Matrix transformations related to $\mathcal{I}$-convergent sequences

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ABSTRACT. Characterized are matrix transformations related to certain subsets of the space of ideal convergent sequences. Obtained here results are connected with the previous investigations of the author on some transformations defined by infinite matrices of bounded linear operators.

1. Introduction and preliminaries

Let $\mathbb{N} = \{1, 2, \dots\}$ and let X, Y be normed spaces over the field $\mathbb{K}$ of real numbers $\mathbb{R}$ or complex numbers $\mathbb{C}$. As usual, a linear subset of the vector space $\omega(X)$ of all X-valued sequences is called a *sequence space*. A subset Φ of X is called *fundamental* if the linear span of Φ is dense in X. By $B(X, Y)$ we denote the space of all bounded linear operators from X to Y. We write $\sup_n$, $\lim_n$ and $\sum_n$ instead of $\sup_{n \in \mathbb{N}}$, $\lim_{n \to \infty}$ and $\sum_{n=1}^{\infty}$, respectively.

Let $\lambda(X)$ be a subset of $\omega(X)$, let $\mu(Y)$ be a subset of $\omega(Y)$, and let $\mathfrak{A} = (A_{nk})$ be an infinite matrix of operators $A_{nk} \in B(X, Y)$ $(n, k \in \mathbb{N})$. We say that $\mathfrak{A}$ *maps* $\lambda(X)$ *into* $\mu(Y)$, and write $\mathfrak{A} : \lambda(X) \to \mu(Y)$, if for all $\mathfrak{x} = (x_k) \in \lambda(X)$ the series $\mathfrak{A}_n \mathfrak{x} = \sum_k A_{nk} x_k$ $(n \in \mathbb{N})$ converge and the sequence $\mathfrak{A}\mathfrak{x} = (\mathfrak{A}_n \mathfrak{x})$ belongs to $\mu(Y)$. For a sequence (A_{nk}) we define so-called *group norms* (cf. [18], p. 5)

$$\|(A_{nk})\|_{n,m} := \sup_r \sup_{\|x_k\| \le 1} \left\| \sum_{k=m}^{r} A_{nk} x_k \right\| \quad (n, m \in \mathbb{N}).$$

It is known that the sets $c(X)$, $c_0(X)$ and $\ell_\infty(X)$ of all convergent, convergent to zero and bounded sequences $\mathfrak{x} = (x_k) \in \omega(X)$ are Banach sequence spaces with the norm $\|\mathfrak{x}\|_\infty = \sup_k \|x_k\|$, and the set $\ell_p(X)$ of

sequences $\mathfrak{x}$ such that $\sum_k \|x_k\|^p < \infty$ is a Banach space with the norm $\|\mathfrak{x}\|_p = (\sum_k \|x_k\|^p)^{1/p}$ if $1 \le p < \infty$.

For $x \in X$ and $k \in \mathbb{N}$, let $\mathfrak{e}(x) = (x, x, \dots)$ be a constant sequence and let $\mathfrak{e}^k(x) = (e_j^k(x))$ be the sequence with $e_j^k(x) = x$ if $j = k$ and $e_j^k(x) = 0$ otherwise. It is not difficult to see that if Φ is a (countable) fundamental set in X, then

$$\mathcal{E}_0(\Phi) := \{\mathfrak{e}^k(\phi) : k \in \mathbb{N},\ \phi \in \Phi\}$$

is a (countable) fundamental set in $c_0(X)$ and $\ell_p(X)$, and $\mathcal{E}_0(\Phi) \bigcup \mathcal{E}(\Phi)$ with

$$\mathcal{E}(\Phi) := \{\mathfrak{e}(\phi) : \phi \in \Phi\}$$

is a (countable) fundamental set in $c(X)$.

If a matrix map $\mathfrak{A}$ is defined on a Banach sequence space $\lambda(X)$, then the operators $\mathfrak{A}_n$ $(n \in \mathbb{N})$ defined above are linear and bounded, i.e., $\mathfrak{A}_n \in B(\lambda(X), Y)$. Therefore, by the investigation of matrix transformations the following two well-known theorems of functional analysis (see, for example, [8] or [17]) are useful.

