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On new extensions of the generalized Hermite matrix polynomials

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Abstract. Various families of generating matrix functions have been established in diverse ways. The objective of the present paper is to investigate these generalized Hermite matrix polynomials, and derive some important results for them, such as, the generating matrix functions, matrix recurrence relations, an expansion of $x^n I$, finite summation formulas, addition theorems, integral representations, fractional calculus operators, and certain other implicit summation formulae.

1. Introduction and preliminaries

Various possible extensions to the matrix framework of the classical families of Laguerre, Hermite, Legendre, Gegenbauer, and Chebyshev polynomials have been widely investigated in the literature (see, for example, [2, 3, 4, 7, 10, 11, 14, 16, 22, 24, 25, 26, 27]). Earlier, the Hermite matrix polynomials and its extensions and generalizations were introduced in [15, 21, 23, 29] for matrices in $\mathbb{C}^{\mathbb{N}\times\mathbb{N}}$, whose eigenvalues are all situated in the right open half-plane.

Since it has been amply demonstrated that the various extended Hermite matrix polynomials of one variable potential have applications in many diverse areas of mathematics, physical, engineering and statistical sciences (see, for details, [1, 12, 13] and the references cited therein), we propose to provide a new extension of the Hermite matrix polynomials in this paper which shall also find applications in the diverse fields mentioned hitherto. The structure of this work is as follows. In Section 2, we deal with important properties of the generalized Hermite matrix polynomials such as addition, multiplication theorems, and summation formula. In Section 3, we obtain

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integral transforms for the generalized Legendre matrix polynomials, the Chebyshev matrix polynomials of the first, the second, and the third kind in terms of the generalized Hermite matrix polynomials introduced by us. In Section 4, we have dealt with the fractional integrals and the fractional derivatives which yield a different view of the generalized Hermite matrix polynomials. Finally, in Section 5, some concluding remarks are given.

Frequently occurring definitions, theorems, notations, and miscellaneous results used throughout this paper are as given below. Throughout this paper, for a matrix A in $\mathbb{C}^{\mathbb{N}\times\mathbb{N}}$, its spectrum $\sigma(A)$ will denote the set of all eigenvalues of A. Furthermore, the unit matrix and the null matrix in $\mathbb{C}^{\mathbb{N}\times\mathbb{N}}$ will be denoted by I and $\boldsymbol{0}$, respectively.

Lemma 1.1 (see [3]). *If $A(k,n)$ and $B(k,n)$ are matrices in $\mathbb{C}^{\mathbb{N}\times\mathbb{N}}$ for $n \geq 0$, $k \geq 0$, then*

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} A(k,n) = \sum_{n=0}^{\infty}\sum_{k=0}^{[\frac{1}{m}n]} A(k,n-mk), \quad m \in \mathbb{N}. \tag{1.1}$$

Similarly to (1.1)*, we can write*

$$\sum_{n=0}^{\infty}\sum_{k=0}^{[\frac{1}{m}n]} A(k,n) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} A(k,n+mk), \quad m \in \mathbb{N}. \tag{1.2}$$

Definition 1.1 (see [9]). A matrix A in $\mathbb{C}^{\mathbb{N}\times\mathbb{N}}$ is said be a positive stable matrix if

$$\Re(\mu) \nleq 0 \quad \text{for every eigenvalue } \mu \in \sigma(A). \tag{1.3}$$

Fact 1.1 (see [9]). If B is a matrix in $\mathbb{C}^{\mathbb{N}\times\mathbb{N}}$ such that

$$B + nI \quad \text{is an invertible matrix for all integers } n \geq 0, \tag{1.4}$$

then

$$\Gamma(B) = \int_{o}^{\infty} e^{-t}\exp\left((B-I)\ln t\right) dt$$

is an invertible matrix in $\mathbb{C}^{\mathbb{N}\times\mathbb{N}}$, and one gets

$$\begin{aligned}(B)_n : &= B(B+I)\dots(B+(n-1)I)\\ &= \Gamma(B+nI)\Gamma^{-1}(B), \quad n \geq 1,\ (B)_0 = I,\end{aligned}$$

where $\Gamma^{-1}(B)$ is the image of $\Gamma^{-1}(z) = 1/\Gamma(z)$ acting on B.

Fact 1.2 (see [8]). For a matrix A in $\mathbb{C}^{\mathbb{N}\times\mathbb{N}}$ we have

$$(1-z)^{-A} = \sum_{n=0}^{\infty} \frac{1}{n!}(A)_n z^n, \quad |z| < 1. \tag{1.5}$$

If $\Phi(z)$ is a holomorphic function at $z = z_0$, $\Phi(z_0) \neq 0$, and if $z = z_0 + w\Phi(z)$ and $f(z)$ is an analytic function, we expanded a power series in w by the Lagrange expansion formula as (see [19])

$$\frac{f(z)}{1 - w\Phi'(z)} = \sum_{n=0}^{\infty} \frac{w^n}{n!} \frac{d^n}{dz^n} \left[f(z) \left(\Phi(z)\right)^n \right] \Big|_{z=z_0}. \tag{1.6}$$

From the definition of the Gamma function, we have (see [28])

$$\int_0^\infty e^{-t^2} t^{\frac{2n-mk}{p}} dt = \frac{1}{2}\Gamma\left(\frac{2n-mk}{2p} + \frac{1}{2}\right) = \frac{\sqrt{\pi}}{2} \left(\frac{1}{2}\right)_{\frac{2n-mk}{2p}} \tag{1.7}$$

and

$$\int_0^\infty e^{-t} t^{\frac{n-(m-p)k}{p}} dt = \Gamma\left(\frac{n-(m-p)k}{p} + 1\right). \tag{1.8}$$

In order to describe more details of our work, we will need some definitions of fractional integrals and fractional derivatives, which are given as below, and can be found in standard works in this field, like, [5, 6, 17, 18, 20].

Definition 1.2. Riemann–Liouville fractional integral of order μ is defined by

$$\mathbb{I}^\mu\{f(x)\} = \frac{1}{\Gamma(\mu)} \int_0^x (x-t)^{\mu-1} f(t) dt, \quad \Re(\mu) > 0. \tag{1.9}$$

Definition 1.3. Let $f(x) \in L(b,c)$, $\alpha \in \mathbb{C}$, and $\Re(\alpha) > 0$. The left-sided operator of Riemann–Liouville fractional integral of order α is defined by

$${}_b\mathbb{I}_x^\alpha\{f(x)\} = \frac{1}{\Gamma(\mu)} \int_b^x (x-t)^{\alpha-1} f(t) dt, \quad x > b. \tag{1.10}$$

Definition 1.4. Let $f(x) \in L(b,c)$, $\alpha \in \mathbb{C}$, and $Re(\alpha) > 0$. The right-sided operator of Riemann–Liouville fractional integral of order α is defined by

$${}_x\mathbb{I}_c^\alpha\{f(x)\} = \frac{1}{\Gamma(\alpha)} \int_x^c (t-x)^{\alpha-1} f(t) dt, \quad x < c. \tag{1.11}$$

Definition 1.5. The Weyl integral of $f(x)$ of order α, denoted by ${}_xW_\infty^\alpha$, is defined by

$${}_xW_\infty^\alpha\{f(x)\} = \frac{1}{\Gamma(\alpha)} \int_x^\infty (t-x)^{\alpha-1} f(t) dt, \quad -\infty < x < \infty, \tag{1.12}$$

where $\alpha \in \mathbb{C}$ and $\Re(\alpha) > 0$.

