1-\lambda)b+\lambda a}\right)f\left(\frac{ab}{(1-\lambda)a+\lambda b}\right)d\lambda\\ &= \int_0^1 f\left(\frac{ab}{(1-\lambda)b+\lambda a}\right)f\left(\frac{ab}{a+b-((1-\lambda)b+\lambda a)}\right)d\lambda\\ &= \int_0^1 f\left(\frac{ab}{(1-\lambda)b+\lambda a}\right)f\left(\frac{1}{\frac{1}{b}+\frac{1}{a}-\frac{((1-\lambda)b+\lambda a)}{ab}}\right)d\lambda.\end{aligned} \tag{2.17}$$

If we change the variable $t=\frac{ab}{(1-\lambda)b+\lambda a}$, then we have

$$\begin{aligned}&\int_0^1 f\left(\frac{ab}{(1-\lambda)b+\lambda a}\right)f\left(\frac{1}{\frac{1}{b}+\frac{1}{a}-\frac{((1-\lambda)b+\lambda a)}{ab}}\right)d\lambda\\ &= \frac{ab}{b-a}\int_a^b \frac{1}{t^2}f(t)f\left(\frac{1}{\frac{1}{b}+\frac{1}{a}-\frac{1}{t}}\right)dt\\ &= \frac{ab}{b-a}\int_a^b \frac{1}{t^2}f(t)f\left(\frac{abt}{(a+b)t-ab}\right)dt.\end{aligned} \tag{2.18}$$

On making use of (2.15)–(2.18) we deduce the desired result (2.12). □

Remark 1. By Cauchy–Bunyakovsky–Schwarz integral inequality we have

$$\begin{aligned}&\int_0^1 f\left(\frac{ab}{(1-\lambda)b+\lambda a}\right)f\left(\frac{ab}{(1-\lambda)a+\lambda b}\right)d\lambda\\ &\le \left(\int_0^1 f^2\left(\frac{ab}{(1-\lambda)b+\lambda a}\right)d\lambda\right)^{1/2}\\ &\quad\times\left(\int_0^1 f^2\left(\frac{ab}{(1-\lambda)a+\lambda b}\right)d\lambda\right)^{1/2}\\ &= \int_0^1 f^2\left(\frac{ab}{(1-\lambda)b+\lambda a}\right)d\lambda = \int_a^b \frac{f^2(t)}{t^2}dt.\end{aligned} \tag{2.19}$$

Now, if $f:[a,b]\to\mathbb{R}$ is an HH-convex function on $[a,b]\subset(0,\infty)$, then by (2.12) and (2.19) we get

$$f\left(\frac{2ab}{a+b}\right) \le \frac{\int_a^b \frac{1}{t^2}f(t)f\left(\frac{abt}{(a+b)t-ab}\right)dt}{\int_a^b \frac{f(t)}{t^2}dt} \le \frac{\int_a^b \frac{f^2(t)}{t^2}dt}{\int_a^b \frac{f(t)}{t^2}dt}. \tag{2.20}$$

The following lemma is of interest as well.

Lemma 5. *If $f:[a,b]\to(0,\infty)$ is HH-convex on $[a,b]\subset(0,\infty)$, then the associated function $R_f:[a,b]\to(0,\infty)$, $R_f(x)=\frac{f(x)}{x}$, is convex on $[a,b]$. The reverse is not true.*

Proof. Let $\alpha,\beta>0$ with $\alpha+\beta=1$ and $x,y\in[a,b]$.
By the HH-convexity of f we have

$$\begin{aligned}R_f(\alpha x+\beta y)&=\frac{f(\alpha x+\beta y)}{\alpha x+\beta y}=\frac{f\left(\frac{1}{\frac{1}{\alpha x+\beta y}}\right)}{\alpha x+\beta y}=\frac{f\left(\frac{1}{\frac{\alpha x\frac{1}{x}+\beta y\frac{1}{y}}{\alpha x+\beta y}}\right)}{\alpha x+\beta y}\\&\leq\frac{\frac{1}{\frac{\alpha x\frac{1}{f(x)}+\beta y\frac{1}{f(y)}}{\alpha x+\beta y}}}{\alpha x+\beta y}=\frac{\alpha x+\beta y}{\alpha x\frac{1}{f(x)}+\beta y\frac{1}{f(y)}}\cdot\frac{1}{\alpha x+\beta y}\\&=\frac{1}{\alpha\frac{x}{f(x)}+\beta\frac{y}{f(y)}}.\end{aligned}\tag{2.21}$$

By the weighted Cauchy–Bunyakovsky–Schwarz inequality we have

$$\begin{aligned}&\left(\alpha\frac{x}{f(x)}+\beta\frac{y}{f(y)}\right)\left(\alpha\frac{f(x)}{x}+\beta\frac{f(y)}{y}\right)=\left(\alpha\left(\sqrt{\frac{x}{f(x)}}\right)^2\right.\\&\qquad\left.+\beta\left(\sqrt{\frac{y}{f(y)}}\right)^2\right)\left(\alpha\left(\sqrt{\frac{f(x)}{x}}\right)^2+\beta\left(\sqrt{\frac{f(y)}{y}}\right)^2\right)\\&\geq(\alpha+\beta)^2=1,\end{aligned}$$

which implies that

$$\frac{1}{\alpha\frac{x}{f(x)}+\beta\frac{y}{f(y)}}\leq\alpha\frac{f(x)}{x}+\beta\frac{f(y)}{y}$$

for any $\alpha,\beta>0$ with $\alpha+\beta=1$ and $x,y\in[a,b]$.