Theorem 1.1 (Principle of uniform boundedness). *Let X, Y be Banach spaces and $A_n \in B(X, Y)$ $(n \in \mathbb{N})$. If $\sup_n \|A_n x\| < \infty$ for every $x \in X$, then*

$$\sup_n \|A_n\| < \infty. \tag{1.1}$$

Theorem 1.2 (Banach–Steinhaus). *Let X, Y be Banach spaces, $A_n \in B(X, Y)$ $(n \in \mathbb{N})$, and let Φ be a fundamental set of X. The limit $\lim_n A_n x$ exists for any $x \in X$ if and only if*

$$\sup_n \|A_n\| < \infty \tag{1.2}$$

and $\lim_n A_n \phi$ exists for every $\phi \in \Phi$. Moreover, the limit operator A, $Ax = \lim_n A_n x$ $(x \in X)$, is bounded and linear, i.e., $A \in B(X, Y)$, and $\|A\| \le \sup_n \|A_n\|$.

The equality $\lim_n A_n x = Ax$ $(x \in X)$ with $A \in B(X, Y)$ is true if and only if (1.2) holds and $\lim_n A_n \phi = A\phi$ $(\phi \in \Phi)$.

Based on Theorems 1.1 and 1.2, Zeller [23] (see also [18]) and Kangro [9] characterized the matrix transformations $\mathfrak{A}$ from $c(X)$, $c_0(X)$ and $\ell_1(X)$ to $c(Y)$ as follows.

Theorem 1.3. *Let X, Y be Banach spaces and let $\mathfrak{A} = (A_{nk})$ be an infinite matrix with $A_{nk} \in B(X, Y)$. Then the following statements hold.*

(i) $\mathfrak{A} : c(X) \to c(Y)$ *if and only if*

$$\||(A_{nk})\||_{n,1} < \infty \quad (n \in \mathbb{N}), \tag{1.3}$$

$$\exists \lim_m \sum_{k=1}^m A_{nk} x \quad (n \in \mathbb{N},\ x \in X), \tag{1.4}$$

$$\|\mathfrak{A}_n\| = O(1), \tag{1.5}$$

$$\exists \lim_n A_{nk}x =: A_k x \quad (k \in \mathbb{N}, \ x \in X), \tag{1.6}$$

$$\exists \lim_n \sum_k A_{nk}x \quad (x \in X). \tag{1.7}$$

(ii) $\mathfrak{A} : c_0(X) \to c(Y)$ *if and only if* (1.3) – (1.6) *hold.*

(iii) $\mathfrak{A} : \ell_1(X) \to c(Y)$ *if and only if* (1.6) *holds and*

$$H_n := \sup_k \|A_{nk}\| < \infty \quad (n \in \mathbb{N}), \tag{1.8}$$

$$H_n = O(1). \tag{1.9}$$

Necessary and sufficient conditions for the matrix transformation $\mathfrak{A} : \ell_\infty(X) \to c(Y)$ are contained in the following theorem of Maddox (see [18], Theorem 4.6; cf. also [9], Theorem 2).

Theorem 1.4. *Let* X, Y *be Banach spaces and let* $\mathfrak{A} = (A_{nk})$ *be an infinite matrix with* $A_{nk} \in B(X, Y)$. *Then* $\mathfrak{A} : \ell_\infty(X) \to c(Y)$ *if and only if* (1.3) – (1.7) *are satisfied and*

$$\lim_m \||(A_{nk})\||_{n,m} = 0 \quad (n \in \mathbb{N}),$$

$$\lim_m \sup_n \||(A_{nk} - A_k)\||_{n,m} = 0.$$

Remark 1.5. It is not difficult to see, using Theorem 1.2, that in Theorems 1.3 and 1.4 it suffices to require the fulfillment of conditions (1.4), (1.6) and (1.7) only for all elements ϕ from a fundamental set Φ of X.

The classical summability theory deals mostly with the transformations defined by infinite matrices of real or complex numbers. Characterizations of such matrix transformations from (and to) various spaces of number sequences may be found, for example, in [22].

As a generalization of usual convergence, Fast [4] (see also [21] and [20]) introduced the statistical convergence of number sequences in terms of asymptotic density of subsets of $\mathbb{N}$. Later several applications and generalizations of this notion have been investigated (for references, see [2] and [3]). For instance, Maddox [19] and Kolk [10, 11] considered the statistical convergence of sequences taking values in a locally convex space and a normed space, respectively. Another extension of statistical convergence is related to generalized densities.

Let $T = (t_{nk})$ be a non-negative regular matrix of scalars (i.e., $t_{nk} \geq 0$ $(n, k \in \mathbb{N})$ and $\lim_n \sum_k t_{nk} u_k = \lim_k u_k$ for any convergent scalar sequence (u_k)). A set $K \subset \mathbb{N}$ is said to have *T-density* $\delta_T(K)$ if the limit

$$\delta_T(K) := \lim_n \sum_{k \in K} t_{nk}$$

exists (cf. [6]).