Definition 1.6. Let $f(x) \in L(b,c)$, $\alpha \in \mathbb{C}$, $\Re(\alpha) \geq 0$, and $n = [\Re(\alpha)] + 1$. The left-sided operator of Riemann–Liouville fractional derivative of order α is defined by

$${}_bD_x^\alpha\{f(x)\} = \frac{1}{\Gamma(n-\alpha)} \left(\frac{\partial}{\partial x}\right)^n \int_b^x \frac{f(t)}{(x-t)^{\alpha-n+1}} dt, \quad x > b. \tag{1.13}$$

Definition 1.7. Let $f(x) \in L(b,c)$, $\alpha \in \mathbb{C}$, $\Re(\alpha) \geq 0$, and $n = [\Re(\alpha)] + 1$. The right-sided operator of Riemann–Liouville fractional derivative of order α is defined by

$$ {}_xD_c^{\alpha}\{f(x)\} = \frac{(-1)^n}{\Gamma(n-\alpha)} \left(\frac{\partial}{\partial\, x}\right)^n \int_x^c \frac{f(t)}{(t-x)^{\alpha-n+1}} dt, \quad x < c. \tag{1.14}$$

Definition 1.8. Let $f(x) \in L(b,c)$, $\alpha \in \mathbb{C}$, $\Re(\alpha) \geq 0$, and $n = [\Re(\alpha)] + 1$. The Weyl fractional derivative of $f(x)$ of order α, denoted by ${}_xD_{\infty}^{\alpha}$, is defined by

$$ {}_xD_{\infty}^{\alpha}\{f(x)\} = \frac{(-1)^m}{\Gamma(m-\alpha)} \left(\frac{\partial}{\partial x}\right)^m \int_x^{\infty} \frac{f(t)}{(t-x)^{\alpha-m+1}} dt, \tag{1.15}$$

where $-\infty < x < \infty$, $m-1 \leq \alpha < m$, and $m \in \mathbb{N}$.

2. The definition of generalized Hermite matrix polynomials and their properties

Let A and B be commutative matrices in $\mathbb{C}^{\mathbb{N}\times\mathbb{N}}$. For any complex number ν let νA be a positive stable matrix in $\mathbb{C}^{\mathbb{N}\times\mathbb{N}}$ satisfying condition (1.3), and let B be a matrix satisfying condition (1.4). We define the generalized Hermite matrix polynomials by means of the matrix generating function

$$ F = \sum_{n=0}^{\infty} H_{n,m,p}(x, A, B; a, b, \nu) \frac{t^n}{n!} = a^{xt^p\sqrt{\nu A}} \left(1 + \frac{t^m}{b}\right)^{-B}, \quad \left|\frac{t^m}{b}\right| < 1, \tag{2.1}$$

where $a > 0$, $a \neq 1$, and p and m are any numbers.

Making use of the exponential matrix function and the binomial expansion (1.5), we obtain

$$ F = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-1)^k (B)_k}{k! b^k \Gamma(n+1)} \left(x \log(a)\sqrt{\nu A}\right)^n t^{pn+km}. \tag{2.2}$$

Using (1.1) and (2.2), we can write

$$ F = \sum_{n=0}^{\infty}\sum_{k=0}^{[\frac{n}{m}]} \frac{(-1)^k (B)_k}{k! b^k \Gamma\left(\frac{n-mk}{p}+1\right)} \left(x \log(a)\sqrt{\nu A}\right)^{\frac{n-mk}{p}} t^n.$$

Comparing the coefficients of t^n, we obtain an explicit representation of the matrix version of generalized Hermite matrix polynomials for $\Re\left(\frac{n-mk}{p}\right) > -1$:

$$ \begin{aligned} H_{n,m,p} &= H_{n,m,p}(x, A, B; a, b, \nu) \\ &= n! \sum_{k=0}^{[\frac{1}{m}n]} \frac{(-1)^k (B)_k}{k! b^k \Gamma\left(\frac{n-mk}{p}+1\right)} \left(x \log(a)\sqrt{\nu A}\right)^{\frac{n-mk}{p}}. \end{aligned} \tag{2.3}$$

In the following theorem, we obtain the matrix recurrence relations for the generalized Hermite matrix polynomials.

Theorem 2.1. *The generalized Hermite matrix polynomials satisfy the relations*

$$\begin{aligned}\frac{d^r}{dx^r}H_{n,m,p} = \frac{n!}{(n-pr)!}&\left(log(a)\sqrt{\nu\, A}\right)^r\\ &\times H_{n-pr,m,p}(x,A,B;a,b,\nu), \quad 0 \le r \le \left[\frac{n}{p}\right],\end{aligned} \tag{2.4}$$

$$\begin{aligned}&\frac{b}{n!}H_{n+1,m,p}(x,A,B;a,b,\nu) + \frac{(n-m)I+mB}{(n-m+1)!}H_{n-m+1,m,p}(x,A,B;a,b,\nu)\\ &= \frac{bpx\log(a)\sqrt{\nu\, A}}{(n-p+1)!}H_{n-p+1,m,p}(x,A,B;a,b,\nu)\\ &\quad + \frac{px\log(a)\sqrt{\nu\, A}}{(n-m-p+1)!}H_{n-m-p+1,m,p}(x,A,B;a,b,\nu), \quad n \ge m+p-1,\end{aligned} \tag{2.5}$$

and

$$\begin{aligned}&\frac{b\log(a)\sqrt{\nu A}}{(n-p)!}H_{n-p+1,m,p}(x,A,B;a,b,\nu)\\ &\quad + \frac{\log(a)\sqrt{\nu\, A}}{(n-p-m)!}\, H_{n-m-p+1,m,p}(x,A,B;a,b,\nu)\\ &= \frac{bpx\log(a)\sqrt{\nu\, A}}{(n-p+1)!}\frac{d}{dx}H_{n-p+1,m,p}(x,A,B;a,b,\nu)\\ &\quad + \frac{px\log(a)\sqrt{\nu\, A}}{(n-m-p+1)!}\frac{d}{dx}H_{n-m-p+1,m,p}(x,A,B;a,b,\nu)\\ &\quad - \frac{mB}{(n-m+1)!}\frac{d}{dx}H_{n-m+1,m,p}(x,A,B;a,b,\nu), \quad n \ge m+p-1.\end{aligned} \tag{2.6}$$

Proof. Differentiating (2.1) with respect to x, we have

$$\frac{\partial F}{\partial x} = \sum_{n=0}^{\infty}\frac{d}{dx}H_{n,m,p}\frac{t^n}{n!} = \log(a)\sqrt{\nu A}\sum_{n=0}^{\infty}H_{n,m,p}\frac{t^{n+p}}{n!}. \tag{2.7}$$

Comparing the coefficients of t^n for $0 \le p \le n$, we obtain

$$\frac{d}{dx}H_{n,m,p} = \frac{n!}{(n-p)!}\log(a)\sqrt{\nu\, A}\; H_{n-p,m,p}(x,A,B;a,b,\nu), \tag{2.8}$$

which is the required matrix differential recurrence relation. The iteration of (2.8) for $0 \le r \le [n/p]$ leads us to (2.4).