By (2.21) we have

$$R_f(\alpha x+\beta y)\leq\alpha\frac{f(x)}{x}+\beta\frac{f(y)}{y}=\alpha R_f(x)+\beta R_f(y)$$

for any $\alpha,\beta>0$ with $\alpha+\beta=1$ and $x,y\in[a,b]$, which shows that R_f is convex on $[a,b]$.

Consider the function $f:[a,b]\to(0,\infty)$, $f(x)=x^p$, $p\neq0$. The function $R_f(x)=x^{p-1}$ is convex if and only if $p\in(-\infty,1)\cup[2,\infty)$. Since $Q_f(x)=x^{1-p}$ is concave if and only if $p\in(0,1)$, by Proposition 1 we have that the function $f:[a,b]\to(0,\infty)$, $f(x)=x^p$, is HH-convex if and only if Q_f is concave, namely $p\in(0,1)$. Therefore, R_f is convex and not HH-convex if $p\in(-\infty,0)\cup[2,\infty)$. □

If we denote by $\mathcal{C}_I[a,b]$ the class of all positive functions f for which R_f is convex, then the class of HH-convex functions $f:[a,b]\to(0,\infty)$ on $[a,b]\subset(0,\infty)$ is strictly enclosed in $\mathcal{C}_I[a,b]$.

We have the following inequalities of Hermite–Hadamard type.

Theorem 5. *If $f\in\mathcal{C}_I[a,b]$, then we have*

$$\frac{f\left(\frac{a+b}{2}\right)}{\frac{a+b}{2}}\le\frac{1}{b-a}\int_a^b\frac{f(t)}{t}dt\le\frac{f(a)\,b+f(b)\,a}{2ab},\tag{2.22}$$

$$\begin{aligned}0&\le\frac{f(a)\,b+f(b)\,a}{2ab}-\frac{1}{b-a}\int_a^b\frac{f(t)}{t}dt\\&\le\frac{1}{8}\left[\frac{f'_-(b)\,b-f(b)}{b^2}-\frac{f'_+(a)\,a-f(a)}{a^2}\right](b-a)\end{aligned}\tag{2.23}$$

and

$$\begin{aligned}0&\le\frac{1}{b-a}\int_a^b\frac{f(t)}{t}dt-\frac{f\left(\frac{a+b}{2}\right)}{\frac{a+b}{2}}\\&\le\frac{1}{8}\left[\frac{f'_-(b)\,b-f(b)}{b^2}-\frac{f'_+(a)\,a-f(a)}{a^2}\right](b-a).\end{aligned}\tag{2.24}$$

Proof. By the Hermite–Hadamard inequalities (1.4) for R_f we have

$$\frac{f\left(\frac{a+b}{2}\right)}{\frac{a+b}{2}}\le\frac{1}{b-a}\int_a^b\frac{f(t)}{t}dt\le\frac{\frac{f(a)}{a}+\frac{f(b)}{b}}{2}$$

and the inequality (2.22) is proved.

We have

$$R'_f(t)=\left(\frac{f(t)}{t}\right)'=\frac{f'(t)\,t-f(t)}{t^2}$$

and then

$$R'_{-f}(b)=\frac{f'_-(b)\,b-f(b)}{b^2}\quad\text{and}\quad R'_{+f}(a)=\frac{f'_+(a)\,a-f(a)}{a^2}.$$

By Lemma 4 we have

$$\begin{aligned}0&\le\frac{\frac{f(a)}{a}+\frac{f(b)}{b}}{2}-\frac{1}{b-a}\int_a^b\frac{f(t)}{t}dt\\&\le\frac{1}{8}\left[\frac{f'_-(b)\,b-f(b)}{b^2}-\frac{f'_+(a)\,a-f(a)}{a^2}\right](b-a)\end{aligned}$$

and

$$\begin{aligned}0&\le\frac{1}{b-a}\int_a^b\frac{f(t)}{t}dt-\frac{f\left(\frac{a+b}{2}\right)}{\frac{a+b}{2}}\\&\le\frac{1}{8}\left[\frac{f'_-(b)\,b-f(b)}{b^2}-\frac{f'_+(a)\,a-f(a)}{a^2}\right](b-a),\end{aligned}$$

which are equivalent to the desired inequalities (2.23) and (2.24). □

3. Applications

Consider the function $f:[a,b]\to(0,\infty)$, $f(x)=x^p$, on $[a,b]\subset(0,\infty)$. Observe that $Q_f(x)=x^{p-1}$ is convex if and only if $p\in(-\infty,1)\cup[2,\infty)$ and concave if and only if $p\in(1,2)$. By Proposition 1 we have that the function f is HH-convex (concave) on $[a,b]$ if and only if Q_f is concave (convex) on $[a,b]$, namely $p\in(1,2)$ $(p\in(-\infty,1)\cup[2,\infty))$.