A sequence $\mathfrak{x} = (x_k) \in \omega(X)$ is called T-*statistically convergent* to a point $l \in X$, briefly $st_T\text{-}\lim_k x_k = l$, if

$$\delta_T(\{k : \|x_k - l\| \geq \varepsilon\}) = 0$$

for every $\varepsilon > 0$ (see [1], Definition 7; [11], p. 44).

If T is the identity matrix, then the T-statistical convergence is just the usual convergence in X, and if T is the Cesàro matrix C_1, then the T-statistical convergence is just the statistical convergence as defined by Fast [4].

A further extension of statistical convergence was given in [16] by means of ideals. Recall that a subfamily $\mathcal{I}$ of the family $2^{\mathbb{N}}$ of all subsets of $\mathbb{N}$ is an *ideal* if for each $K, L \in \mathcal{I}$ we have $K \cup L \in \mathcal{I}$ and for each $K \in \mathcal{I}$ and each $L \subset K$ we have $L \in \mathcal{I}$. An ideal $\mathcal{I}$ is called *non-trivial* if $\mathcal{I} \neq \emptyset$ and $\mathbb{N} \notin \mathcal{I}$. A non-trivial ideal $\mathcal{I}$ is called *admissible* if $\mathcal{I}$ contains all finite subsets of $\mathbb{N}$. Any non-trivial ideal $\mathcal{I}$ defines a *filter*

$$\mathcal{F}(\mathcal{I}) := \{K \subset \mathbb{N} : \mathbb{N} \setminus K \in \mathcal{I}\}.$$

A sequence $\mathfrak{x} = (x_k) \in \omega(X)$ is said to be $\mathcal{I}$-*convergent* to $l \in X$, briefly $\mathcal{I}\text{-}\lim_k x_k = l$, if for each $\varepsilon > 0$ the set $\{k \in \mathbb{N} : \|x_k - l\| \geq \varepsilon\}$ belongs to $\mathcal{I}$ (see [16], Definition 3.1). For example,

$$\mathcal{I}_T := \{K \subset \mathbb{N} : \delta_T(K) = 0\}$$

is an admissible ideal and the $\mathcal{I}_T$-convergence coincides with the T-statistical convergence.

The following two notions are closely related with the $\mathcal{I}$-convergence. A sequence $\mathfrak{x} = (x_k) \in \omega(X)$ is said to be $\mathcal{I}^*$-*convergent* to $l \in X$, briefly $\mathcal{I}^*\text{-}\lim x_k = l$, if there exists an index set $K = (k_i)$ such that $K \in \mathcal{F}(\mathcal{I})$ and $\lim_i x_{k_i} = l$ in X (see [16], Definition 3.2). A sequence $\mathfrak{x} = (x_k) \in \omega(X)$ is said to be $\mathcal{I}$-*bounded*, briefly $\|x_k\| = O_{\mathcal{I}}(1)$, if for some $K = (k_i) \in \mathcal{F}(\mathcal{I})$ the subsequence (x_{k_i}) is bounded in X (cf. [7]).

We remark that the $\mathcal{I}^*$-convergence of number sequences was introduced already by Freedman [5] as $\mathcal{I}$-near convergence.

It is easy to see that $\mathcal{I}^*$-convergence implies $\mathcal{I}$-convergence and every $\mathcal{I}^*$-convergent sequence is $\mathcal{I}$-bounded.

An admissible ideal $\mathcal{I} \subset 2^{\mathbb{N}}$ is said to have *property* (AP) if for every countable family of mutually disjoint sets $K_1, K_2, \ldots$ from $\mathcal{I}$ there exist sets $L_1, L_2, \ldots$ from $2^{\mathbb{N}}$ such that the symmetric differences $K_i \Delta L_i$ $(i \in \mathbb{N})$ are finite and $L = \cup_i L_i \in \mathcal{I}$. It is known that the ideal $\mathcal{I}_T$ defined above has the property (AP) (see [6], Proposition 3.2).

The following characterization of $\mathcal{I}$-convergence is important for us (see [16], Theorem 3.2).

Proposition 1.6. *Let $\mathcal{I}$ be an admissible ideal with property* (AP). *If $\mathcal{I}$-$\lim x_k = l$ in a Banach space X, then $\mathcal{I}^*$-$\lim x_k = l$ in X.*

By the investigation of matrix transformations related to the $\mathcal{I}$-convergence, the sets $c^{\mathcal{I}}(X)$, $c_0^{\mathcal{I}}(X)$ and $\ell_\infty^{\mathcal{I}}(X)$ of all $\mathcal{I}$-convergent, $\mathcal{I}$-convergent to zero and $\mathcal{I}$-bounded sequences $\mathfrak{x} \in \omega(X)$ appear instead of $c(X)$, $c_0(X)$ and $\ell_\infty(X)$, respectively. For $X = \mathbb{K}$ we omit the symbol X in notations.