Again, by differentiating (2.1) with respect to t, we have

$$\begin{aligned}\frac{\partial F}{\partial t} &= \sum_{n=1}^{\infty} H_{n,m,p}\frac{t^{n-1}}{(n-1)!} = px\log(a)t^{p-1}\sqrt{\nu A}a^{xt^p\sqrt{\nu A}}\left(1+\frac{t^m}{b}\right)^{-B}\\ &\quad - \frac{m}{b}Bt^{m-1}a^{xt^p\sqrt{\nu A}}\left(1+\frac{t^m}{b}\right)^{-B-I},\end{aligned} \tag{2.9}$$

and we can write

$$\begin{aligned}& b\sum_{n=1}^{\infty} H_{n,m,p}\frac{t^{n-1}}{(n-1)!} + \sum_{n=1}^{\infty} H_{n,m,p}\frac{t^{n+m-1}}{(n-1)!}\\ &= bpx\log(a)\sqrt{\nu A}\sum_{n=0}^{\infty} H_{n,m,p}\frac{t^{n+p-1}}{n!} + px\log(a)\sqrt{\nu\ A}\sum_{n=0}^{\infty} H_{n,m,p}\frac{t^{n+m+p-1}}{n!}\\ &\quad - mB\sum_{n=0}^{\infty} H_{n,m,p}\frac{t^{n+m-1}}{n!}.\end{aligned}$$

Equating the coefficients of t^n, we get

$$\begin{aligned}&\frac{b}{n!}H_{n+1,m,p}(x,A,B;a,b,\nu) + \frac{(n-m)I+mB}{(n-m+1)!}H_{n-m+1,m,p}(x,A,B;a,b,\nu)\\ &= \frac{bpx\log(a)\sqrt{\nu\ A}}{(n-p+1)!}H_{n-p+1,m,p}(x,A,B;a,b,\nu)\\ &\quad + \frac{px\log(a)\sqrt{\nu\ A}}{(n-m-p+1)!}H_{n-m-p+1,m,p}(x,A,B;a,b,\nu),\end{aligned}$$

which is the required pure matrix recurrence relation (2.5).

From (2.7) and (2.9), we observe that

$$\begin{aligned}b\log(a)t^p\sqrt{\nu A}\frac{\partial F}{\partial t} + \log(a)\sqrt{\nu A}t^{m+p}\frac{\partial F}{\partial t} &= bpx\log(a)t^{p-1}\sqrt{\nu A}\frac{\partial F}{\partial x}\\ + px\log(a)\sqrt{\nu A}t^{m+p-1}\frac{\partial F}{\partial x} - mBt^{m-1}\frac{\partial F}{\partial x}.\end{aligned}$$

This, again in view of (2.1), by comparing the coefficients of t^n, yields (2.6). The proof is complete. □

Theorem 2.2. *For any complex number ν, let νA be a positive stable matrix in $\mathbb{C}^{N\times N}$ satisfying* (1.3). *Then we have the expansion of $x^n I$ in the form*

$$\left(x\log(a)\sqrt{\nu\ A}\right)^n = n!\sum_{k=0}^{[\frac{np}{m}]}\frac{(-1)^k(-B)_k}{k!(np-mk)!b^k}H_{np-mk,m,p}(x,A,B;a,b,\nu), \tag{2.10}$$

where B is a matrix in $\mathbb{C}^{N\times N}$ satisfying (1.4).

Proof. By (2.1), we can write

$$a^{xt^p\sqrt{\nu A}} = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-1)^k(-B)_k}{k!n!b^k} H_{n,m,p}(x, A, B; a, b, \nu)t^{n+mk}. \tag{2.11}$$

Replacing n by $np - mk$ in the right hand side of (2.11), we get

$$\begin{aligned}
&\sum_{n=0}^{\infty} \frac{1}{n!}\left(x\log(a)\sqrt{\nu\, A}\right)^n t^{np} \\
&\quad = \sum_{n=0}^{\infty}\sum_{k=0}^{[\frac{np}{m}]} \frac{(-1)^k(-B)_k}{k!(np-mk)!b^k} H_{np-mk,m,p}(x, A, B; a, b, \nu)t^{np},
\end{aligned}$$

and, by comparing the coefficients of t^n in the above equation, we arrive at (2.10). □

Now, we give the multiplication, addition, and summation formulae for the generalized Hermite matrix polynomials in the following theorems.

Theorem 2.3. *For any complex number* ν*, let* νA *be a positive stable matrix in* $\mathbb{C}^{N\times N}$ *satisfying the condition* (1.3)*. For commutative matrices* B*,* D*, and* $B - D$ *in* $\mathbb{C}^{N\times N}$ *satisfying* (1.4)*, the generalized Hermite matrix polynomials satisfy the finite summation formula*

$$H_{n,m,p} = n!\sum_{k=0}^{[\frac{n}{m}]} \frac{(B-D)_k}{k!b^k\Gamma(n-mk+1)} H_{n-mk,m,p}(x, A, D; a, b, \nu). \tag{2.12}$$

Proof. From (2.1) and (1.1), we have

$$\begin{aligned}
\sum_{n=0}^{\infty} H_{n,m,p}\frac{t^n}{n!} &= a^{xt^p\sqrt{\nu A}}\left(1+\frac{t^m}{b}\right)^{-D}\left(1+\frac{t^m}{b}\right)^{D-B} \\
&= \left(1+\frac{t^m}{b}\right)^{D-B}\sum_{n=0}^{\infty} H_{n,m,p}(x, A, D; a, b, \nu)\frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{(B-D)_k}{n!k!b^k} H_{n,m,p}(x, A, D; a, b, \nu)t^{n+mk} \\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{[\frac{n}{m}]} \frac{(B-D)_k}{k!b^k\Gamma(n-mk+1)} H_{n-mk,m,p}(x, A, D; a, b, \nu)t^n.
\end{aligned}$$

Comparing the coefficients of t^n in the above equation leads us to (2.12). □

Theorem 2.4. *The generalized Hermite matrix polynomials satisfy the multiplication formula*

$$H_{n,m,p}(\alpha\, x, A, B; a, b, \nu) = n! \sum_{k=0}^{[\frac{n}{p}]} \frac{\Big((\alpha-1)x\log(a)\sqrt{\nu A}\Big)^k H_{n-kp,m,p}(x, A, B; a, b, \nu)}{k!\Gamma(n-kp+1)}, \tag{2.13}$$

where α is constant.