We introduce the L_q-*harmonic mean* for $q\neq 0,-1$ by

$$L_q(a,b):=\begin{cases}\left(\frac{b^{q+1}-a^{q+1}}{(q+1)(b-a)}\right)^{\frac{1}{q}} & \text{if } b\neq a,\\ b & \text{if } b=a,\end{cases}$$

the *logarithmic mean* by

$$L(a,b):=\begin{cases}\frac{b-a}{\ln b-\ln a} & \text{if } b\neq a,\\ b & \text{if } b=a,\end{cases}$$

and the *identric mean* by

$$I(a,b):=\begin{cases}\frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}} & \text{if } b\neq a,\\ b & \text{if } b=a.\end{cases}$$

If we set $L_0(a,b):=I(a,b)$ and $L_{-1}(a,b):=L(a,b)$, then we have that the function $\mathbb{R}\ni q\mapsto L_q(a,b)$ is monotonic increasing as a function of q. We also have the inequalities

$$H(a,b)\leq G(a,b)\leq L(a,b)\leq I(a,b)\leq A(a,b).$$

By making use of Theorem 1 we have for $p\in(1,2)$ $(p\in(-\infty,1)\cup[2,\infty))$ that

$$\frac{H^p(a,b)}{G^2(a,b)}\geq(\leq)\,L_{-p-2}^{-p-2}(a,b)\geq(\leq)\,\frac{A(a^p,b^p)}{G^2(a,b)},\ p\neq 0 \tag{3.1}$$

and

$$A^{1-p}(a,b)\geq(\leq)\,L_{1-p}^{1-p}(a,b)\geq(\leq)\,A\left(a^{p-1},b^{p-1}\right). \tag{3.2}$$

If we take $p=-1$ in (3.1), then we get

$$\frac{1}{G^2(a,b)\,H(a,b)}\leq\frac{1}{L(a,b)}\leq\frac{A\left(a^{-1},b^{-1}\right)}{G^2(a,b)}.$$

By Theorem 3 we have for $p\in(1,2)$ $(p\in(-\infty,1)\cup[2,\infty))$

$$G^2(a,b)\,L_{p-2}^{p-2}(a,b)\leq(\geq)\,\frac{G^2(a^p,b^p)}{L(a^p,b^p)}. \tag{3.3}$$

Observe that

$$
\begin{aligned}
L\left(a^p, b^p\right) &= \frac{b^p - a^p}{p\left(\ln b - \ln a\right)} = \frac{b^p - a^p}{p\left(b - a\right)} \cdot \frac{b - a}{\ln b - \ln a} \\
&= L_{p-1}^{p-1}\left(a, b\right) L\left(a, b\right).
\end{aligned}
$$

By (3.3) we get that

$$
L_{p-2}^{p-2}\left(a, b\right) L_{p-1}^{p-1}\left(a, b\right) \leq (\geq) \frac{G^2\left(a^p, b^p\right)}{G^2\left(a, b\right) L\left(a, b\right)}, \tag{3.4}
$$

for $p \in (1, 2)$ $(p \in (-\infty, 1) \cup (2, \infty))$.

Now, consider the function $f : [a, b] \to (0, \infty)$, $f(t) = \frac{t}{\ln t}$, on $[a, b] \subset (1, \infty)$. Then $Q_f(x) = x / \frac{x}{\ln x} = \ln x$ is concave on $[a, b]$, therefore f is *HH*-convex on $[a, b] \subset (1, \infty)$. If we use the inequality (2.2), then we get the well-known inequality

$$
A\left(a, b\right) \geq I\left(a, b\right) \geq G\left(a, b\right).
$$

If we use the inequality (2.10) for $f(t) = \frac{t}{\ln t}$, then we get

$$
\frac{ab}{b - a} \int_a^b \frac{1}{t \ln t} dt \leq \frac{G^2\left(\frac{a}{\ln a}, \frac{b}{\ln b}\right)}{L\left(\frac{a}{\ln a}, \frac{b}{\ln b}\right)}. \tag{3.5}
$$

Since

$$
\int_a^b \frac{1}{t \ln t} dt = \int_a^b \frac{1}{\ln t} d\left(\ln t\right) = \ln\left(\ln b\right) - \ln\left(\ln a\right),
$$

and

$$
G^2\left(\frac{a}{\ln a}, \frac{b}{\ln b}\right) = \frac{G^2\left(a, b\right)}{G^2\left(\ln a, \ln b\right)},
$$

from (3.5) we have

$$
G^2\left(\ln a, \ln b\right) L\left(\frac{a}{\ln a}, \frac{b}{\ln b}\right) \leq L\left(a, b\right) L\left(\ln b, \ln a\right). \tag{3.6}
$$

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