In the following two sections we characterize matrix transformations $\mathfrak{A}$ related to some subsets of $c^{\mathcal{I}}(X)$.

2. Matrix transformations of $c^{\mathcal{I}}(X)$

Let X, Y be Banach spaces, and let $\mathfrak{A} = (A_{nk})$ be an infinite matrix of operators $A_{nk} \in B(X, Y)$ $(n, k \in \mathbb{N})$. For a set $K = (k_i) \subset \mathbb{N}$ we define the K-*column-section* of $\mathfrak{A}$ as $\mathfrak{A}^{[K]} = (A_{nk}^{[K]})$, where, for any $n \in \mathbb{N}$, $A_{nk}^{[K]} = A_{nk}$ if $k \in K$ and $A_{nk}^{[K]} = 0$ otherwise. Analogously, the K-*section* of a sequence $\mathfrak{x} = (x_k)$ is defined by $\mathfrak{x}^{[K]} = (z_k)$, where $z_k = x_k$ if $k \in K$ and $z_k = 0$ otherwise.

Let $\mathcal{I} \subset 2^{\mathbb{N}}$ be an ideal. We say that a sequence space $\lambda(X)$ is $\mathcal{I}$-*section-closed* if for every $\mathfrak{x} \in \lambda(X)$ and for any $K \in \mathcal{I} \cup \mathcal{F}(\mathcal{I})$ we have $\mathfrak{x}^{[K]} \in \lambda(X)$.

Theorem 2.1. *Let $\mathcal{I}$ be an admissible ideal. Let $\lambda(X)$ be an $\mathcal{I}$-section-closed sequence space containing the set $\mathcal{E}(X) = \{\mathfrak{e}(x) : x \in X\}$, and let $\mu(Y)$ be an arbitrary sequence space. If $\mathfrak{A} : c^{\mathcal{I}}(X) \cap \lambda(X) \to \mu(Y)$, then*

$$\mathfrak{A} : c(X) \cap \lambda(X) \to \mu(Y), \tag{2.1}$$

$$\mathfrak{A}^{[K]} : \lambda(X) \to \mu(Y) \quad (K \in \mathcal{I}). \tag{2.2}$$

If $\mathcal{I}$ has property (AP), then (2.1) and (2.2) imply that $\mathfrak{A} : c^{\mathcal{I}}(X) \cap \lambda(X) \to \mu(Y)$.

Proof. Let $\mathfrak{A} : c^{\mathcal{I}}(X) \cap \lambda(X) \to \mu(Y)$. Then (2.1) holds in view of $c(X) \subset c^{\mathcal{I}}(X)$ because the ideal $\mathcal{I}$ is admissible.

Now, let $K \in \mathcal{I}$ and $\mathfrak{x} \in \lambda(X)$. The sequence $\mathfrak{y} = \mathfrak{x}^{[K]}$ is obviously $\mathcal{I}^*$-convergent to 0 and so, $\mathfrak{y} \in c^{\mathcal{I}}(X)$. Moreover, since $\lambda(X)$ is $\mathcal{I}$-section-closed, we have that $\mathfrak{y} \in \lambda(X)$. Thus $\mathfrak{y} \in c^{\mathcal{I}}(X) \cap \lambda(X)$ and so, $\mathfrak{A}\mathfrak{y} \in \mu(Y)$. By $\mathfrak{A}_n^{[K]}\mathfrak{x} = \mathfrak{A}_n\mathfrak{y}$ $(n \in \mathbb{N})$ we get $\mathfrak{A}^{[K]}\mathfrak{x} \in \mu(Y)$, i.e., (2.2) holds.

Conversely, suppose that (2.1) and (2.2) hold and $\mathcal{I}$ has property (AP). If $\mathfrak{x} \in c^{\mathcal{I}}(X) \cap \lambda(X)$, then for some $l \in X$ the sequence $\mathfrak{y} = (y_k)$ with $y_k = x_k - l$ is $\mathcal{I}$-convergent to 0 and, by Proposition 1.6, $\mathcal{I}^*$-$\lim y_k = 0$. Thus, for some $K \in \mathcal{I}$, the sequence $\mathfrak{z} = \mathfrak{y}^{[\mathbb{N}/K]}$ belongs to $c_0(X)$ which gives $\mathfrak{A}\mathfrak{z} \in \mu(Y)$ by (2.1). Further, since $\mathfrak{y} \in \lambda(X)$ because of $\mathcal{E}(X) \subset \lambda(X)$, by (2.2) we get $\mathfrak{A}^{[K]}\mathfrak{y} \in \mu(Y)$. Now, using the equality $\mathfrak{A}\mathfrak{y} = \mathfrak{A}\mathfrak{z} + \mathfrak{A}^{[K]}\mathfrak{y}$, we get $\mathfrak{A}\mathfrak{y} \in \mu(Y)$. But this shows that $\mathfrak{A}\mathfrak{x} \in \mu(Y)$ with $\mathcal{I}$-$\lim x_k = l$. $\square$