Proof. By (2.1) and (1.1), we have

$$\sum_{n=0}^{\infty} \frac{H_{n,m,p}(\alpha\, x, A, B; a, b, \nu)t^n}{n!} = a^{(\alpha-1)t^p\sqrt{\nu A}} a^{xt^p\sqrt{\nu A}} \left(1+\frac{t^m}{b}\right)^{-B}$$
$$= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{\Big((\alpha-1)\log(a)\sqrt{\nu A}\Big)^k H_{n,m,p}(x, A, B; a, b, \nu)t^{n+kp}}{k!n!}$$
$$= \sum_{n=0}^{\infty}\sum_{k=0}^{[\frac{n}{p}]} \frac{\Big((\alpha-1)x\log(a)\sqrt{\nu A}\Big)^k H_{n-kp,m,p}(x, A, B; a, b, \nu)t^n}{k!\Gamma(n-kp+1)}.$$

Thus, comparing the coefficients of t^n, we get (2.13). □

Theorem 2.5. *For commutative matrices B, D and $B+D$ in $\mathbb{C}^{N\times N}$, satisfying the condition* (1.4)*, the finite summation formula for the generalized Hermite matrix polynomials is as follows:*

$$H_{n,m,p}(\alpha x+\beta z, A, B+D; a, b, \nu) = n! \sum_{k=0}^{n} \frac{H_{k,m,p}(\beta z, A, B; a, b, \nu)H_{n-k,m,p}(\alpha x, A, D; a, b, \nu)}{k!(n-k)!}, \tag{2.14}$$

where α and β are constants.

Proof. Using (1.2), we consider the series

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{H_{n-k,m,p}(\beta z, A, B; a, b, \nu)H_{k,m,p}(\alpha x, A, D; a, b, \nu)t^n}{k!(n-k)!}$$
$$= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{H_{n,m,p}(\beta z, A, B; a, b, \nu)H_{k,m}(\alpha x, A, D; a, b, \nu)t^{n+k}}{k!n!}$$
$$= a^{(\alpha x+\beta z)t^p\sqrt{\nu A}} \left(1+\frac{t^m}{b}\right)^{-B-D} = \sum_{n=0}^{\infty} \frac{H_{n,m,p}(\alpha x+\beta z, A, B+D; a, b, \nu)t^n}{n!}.$$

By comparing the coefficients of t^n, we get (2.14). □

Theorem 2.6. *The generalized Hermite matrix polynomials satisfy the addition formula*

$$\begin{aligned} &H_{n,m,p}(x+y,A,B;a,b,\nu) \\ &= n!\sum_{k=0}^{[\frac{n}{p}]}\frac{y^k}{k!(n-pk)!}\left(\log(a)\sqrt{\nu\,A}\right)^k H_{n-pk,m,p}(x,A,B;a,b,\nu). \end{aligned} \tag{2.15}$$

Proof. Rewriting (2.1) in the form

$$\left(1+\frac{t^m}{b}\right)^{-B} = a^{-xt^p\sqrt{\nu A}}\sum_{n=0}^{\infty}H_{n,m,p}(x,A,B;a,b,\nu)\frac{t^n}{n!}$$

and replacing x by y, we have

$$\left(1+\frac{t^m}{b}\right)^{-B} = a^{-yt^p\sqrt{\nu A}}\sum_{n=0}^{\infty}H_{n,m,p}(y,A,B;a,b,\nu)\frac{t^n}{n!}.$$

By comparing, we get

$$a^{(y-x)t^p\sqrt{\nu A}}\sum_{n=0}^{\infty}H_{n,m,p}(x,A,B;a,b,\nu)\frac{t^n}{n!} = \sum_{n=0}^{\infty}H_{n,m,p}(y,A,B;a,b,\nu)\frac{t^n}{n!}. \tag{2.16}$$

Futher, by expanding the exponential matrix function in (2.16), we have

$$\begin{aligned} &\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(y-x)^k}{n!k!}\left(\log(a)\sqrt{\nu\,A}\right)^k H_{n,m,p}(x,A,B;a,b,\nu)t^{n+pk} \\ &= \sum_{n=0}^{\infty}H_{n,m,p}(y,A,B;a,b,\nu)\frac{t^n}{n!}. \end{aligned} \tag{2.17}$$

Replacing n by $n-pk$ and comparing the coefficients of t^n in (2.17), we get

$$\begin{aligned} &n!\sum_{k=0}^{[\frac{n}{p}]}\frac{(y-x)^k}{k!(n-pk)!}\left(\log(a)\sqrt{\nu\,A}\right)^k H_{n-pk,m,p}(x,A,B;a,b,\nu) \\ &= H_{n,m,p}(y,A,B;a,b,\nu). \end{aligned} \tag{2.18}$$

Replacing y by $y+x$ in (2.18), we get the addition formula (2.15). $\square$

Theorem 2.7. *For matrices A and B in $\mathbb{C}^{N\times N}$ with commutative matrices, the matrix generating function for the generalized Hermite matrix polynomials can be given as*

$$\begin{aligned} \sum_{n=0}^{\infty}H_{n,m,p}(x+ny,A,B;a,b,\nu)\frac{t^n}{n!} &= a^{xt^p\sqrt{\nu A}}\left(1+\frac{t^m}{b}\right)^{-B} \\ &\times\left[I-yt^p\log(a)\sqrt{\nu\,A}a^{xt^p\sqrt{\nu\,A}}\right]^{-1}. \end{aligned} \tag{2.19}$$

Proof. Applying Taylor's expansion in (2.1), we have

$$\sum_{n=0}^{\infty} H_{n,m,p}(x,A,B;a,b,\nu)\frac{t^n}{n!} = \sum_{n=0}^{\infty} \frac{d^n}{dx^n}\left[a^{xt^p\sqrt{\nu A}}\left(1+\frac{t^m}{b}\right)^{-B}\right]\Big|_{t=0}\frac{t^n}{n!}. \tag{2.20}$$

Equating the coefficients of t^n on both sides in (2.20), we have

$$H_{n,m,p}(x,A,B;a,b,\nu) = \frac{d^n}{dx^n}\left[a^{xt^p\sqrt{\nu A}}\left(1+\frac{t^m}{b}\right)^{-B}\right]\Big|_{t=0}. \tag{2.21}$$

Replacing x by $x+ny$ in (2.21), we have

$$H_{n,m,p}(x+ny,A,B;a,b,\nu) = \frac{d^n}{dx^n}\left[a^{(x+ny)t^p\sqrt{\nu A}}\left(1+\frac{t^m}{b}\right)^{-B}\right]\Big|_{t=0},$$

and thus

$$\begin{aligned}\sum_{n=0}^{\infty} & H_{n,m,p}(x+ny,A,B;a,b,\nu)\frac{t^n}{n!}\\ &= \sum_{n=0}^{\infty} \frac{d^n}{dx^n}\left[a^{xt^p\sqrt{\nu A}}\left(1+\frac{t^m}{b}\right)^{-B} a^{nyt^p\sqrt{\nu A}}\right]\Big|_{t=0}\frac{t^n}{n!}.\end{aligned}$$