It is not difficult to see that $\ell_\infty(X)$ and $\ell_\infty^{\mathcal{I}}(X)$ are examples of $\mathcal{I}$-section-closed sequence spaces which contain $\mathcal{E}(X)$.

Let $st_T(X)$ denote the set of all T-statistically convergent sequences $\mathfrak{x} \in \omega(X)$. Since the ideal $\mathcal{I}_T$ is admissible and has property (AP), from Theorem 2.1 we immediately get a generalization of Theorem 4.1 from [12].

Proposition 2.2. *Let $T = (t_{nk})$ be a non-negative regular matrix of scalars. Assume that $\lambda(X)$ is an $\mathcal{I}_T$-section-closed sequence space containing $\mathcal{E}(X)$. Then, for an arbitrary sequence space $\mu(Y)$, $\mathfrak{A} : st_T(X) \cap \lambda(X) \to \mu(Y)$ if and only if (2.1) holds and*

$$\mathfrak{A}^{[K]} : \lambda(X) \to \mu(Y) \quad (\delta_T(K) = 0).$$

Theorem 2.1 reduces, for $\lambda(X) = \ell_\infty(X)$, to the following result.

Proposition 2.3. *Let $\mu(Y) \subset \omega(Y)$ be a sequence space and let $\mathcal{I}$ be an admissible ideal. If $\mathfrak{A} : c^{\mathcal{I}}(X) \cap \ell_\infty(X) \to \mu(Y)$, then (2.1) and*

$$\mathfrak{A}^{[K]} : \ell_\infty(X) \to \mu(Y) \quad (K \in \mathcal{I}) \tag{2.3}$$

are satisfied. If $\mathcal{I}$ has property (AP), then (2.1) and (2.3) imply $\mathfrak{A} : c^{\mathcal{I}}(X) \cap \ell_\infty(X) \to \mu(Y)$.

Proposition 2.3 together with Theorems 1.3(i) and 1.4 gives the following characterization of the matrix transformation $\mathfrak{A} : c^{\mathcal{I}}(X) \cap \ell_\infty(X) \to c(Y)$.

Corollary 2.4. *Let X, Y be Banach spaces, $\mathfrak{A} = (A_{nk})$ be an infinite matrix with $A_{nk} \in B(X, Y)$, and let $\mathcal{I}$ be an admissible ideal. If $\mathfrak{A}$ maps $c^{\mathcal{I}}(X) \cap \ell_\infty(X)$ to $c(Y)$, then (1.3)–(1.7) hold and, for any $K = (k_i) \in \mathcal{I}$, the matrix $\mathfrak{A}^{[K]} = (A_{nk}^{[K]})$ satisfies the conditions*

$$\lim_m \||(A_{nk}^{[K]})\||_{n,m} = 0 \quad (n \in \mathbb{N}), \tag{2.4}$$

$$\limsup_m \sup_n \||(A_{nk}^{[K]} - A_k^{[K]})\||_{n,m} = 0, \tag{2.5}$$

If $\mathcal{I}$ has property (AP), then (1.3)–(1.7), (2.4) and (2.5) are also sufficient for $\mathfrak{A} : c^{\mathcal{I}}(X) \cap \ell_\infty(X) \to c(Y)$.

Let $B = (b_{nk})$ be an infinite matrix of scalars. Using the known characterizations of matrix transformations $B : c \to c$ and $B : \ell_\infty \to c$ (see, for example, [22]), Proposition 2.3 permits to formulate also an extension of Corollary 5.1 from [12].

Corollary 2.5. *Let $\mathcal{I}$ be an admissible ideal. If $B : c^{\mathcal{I}} \cap \ell_\infty \to c$, then*

$$\sup_n \sum_k |b_{nk}| < \infty, \tag{2.6}$$

$$\exists \lim_n b_{nk} =: b_k \quad (k \in \mathbb{N}), \tag{2.7}$$

$$\exists \lim_n \sum_k b_{nk}, \tag{2.8}$$

$$\lim_n \sum_{k \in K} |b_{nk} - b_k| = 0 \quad (K \in \mathcal{I}). \tag{2.9}$$

If $\mathcal{I}$ has property (AP), then (2.6)–(2.9) are also sufficient for $B : c^{\mathcal{I}} \cap \ell_\infty \to c$.

3. Matrix transformations to $c^{\mathcal{I}}(Y)$

Let $\mathcal{I}$ be an admissible ideal and let X, Y and $\mathfrak{A}$ be the same as in Section 2. The following characterizations of matrix transformations $\mathfrak{A}$ to the space of $\mathcal{I}$-convergent sequences are known.

Theorem 3.1 (see [14, 15]). *Let Φ X be a fundamental set of X. The following statements are true.*
  (i) *If $\mathfrak{A} : c(X) \to c^{\mathcal{I}}(Y) \cap \ell_\infty(Y)$, then (1.3), (1.5) hold and*

$$\exists \lim_m \sum_{k=1}^m A_{nk}\phi \quad (n \in \mathbb{N}, \ \phi \in \Phi), \tag{3.1}$$

$$\exists \mathcal{I}\text{-}\lim_n A_{nk}\phi =: A_k\phi \quad (k \in \mathbb{N}, \ \phi \in \Phi), \tag{3.2}$$

$$\exists \mathcal{I}\text{-}\lim_n \sum_k A_{nk}\phi \quad (\phi \in \Phi). \tag{3.3}$$

Conversely, if Φ is countable and $\mathcal{I}$ has property (AP), then (1.3), (1.5) and (3.1)–(3.3) are also sufficient for $\mathfrak{A} : c(X) \to c^{\mathcal{I}}(Y) \cap \ell_\infty(Y)$.
  (ii) *If $\mathfrak{A} : c_0(X) \to c^{\mathcal{I}}(Y) \cap \ell_\infty(Y)$, then (1.3), (1.5) and (3.2) hold. If Φ is countable and $\mathcal{I}$ has property (AP), then (1.3), (1.5) and (3.2) imply $\mathfrak{A} : c_0(X) \to c^{\mathcal{I}}(Y) \cap \ell_\infty(Y)$.*
  (iii) *If $\mathfrak{A} : \ell_1(X) \to c^{\mathcal{I}}(Y) \cap \ell_\infty(Y)$, then (1.8), (1.9) and (3.2) are satisfied. Conversely, if Φ is countable and $\mathcal{I}$ has property (AP), then (1.8), (1.9) and (3.2) are also sufficient for $\mathfrak{A} : \ell_1(X) \to c^{\mathcal{I}}(Y) \cap \ell_\infty(Y)$.*
  If $O(1)$ is replaced by $O_{\mathcal{I}}(1)$ in (1.5) and (1.9), then (i)−(iii) give the characterizations of matrix maps $\mathfrak{A} : c(X) \to c^{\mathcal{I}}(Y) \cap \ell_\infty^{\mathcal{I}}(Y)$, $\mathfrak{A} : c_0(X) \to c^{\mathcal{I}}(Y) \cap \ell_\infty^{\mathcal{I}}(Y)$ and $\mathfrak{A} : \ell_1(X) \to c^{\mathcal{I}}(Y) \cap \ell_\infty^{\mathcal{I}}(Y)$, respectively.

Our purpose is to consider the same type transformations $\mathfrak{A}$ without the separability assumption of the space X. As one may expect, results obtained in this case are in some respects weaker in comparison with the results in Theorem 3.1.

Let $N = (n_i)$ be a set from the filter $\mathcal{F}(\mathcal{I})$. We say that a sequence $\mathfrak{x} \in \omega(X)$ is $\mathcal{I},N$-*bounded* if $(x_{n_i}) \in \ell_\infty(X)$. If $\ell_\infty^{\mathcal{I},N}(X)$ denotes the set of all $\mathcal{I},N$-bounded sequences $\mathfrak{x} \in \omega(X)$, then it is clear that

$$\ell_\infty(X) \subset \ell_\infty^{\mathcal{I},N}(X) \ \text{and} \ \ell_\infty^{\mathcal{I}}(X) = \bigcup_{N \in \mathcal{F}(\mathcal{I})} \ell_\infty^{\mathcal{I},N}(X).$$