Using Lagrange's expansion formula (1.6), we obtain the matrix generating function (2.19). □

Theorem 2.8. *For $r \in \mathbb{N}$ and for complex numbers $\nu_1,\nu_2,\ldots,\nu_r$, let $\nu_1 A_1, \nu_2 A_2, \ldots, \nu_r A_r$ be commutative matrices in $\mathbb{C}^{N\times N}$ satisfying* (1.3)*, and let $B_1, B_2, \ldots, B_r$ be commutative matrices in $\mathbb{C}^{N\times N}$ satisfying* (1.4)*. Then*

$$\begin{aligned}&\sum_{k=0}^{\left[\frac{n}{m}\right]} \frac{(-1)^k (B_1+\ldots+B_r)_k}{k! b^k \Gamma\left(\frac{n-mk}{p}+1\right)}\left(\log(a)(x_1\sqrt{\nu_1 A_1}+\ldots+x_r\sqrt{\nu_r A_r})\right)^{\frac{n-mk}{p}}\\ &= \sum_{n_1+n_2+\ldots+n_r=n} \frac{H_{n_1,m,p}(x_1,A_1,B_1;a,b,\nu_1)\ H_{n_2,m,p}(x_2,A_2,B_2;a,b,\nu_2)}{n_1! n_2! \times \ldots\ n_r!}\\ &\quad \ldots \times H_{n_r,m,p}(x_r,A_r,B_r;a,b,\nu_r).\end{aligned} \tag{2.22}$$

Proof. We have

$$
\begin{aligned}
&a^{t^p(x_1\sqrt{\nu_1 A_1}+\ldots+x_r\sqrt{\nu_r A_k})}\left(1+\frac{t^m}{b}\right)^{-B_1-\ldots-B_r}\\
&= a^{t^p x_1\sqrt{\nu_1 A_1}}\left(1+\frac{t^m}{b}\right)^{-B_1}\ldots a^{t^p x_r\sqrt{\nu_k A_r}}\left(1+\frac{t^m}{b}\right)^{-B_r}\\
&= \sum_{n_1=0}^{\infty}\frac{H_{n_1,m,p}(x_1,A_1,B_1;a,b,\nu_1)t^{n_1}}{n_1}\\
&\quad \ldots\times\sum_{n_k=0}^{\infty}\frac{H_{n_k,m,p}(x_k,A_k,B_k;a,b,\nu_k)t^{n_k}}{n_k!}\\
&= \sum_{n=0}^{\infty}\left[\sum_{n_1+\ldots+n_r=n}\frac{H_{n_1,m,p}(x_1,A_1,B_1;a,b,\nu_1)\ H_{n_2,m,p}(x_2,A_2,B_2;a,b,\nu_2)}{n_1!\ldots\ n_r!}\right]t^n\\
&\quad \ldots\times H_{n_r,m,p}(x_r,A_r,B_2;a,b,\nu_r)=\sum_{n=0}^{\infty}\sum_{k=0}^{[\frac{n}{m}]}\frac{(-1)^k(B_1+B_2+\ldots+B_r)_k}{k!b^k\Gamma\left(\frac{n-mk}{p}+1\right)}\\
&\quad \times\left(\log(a)(x_1\sqrt{\nu_1 A_1}+\ldots+x_r\sqrt{\nu_r A_r})\right)^{\frac{n-mk}{p}}t^n.
\end{aligned}
$$

Comparing the coefficients of t^n, we have the desired relation (2.22). □

3. Integral representations

The aim of this section is to introduce a generalization for Chebyshev, Legendre, and Gegenbauer matrix polynomials by modifying the integral transform, which can be easily established by the application of beta and gamma function formulae, generalized Hermite matrix polynomials, and other techniques in the following theorem.

Theorem 3.1. *Let A and B be commutative matrices in $\mathbb{C}^{N\times N}$. For any complex number ν, let νA be a positive stable matrix in $\mathbb{C}^{N\times N}$ satisfying* (1.3)*, and let B be a matrix in $\mathbb{C}^{N\times N}$ satisfying* (1.4)*. Then the generalized Chebyshev, Legendre, and Gegenbauer matrix polynomials are given by modifying the integral transforms involving generalized Hermite matrix polynomials as follows:*

$$P_{n,m,p}(x,A,B;a,b,\nu)=\frac{2}{n!\sqrt{\pi}}\int_0^{\infty}e^{-t^2}t^nH_{n,m,p}(xt,A,B;a,b,\nu)dt, \quad (3.1)$$

$$U_{n,m,p}(x,A,B;a,b,\nu)=\frac{1}{n!}\int_0^{\infty}e^{-t}t^{\frac{n}{m}}H_{n,m,p}(xt^{\frac{m-p}{m}},A,B;a,b,\nu)dt, \quad (3.2)$$

$$T_{n,m,p}(x,A,B;a,b,\nu) = \frac{(\sqrt{\nu A})^{-1}}{(n-1)!}\int_0^\infty e^{-t}t^{\frac{n}{m}-1}H_{n,m,p}(xt^{\frac{m-p}{m}},A,B;a,b,\nu)dt, \quad n\geq 1, \tag{3.3}$$

where $T_{0,m,p}(x,A,B;a,b,\nu)=\boldsymbol{0}$,

$$W_{n,m,p}(x,A,B;a,b,\nu) = \frac{\sqrt{\nu A}}{(n+1)!}\int_0^\infty e^{-t}t^{\frac{n}{m}+1}H_{n,m,p}(xt^{\frac{m-p}{m}},A,B;a,b,\nu)dt, \tag{3.4}$$

and

$$\mathbf{C}^v_{n,m,p}(x,A,B;a,b,\nu) = \frac{1}{n!\Gamma(v)}\int_0^\infty e^{-t}t^{\frac{n}{m}+v-1}H_{n,m,p}(xt^{\frac{m-p}{m}},A,B;a,b,\nu)dt. \tag{3.5}$$

Proof. To prove (3.1), by using (2.3), (1.7), and (1.8), it follows that

$$\begin{aligned}
&\frac{2}{n!\sqrt{\pi}}\int_0^\infty e^{-t^2}t^{\frac{n}{p}}H_{n,m,p}(xt,A,B;a,b,\nu)dt \\
&= \frac{2}{\sqrt{\pi}}\sum_{k=0}^{[\frac{n}{m}]}\frac{(-1)^k(B)_k\left(x\log(a)t\sqrt{\nu\ A}\right)^{\frac{n-mk}{p}}}{k!b^k\Gamma\left(fracn-mkp+1\right)}\int_0^\infty e^{-t^2}t^{\frac{2n-mk}{p}}dt \\
&= \sum_{k=0}^{[\frac{n}{m}]}\frac{(-1)^k(B)_k\Gamma\left(\frac{2n-mk}{p}+1\right)\left(x\log(a)\sqrt{\nu\ A}\right)^{\frac{n-mk}{p}}}{2^{\frac{2n-mk}{p}}k!b^k\ \Gamma\left(\frac{n-mk}{p}+1\right)\Gamma\left(\frac{2n-mk}{2p}\right)}.
\end{aligned}$$