Theorem 3.2. *Let $\lambda(X) \subset \omega(X)$ be a Banach sequence space with the norm $\|\cdot\|_\lambda$ and a fundamental set E. Let $\mathcal{I}$, $\mathcal{J}$ be admissible ideals and $N = (n_i) \in \mathcal{F}(\mathcal{J})$. If $\mathfrak{A} : \lambda(X) \to c^{\mathcal{I}}(Y) \cap \ell_\infty^{\mathcal{J},N}(Y)$, then*

$$\sup_m \sup_{\|\mathfrak{x}\|_\lambda \leq 1} \left\| \sum_{k=1}^m A_{nk} x_k \right\| < \infty \quad (n \in \mathbb{N}), \tag{3.4}$$

$$\exists \lim_m \sum_{k=1}^m A_{nk} x_k \quad (n \in \mathbb{N}, \ \mathfrak{x} \in E), \tag{3.5}$$

$$\|\mathfrak{A}_{n_i}\| = O(1), \tag{3.6}$$

$$\exists \mathcal{I}\text{-}\lim_n \mathfrak{A}_n \mathfrak{x} \quad (\mathfrak{x} \in E). \tag{3.7}$$

Conversely, if $\mathcal{J} \subset \mathcal{I}$, conditions (3.4)–(3.6) hold, and there exists a set $K = (k_i) \in \mathcal{F}(\mathcal{I})$ such that

$$\exists \lim_i \mathfrak{A}_{k_i} \mathfrak{x} \quad (\mathfrak{x} \in E), \tag{3.8}$$

then $\mathfrak{A} : \lambda(X) \to c^{\mathcal{I}}(Y) \cap \ell_\infty^{\mathcal{J},N}(Y)$.

Proof. Assume that $\mathfrak{A} : \lambda(X) \to c^{\mathcal{I}}(Y) \cap \ell_\infty^{\mathcal{J},N}(Y)$. Then the series $\mathfrak{A}_n \mathfrak{x} = \sum_k A_{nk} x_k$ converge for all $\mathfrak{x} \in \lambda(X)$. Thus, because of Theorem 1.2, conditions (3.4), (3.5) hold and $\mathfrak{A}_n \in B(\lambda(X), Y)$ $(n \in \mathbb{N})$. Further, since $\mathfrak{A}\mathfrak{x} \in \ell_\infty^{\mathcal{I},N}(Y)$, $(\mathfrak{A}_{n_i} \mathfrak{x})$ is bounded for any $\mathfrak{x} \in \lambda(X)$. So, in view of Theorem 1.1, condition (3.6) must be satisfied. Condition (3.7) holds by $\mathfrak{A}\mathfrak{x} \in c^{\mathcal{I}}(Y)$ $(\mathfrak{x} \in E)$.

Now. assume that $\mathcal{J} \subset \mathcal{I}$ and conditions (3.4)–(3.6) and (3.8) are satisfied. Then the operator $\mathfrak{A}$ is determined on $\lambda(X)$ and $\mathfrak{A}_n \in B(\lambda(X), Y)$ $(n \in \mathbb{N})$ in view of (3.4) and (3.5). Condition (3.6) shows, by Theorem 1.1, that the sequences $\mathfrak{A}\mathfrak{x}$ $(\mathfrak{x} \in \lambda(X))$ are in $\ell_\infty^{\mathcal{J},N}(Y)$. Moreover, since $\mathcal{J} \subset \mathcal{I}$ implies $N \in \mathcal{F}(\mathcal{I})$, we have $M := N \cap K \subset \mathcal{F}(\mathcal{I})$. Consequently, the sequence of operators $(\mathfrak{A}_{m_i})$, where $M = (m_i)$, satisfies the conditions of Theorem 1.2 with respect to the fundamental set E of $\lambda(X)$. Thus the sequences $\mathfrak{A}\mathfrak{x}$ $(\mathfrak{x} \in \lambda(X))$ must be I^*-convergent, hence also I-convergent in Y. $\qquad\square$

In the case $\lambda \in \{c, c_0, \ell_1\}$, from Theorem 3.2 we get the following result.

Proposition 3.3. *Let $\mathcal{I}$, $\mathcal{J}$ and N be the same as in Theorem 3.2.*

(i)   *If $\mathfrak{A} : c(X) \to c^{\mathcal{I}}(Y) \cap \ell_\infty^{\mathcal{J},N}(Y)$, then conditions (1.3), (1.4), (3.6) hold and*

$$\exists \mathcal{I}\text{-}\lim_n A_{nk} x \quad (k \in \mathbb{N}, \ x \in X), \tag{3.9}$$

$$\exists \mathcal{I}\text{-}\lim_n \sum_k A_{nk} x \quad (x \in X). \tag{3.10}$$

Conversely, if $\mathcal{J} \subset \mathcal{I}$, conditions (1.3), (1.4), (3.6) are satisfied, and there exists a set $K = (k_i) \in \mathcal{F}(\mathcal{I})$ such that

$$\exists \lim_i A_{k_i,k}x \quad (k \in \mathbb{N}, \; x \in X), \tag{3.11}$$

$$\exists \lim_i \sum_k A_{k_i,k}x \quad (x \in X), \tag{3.12}$$

then $\mathfrak{A} : c(X) \to c^{\mathcal{I}}(Y) \cap \ell_\infty^{\mathcal{J},N}(Y)$.

(ii)   *If $\mathfrak{A} : c_0(X) \to c^{\mathcal{I}}(Y) \cap \ell_\infty^{\mathcal{J},N}(Y)$, then conditions (1.3), (3.6) and (3.9) hold. Conversely, if $\mathcal{J} \subset \mathcal{I}$ and conditions (1.3), (3.6), (3.11) are satisfied, then $\mathfrak{A} : c_0(X) \to c^{\mathcal{I}}(Y) \cap \ell_\infty^{\mathcal{J},N}(Y)$.*

(iii)   *If $\mathfrak{A} : \ell_1(X) \to c^{\mathcal{I}}(Y) \cap \ell_\infty^{\mathcal{J},N}(Y)$, then (1.8), (3.9) hold and*

$$H_n = O_{\mathcal{J}}(1). \tag{3.13}$$

If $\mathcal{J} \subset \mathcal{I}$ and conditions (1.8), (3.11) and (3.13) are satisfied, then $\mathfrak{A} : \ell_1(X) \to c^{\mathcal{I}}(Y) \cap \ell_\infty^{\mathcal{J},N}(Y)$.

Proof. Our statements follow from Theorem 3.2 by reason of the following remarks.

(i). Since $c(X)$ has the fundamental set $\mathcal{E}_0(X) \bigcup \mathcal{E}(X)$, conditions (3.4), (3.5) reduce, respectively, to (1.3), (1.4). Moreover, (3.7) takes the form (3.9) if $\mathfrak{x} \in \mathcal{E}_0(X)$, and the form (3.10) if $\mathfrak{x} \in \mathcal{E}(X)$. Similarly, (3.8) reduces to (3.11) and (3.12), respectively

(ii). We argue as above using only the fact that $c_0(X)$ has the fundamental set $\mathcal{E}_0(X)$.

(iii). The proof is quite similar if we observe that $\ell_1(X)$ has the fundamental set $\mathcal{E}_0(X)$ and $\|\mathfrak{A}_n\| = \sup_k \|A_{nk}\|$ $(n \in \mathbb{N})$ (see [9], p. 113). $\square$

If $\mathcal{J} = \mathcal{I}_f$, the ideal of all finite subsets of $\mathbb{N}$, then $\ell_\infty^{\mathcal{J},N}(Y) = \ell_\infty(Y)$. Consequently, Proposition 3.3 gives the following extension of Theorem 3.1.

Proposition 3.4. *The following statements hold.*

(i)   *If $\mathfrak{A} : c(X) \to c^{\mathcal{I}}(Y) \cap \ell_\infty(Y)$, then conditions (1.3)−(1.5), (3.9) and (3.10) hold. Conversely, if conditions (1.3)−(1.5), (3.11) and (3.12) are satisfied, then $\mathfrak{A} : c(X) \to c^{\mathcal{I}}(Y) \cap \ell_\infty(Y)$.*

(ii)   *If $\mathfrak{A} : c_0(X) \to c^{\mathcal{I}}(Y) \cap \ell_\infty(Y)$, then conditions (1.3), (1.5) and (3.9) hold. Conversely, if conditions (1.3), (1.5) and (3.11) are satisfied, then $\mathfrak{A} : c_0(X) \to c^{\mathcal{I}}(Y) \cap \ell_\infty(Y)$.*

(iii)   *If $\mathfrak{A} : \ell_1(X) \to c^{\mathcal{I}}(Y) \cap \ell_\infty(Y)$, then conditions (1.8), (1.9) and (3.9) hold. If conditions (1.8), (1.9) and (3.11) are satisfied, then $\mathfrak{A} : \ell_1(X) \to c^{\mathcal{I}}(Y) \cap \ell_\infty(Y)$.*

This proposition gives, in special case $\mathcal{I} = \mathcal{I}_T$, the characterizations of matrix transformations $\mathfrak{A} : c(X) \to st_T(Y) \cap \ell_\infty(Y)$, $\mathfrak{A} : c_0(X) \to st_T(Y) \cap$

$\ell_\infty(Y)$ and $\mathfrak{A} : \ell_1(X) \to st_T(Y) \cap \ell_\infty(Y)$. Similar characterizations are proved, for separable X, in [14] and, for $X = Y = \mathbb{K}$, in [13].

Acknowledgements

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