Hence, the generalized Legendre matrix polynomials can be given by

$$P_{n,m,p}(x,A,B;a,b,\nu) = \sum_{k=0}^{[\frac{n}{m}]}\frac{(-1)^k(B)_k\Gamma\left(\frac{2n-mk}{p}+1\right)\left(x\log(a)\sqrt{\nu\ A}\right)^{\frac{n-mk}{p}}}{2^{\frac{2n-mk}{p}}k!b^k\ \Gamma\left(\frac{n-mk}{p}+1\right)\Gamma\left(\frac{2n-mk}{2p}\right)},$$

or

$$P_{n,m,p}(x,A,B;a,b,\nu) = \sum_{k=0}^{[\frac{n}{m}]}\frac{(-1)^k(B)_k\left(\frac{1}{2}\right)_{\frac{2n-mk}{2p}}}{k!b^k\ \Gamma\left(\frac{n-mk}{p}+1\right)}\left(x\log(a)\sqrt{\nu\ A}\right)^{\frac{n-mk}{p}},$$

which completes the proof of (3.1).

Using (1.8), (2.3) and (3.2), we can write

$$\frac{1}{n!}\int_0^\infty \exp(-t)t^{\frac{n}{m}} H_{n,m,p}(xt^{\frac{m-p}{p}}, A, B; a, b, \nu)dt$$
$$= \sum_{k=0}^{\left[\frac{n}{m}\right]} \frac{(-1)^k \Gamma\left(\frac{n-(m-p)k}{p}+1\right) y^k}{k!\Gamma\left(\frac{n-mk}{p}+1\right)} (x\sqrt{mA})^{\frac{n-mk}{p}}.$$

Hence, the generalized Chebyshev matrix polynomials of the second kind can be defined by

$$U_{n,m,p}(x, A, B; a, b, \nu) = \sum_{k=0}^{\left[\frac{n}{m}\right]} \frac{(-1)^k (B)_k \Gamma\left(\frac{n-(m-p)k}{p}+1\right)}{k! b^k \Gamma\left(\frac{n-mk}{p}+1\right)} \left(x\log(a)\sqrt{\nu\, A}\right)^{\frac{n-mk}{p}}.$$

In a similar way, we define the generalized Chebyshev matrix polynomials of the first kind

$$T_{n,m,p}(x, A, B; a, b, \nu)$$
$$= n\sum_{k=0}^{\left[\frac{n}{m}\right]} \frac{(-1)^k (B)_k \Gamma\left(\frac{n-(m-p)k}{p}\right)}{k! b^k \Gamma\left(\frac{n-mk}{p}+1\right)} \left(x\log(a)\sqrt{\nu\, A}\right)^{\frac{n-mk}{p}}, \quad n > 0,$$

the generalized Chebyshev matrix polynomials of the third kind

$$W_{n,m,p}(x, A, B; a, b, \nu)$$
$$= \frac{1}{n+1}\sum_{k=0}^{\left[\frac{n}{m}\right]} \frac{(-1)^k (B)_k \Gamma\left(\frac{n-(m-p)k}{p}+2\right)}{k! b^k \Gamma\left(\frac{n-mk}{p}+1\right)} \left(x\log(a)\sqrt{\nu\, A}\right)^{\frac{n-mk}{p}}, \quad n > -1,$$

and the generalized Gegenbauer matrix polynomials in the form

$$\mathbf{C}^{v}_{n,m,p}(x, A, B; a, b, \nu)$$
$$= \frac{1}{\Gamma(v)}\sum_{k=0}^{\left[\frac{1}{2}n\right]} \frac{(-1)^k (B)_k \Gamma\left(\frac{n-(m-p)k}{p}+v\right)}{k! b^k \Gamma\left(\frac{n-mk}{p}+1\right)} \left(x\log(a)\sqrt{\nu\, A}\right)^{\frac{n-mk}{p}}.$$

Using (1.7) and (1.8), we get the explicit expressions (3.3), (3.4), and (3.5) in a similar manner. □

4. Fractional integrals and derivatives for the generalized Hermite matrix polynomials

In this section, we determine the fractional integrals and fractional derivatives for the generalized Hermite matrix polynomials $H_{n,m,p}(x, A, B; a, b, \nu)$.

Theorem 4.1. *The generalized Hermite matrix polynomials satisfy the formula*

$$\begin{aligned}\mathbb{I}^{\mu}\{H_{n,m,p}\} =& \frac{1}{(n+1)_{p\mu}}\left(\log(a)\sqrt{\nu\, A}\right)^{-\mu}\\ &\times H_{n+p\mu,m,p}(x,A,B;a,b,\nu), \quad n+p\mu \geq 0.\end{aligned} \tag{4.1}$$

Proof. From (2.3) and (1.9), we have

$$\begin{aligned}\mathbb{I}^{\mu}\{H_{n,m,p}\} &= \frac{1}{\Gamma(\mu)}\int_0^x (x-t)^{\mu-1}H_{n,m,p}(x,A,B;a,b,\nu)dt\\ &= \frac{n!}{\Gamma(\mu)}\sum_{k=0}^{\left[\frac{1}{m}n\right]}\frac{(-1)^k(B)_k\left(\log(a)\sqrt{\nu A}\right)^{\frac{n-mk}{p}}}{k!b^k\Gamma\left(\frac{n-mk}{p}+1\right)}\int_0^x (x-t)^{\mu-1}t^{\frac{n-mk}{p}}dt.\end{aligned}$$

Putting $t=xu$, $dt=xdu$, $t=0$, $u=0$, and $t=x$, $u=1$, we get

$$\int_0^x (x-t)^{\mu-1}t^{\frac{n-mk}{p}}dt = x^{\mu+\frac{n-mk}{p}}\frac{\Gamma(\mu)\Gamma\left(\frac{n-mk}{p}+1\right)}{\Gamma\left(\mu+\frac{n-mk}{p}+1\right)},$$

and we can write

$$\begin{aligned}\mathbb{I}^{\mu}\{H_{n,m,p}\} &= n!\sum_{k=0}^{\left[\frac{1}{m}n\right]}\frac{(-1)^k(B)_k\left(\log(a)\sqrt{\nu A}\right)^{\frac{n-mk}{p}}}{k!b^k\Gamma\left(\mu+\frac{n-mk}{p}+1\right)}x^{\mu+\frac{n-mk}{p}}\\ &= \frac{\left(\log(a)\sqrt{\nu\, A}\right)^{-\mu}}{(n+1)_{p\mu}}H_{n+p\mu,m,p}(x,A,B;a,b,\nu),\end{aligned}$$

which gives (4.1). □

Theorem 4.2. *The generalized Hermite matrix polynomials have the left-sided operator of Riemann–Liouville fractional integral*

$$\begin{aligned}{}_b\mathbb{I}_x^{\alpha}\{H_{n,m,p}(x-b,A,B;a,b,\nu)\} =& \frac{1}{(n+1)_{p\alpha}}\left(\log(a)\sqrt{\nu\, A}\right)^{-\alpha}\\ &\times H_{n+p\alpha,m,p}(x-b,A,B;a,b,\nu), \quad n+p\alpha \geq 0.\end{aligned} \tag{4.2}$$

Proof. Using (2.3) in the right hand side of (1.10), we have

$$\begin{aligned}&{}_b\mathbb{I}_x^{\alpha}\{H_{n,m,p}(x-b,A,B;a,b,\nu)\}\\ &= \frac{1}{\Gamma(\alpha)}\int_b^x (x-t)^{\alpha-1}H_{n,m,p}(t-b,A,B;a,b,\nu)dt\\ &= \frac{n!}{\Gamma(\alpha)}\sum_{k=0}^{\left[\frac{n}{m}\right]}\frac{(-1)^k(B)_k\left(\log(a)\sqrt{\nu A}\right)^{\frac{n-mk}{p}}}{k!b^k\Gamma\left(\frac{n-mk}{p}+1\right)}\int_b^x (x-t)^{\alpha-1}(t-b)^{\frac{n-mk}{p}}dt.\end{aligned}$$

Putting $u = \frac{t-b}{x-b}$, $t - b = (x-b)u$, $dt = (x-b)du$ $t = b$, $u = 0$, and $t = x$, $u = 1$, we get

$$\int_b^x (x-t)^{\alpha-1}(t-b)^{\frac{n-mk}{p}}dt = (x-b)^{\alpha+\frac{n-mk}{p}} \frac{\Gamma(\alpha)\Gamma\left(\frac{n-mk}{p}+1\right)}{\Gamma\left(\alpha+\frac{n-mk}{p}+1\right)},$$

and we can write

$$\begin{aligned}
&{}_b\mathbb{I}_x^\alpha\{H_{n,m,p}(x-b,A,B;a,b,\nu)\} \\
&= n!\sum_{k=0}^{\left[\frac{n}{m}\right]} \frac{(-1)^k(B)_k\left(\log(a)\sqrt{\nu A}\right)^{\frac{n-mk}{p}}}{k!b^k\Gamma\left(\alpha+\frac{n-mk}{p}+1\right)}(x-b)^{\alpha+\frac{n-mk}{p}} \\
&= \frac{\left(\log(a)\sqrt{\nu\ A}\right)^{-\alpha}}{(n+1)_{p\alpha}} H_{n+p\alpha,m,p}(x-b,A,B;a,b,\nu).
\end{aligned}$$

Thus, we get the desired result (4.2). □

Theorem 4.3. *For the generalized Hermite matrix polynomials, one has*

$$\begin{aligned}
{}_x\mathbb{I}_c^\alpha\{H_{n,m,p}(c-x,A,B;a,b,\nu)\} &= \frac{1}{(n+1)_{p\alpha}}\left(\log(a)\sqrt{\nu\ A}\right)^{-\alpha} \\
&\times H_{n+p\alpha,m,p}(c-x,A,B;a,b,\nu), \quad n+p\alpha\ge 0.
\end{aligned} \tag{4.3}$$

Proof. With the help of (1.11) and (2.3), one obtains (4.3). □

Theorem 4.4. *The Weyl integral of the generalized Hermite matrix polynomials of order α satisfies the formula*

$$\begin{aligned}
{}_xW_\infty^\alpha\{H_{n,m,p}(x,A,B;a,b,\nu)\} &= \frac{(-1)^\alpha}{(n+1)_{p\alpha}}\left(\log(a)\sqrt{\nu\ A}\right)^{-\alpha} \\
&\times H_{n+p\alpha,m,p}(x,A,B;a,b,\nu), \quad n+p\alpha\ge 0.
\end{aligned} \tag{4.4}$$

Proof. From (2.3) and (1.12), we have

$$\begin{aligned}
{}_xW_\infty^\alpha\{H_{n,m,p}\} &= \frac{1}{\Gamma(\alpha)}\int_x^\infty (t-x)^{\alpha-1}H_{n,m,p}(t,A,B;a,b,\nu)dt \\
&= \frac{n!}{\Gamma(\alpha)}\sum_{k=0}^{\left[\frac{n}{m}\right]} \frac{(-1)^k(B)_k\left(\log(a)\sqrt{\nu A}\right)^{\frac{n-mk}{p}}}{k!b^k\Gamma\left(\frac{n-mk}{p}+1\right)}\int_x^\infty (t-x)^{\alpha-1}t^{\frac{n-mk}{p}}dt.
\end{aligned}$$

Putting $u = \frac{x}{t}$, $t = \frac{x}{u}$, $dt = -\frac{x}{u^2}du$, $t = \infty$, $u = 0$ and $t = x$, $u = 1$, we get

$$\int_x^\infty (t-x)^{\alpha-1}t^{\frac{n-mk}{p}}dt = x^{\alpha+\frac{n-mk}{p}} \frac{\Gamma(\alpha)\Gamma\left(\frac{mk-n}{p}-\alpha\right)}{\Gamma\left(\frac{mk-n}{p}\right)},$$

and

$$
\begin{aligned}
{}_xW_\infty^\alpha\{H_{n,m,p}\} &= n!\sum_{k=0}^{\left[\frac{n}{m}\right]}\frac{(-1)^k(B)_k\left(\log(a)\sqrt{\nu A}\right)^{\frac{n-mk}{p}}}{k!b^k\Gamma\left(\alpha+\frac{n-mk}{p}+1\right)}x^{\alpha+\frac{n-mk}{p}}\\
&= \frac{(-1)^\alpha\left(\log(a)\sqrt{\nu\, A}\right)^{-\alpha}}{(n+1)_{p\alpha}}H_{n+p\alpha,m,p}(x,A,B;a,b,\nu),
\end{aligned}
$$

which gives (4.4). □

Theorem 4.5. *Let $\alpha\in\mathbb{C}$, $Re(\alpha)\geq 0$, and let $n=[Re(\alpha)]+1$. Let ${}_xD_c^\alpha$ be the right sided Riemann–Liouville fractional derivative. Then for the generalized Hermite matrix polynomials, one has*

$$
\begin{aligned}
{}_xD_c^\alpha\{H_{n,m,p}(c-x,A,B;a,b,\nu)\} &= \frac{\Gamma(n+1)}{\Gamma(n+1-p\alpha)}\left(\log(a)\sqrt{\nu\, A}\right)^\alpha\\
&\times H_{n-p\alpha,m,p}(c-x,A,B;a,b,\nu),\quad n-p\alpha\geq 0.
\end{aligned}
\tag{4.5}
$$

Proof. Using (2.3) and (1.14), we have

$$
\begin{aligned}
{}_xD_c^\alpha\{H_{n,m,p}(c-x,A,B;a,b,\nu)\} &= \frac{n!(-1)^n}{\Gamma(n-\alpha)}\sum_{k=0}^{\left[\frac{n}{m}\right]}\frac{(-1)^k(B)_k}{k!b^k\Gamma\left(\frac{n-mk}{p}+1\right)}\\
\times\left(\log(a)\sqrt{\nu A}\right)^{\frac{n-mk}{p}}&\left(\frac{\partial}{\partial x}\right)^n\int_x^c\frac{(c-t)^{\frac{n-mk}{p}}}{(t-x)^{\alpha-n+1}}dt.
\end{aligned}
$$

Putting $u=\frac{c-t}{c-x}$, $c-t=(c-x)u$, $dt=(c-x)du$, $t=c$, $u=0$, and $t=x$, $u=1$, we get

$$
\int_x^c\frac{(c-t)^{\frac{n-mk}{p}}}{(t-x)^{\alpha-n+1}}dt=(c-x)^{n-\alpha+\frac{n-mk}{p}}\frac{\Gamma(n-\alpha)\Gamma\left(\frac{n-mk}{p}+1\right)}{\Gamma\left(n+\frac{n-mk}{p}-\alpha+1\right)},
$$

and we can write

$$
\begin{aligned}
&{}_xD_c^\alpha\{H_{n,m,p}(c-x,A,B;a,b,\nu)\}\\
&= n!\sum_{k=0}^{\left[\frac{n}{m}\right]}\frac{(-1)^k(B)_k\left(\log(a)\sqrt{\nu A}\right)^{\frac{n-mk}{p}}}{k!b^k\Gamma\left(1-\alpha+\frac{n-mk}{p}\right)}(c-x)^{\frac{n-mk}{p}-\alpha}\\
&= \frac{\left(\log(a)\sqrt{\nu\, A}\right)^\alpha\Gamma(n+1)}{\Gamma\left(n-p\alpha+1\right)}H_{n-p\alpha,m,p}(c-x,A,B;a,b,\nu),
\end{aligned}
$$

which gives (4.5). □

Theorem 4.6. *The left-sided operator of Riemann–Liouville fractional derivative for the generalized Hermite matrix polynomials satisfies the formula*

$$\begin{aligned} {}_bD_x^{\alpha}\{H_{n,m,p}(x-b,A,B;a,b,\nu)\} &= \frac{\Gamma(n+1)}{\Gamma(n+1-p\alpha)}\left(\log(a)\sqrt{\nu\, A}\right)^{\alpha} \\ &\times H_{n-p\alpha,m,p}(x-b,A,B;a,b,\nu), \quad n-p\alpha \geq 0. \end{aligned} \tag{4.6}$$

Proof. With the help of (1.13) and (2.3), one can obtains (4.6). □

Theorem 4.7. *The Weyl fractional derivative of the generalized Hermite matrix polynomials of order α satisfies the formula*

$$\begin{aligned} {}_xD_{\infty}^{\alpha}\{H_{n,m,p}\} &= \left(\log(a)\sqrt{\nu\, A}\right)^{\alpha} \\ &\times (-n)_{p\alpha} H_{n-p\alpha,m,p}(x,A,B;a,b,\nu), \quad n-p\alpha \geq 0. \end{aligned} \tag{4.7}$$

Proof. Using (2.3) and (1.15), we have

$$\begin{aligned} {}_xD_{\infty}^{\alpha}\{H_{n,m,p}\} &= \frac{n!(-1)^n}{\Gamma(m-\alpha)}\sum_{k=0}^{\left[\frac{n}{m}\right]}\frac{(-1)^k(B)_k\left(\log(a)\sqrt{\nu A}\right)^{\frac{n-mk}{p}}}{k!b^k\Gamma\left(\frac{n-mk}{p}+1\right)} \\ &\times\left(\frac{\partial}{\partial x}\right)^m\int_x^{\infty}\frac{t^{\frac{n-mk}{p}}}{(t-x)^{\alpha-n+1}}dt. \end{aligned}$$

Putting $u=\frac{x}{t}$, $t=\frac{x}{u}$, $dt=-\frac{x}{u^2}du$, $t=\infty$, $u=0$, and $t=x$, $u=1$, we get

$$\int_x^{\infty}\frac{t^{\frac{n-mk}{p}}}{(t-x)^{\alpha-m+1}}dt = \frac{(-1)^{\alpha-m-\frac{n-mk}{p}}}{(-1)^{-\frac{n-mk}{p}}}x^{m-\alpha+\frac{n-mk}{p}}\frac{\Gamma(m-\alpha)\Gamma\left(\frac{n-mk}{p}+1\right)}{\Gamma\left(m-\alpha+\frac{n-mk}{p}+1\right)},$$

and we can write

$${}_xD_{\infty}^{\alpha}\{H_{n,m,p}\} = \frac{(-1)^{\alpha}\left(\log(a)\sqrt{\nu\, A}\right)^{\alpha}\Gamma\big(n+1\big)}{\Gamma\left(n-p\alpha+1\right)}H_{n-p\alpha,m,p}(x,A,B;a,b,\nu),$$

which gives (4.7). □

5. Concluding remarks

The generalized Hermite matrix polynomials discussed in this paper are be useful for investigators in various problems of physics, applied sciences and engineering, and comprise an emerging field of study with important results in the literature. In this paper, we extend the generalized Hermite matrix polynomials of one variable to two variables. We define the generalized

Hermite matrix polynomials of two variables in a series as follows:

$$H_{n,m,p}(x,y,A,B;a,b,\nu)=n!\sum_{k=0}^{\left[\frac{n}{m}\right]}\frac{(-1)^k(B)_k y^k}{k!b^k\Gamma\left(\frac{n-mk}{p}+1\right)}\left(x\log(a)\sqrt{\nu A}\right)^{\frac{n-mk}{p}}.$$

We also consider the sum

$$\begin{aligned}\sum_{n=0}^{\infty}&H_{n,m,p}(x,y,A,B;a,b,\nu)\frac{t^n}{n!}\\&=\sum_{n=0}^{\infty}\sum_{k=0}^{\left[\frac{n}{m}\right]}\frac{(-1)^k(B)_k y^k}{k!b^k\Gamma\left(\frac{n-mk}{p}+1\right)}\left(x\log(a)\sqrt{\nu A}\right)^{\frac{n-mk}{p}}t^n\\&=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^k(B)_k y^k}{k!b^k\Gamma(n+1)}\left(x\log(a)\sqrt{\nu A}\right)^{n}t^{np+mk}\\&=a^{xt^p\sqrt{\nu A}}\left(1+\frac{yt^m}{b}\right)^{-B},\quad\left|\frac{yt^m}{b}\right|<1.\end{aligned}$$

Using these equalities, we obtain the matrix generating function for the generalized Hermite matrix polynomials of two variables in the form

$$\sum_{n=0}^{\infty}H_{n,m,p}(x,y,A,B;a,b,\nu)\frac{t^n}{n!}=a^{xt^p\sqrt{\nu A}}\left(1+\frac{yt^m}{b}\right)^{-B}.$$

The results established in this paper express a clear idea that the use of fractional integrals and fractional derivatives techniques provide a simple and straightforward method to get new relations for special matrix polynomials and matrix functions.

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