ACTA ET COMMENTATIONES UNIVERSITATIS TARTUENSIS DE MATHEMATICA
Volume 22, Number 2, December 2018
Available online at http://acutm.math.ut.ee

Boundedness of the L-index in a direction of entire solutions of second order partial differential equation

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Abstract. We construct a continuous function $L : \mathbb{C}^2 \to (0, +\infty)^2$ such that every entire solution of a certain second order partial differential equation has bounded L-index in a direction $\mathbf{b} = (b_1, b_2) \in \mathbb{C}^2 \setminus \{0\}$. On the other hand, the entire bivariate function $F(z_1, z_2) = \cos\sqrt{z_1 z_2}$ is a solution of this equation. The function F has unbounded index in each direction $\mathbf{b} \in \mathbb{C}^2 \setminus \{0\}$. The constructed function L is a full solution of a problem posed by A. A. Kondratyuk's in 2007 about the existence of the function L with specified properties. We also suggest a continuous function $L_1 : \mathbb{C}^2 \to (0, +\infty)^2$ such that the function F has bounded L_1-index in every direction $\mathbf{b} \in \mathbb{C}^n \setminus \{0\}$.

1. Introduction

Entire functions of bounded index have been used in the theory of value distribution and differential equations [2, 4, 8, 11, 14–16]. A full bibliography can be found in [6]. In particular, Strelitz [13] developed a Wiman–Valiron method for a description of the asymptotic behavior of analytical solutions of ordinary differential equations. He [14] also used the method to investigate index boundedness of entire solutions of algebraic differential equations. In this paper, we apply a partial differential equation to establish the boundedness of the L-index in a direction of a function F. Namely, the present work is devoted to the construction of a function L such that the function

$$F(z_1, z_2) = \cos\sqrt{z_1 z_2} = \sum_{p=0}^{\infty} \frac{(-1)^p (z_1 z_2)^p}{(2p)!}$$

Received May 15, 2017.
2010 *Mathematics Subject Classification.* 32A15; 32W50; 32A60; 30D15.
Key words and phrases. Entire bivariate function; bounded L-index in direction; linear partial differential equation.
http://dx.doi.org/10.12697/ACUTM.2018.22.18

has bounded L-index in a direction $\mathbf{b} = (b_1, b_2)$ (see the definition below). It is an interesting problem in view of the properties of the function $F(z_1, z_2)$: for a given $z^0 = (z_1^0, z_2^0) \in \mathbb{C}^2$, the function $g_{z^0}(t) = F(z_1^0 + tb_1, z_2^0 + tb_2)$ has bounded index as a function of the variable t, but $F(z_1, z_2)$ is of unbounded index in the direction $\mathbf{b} = (b_1, b_2)$, i.e., indices of $F(z_1^0 + tb_1, z_2^0 + tb_2)$ are uniformly unbounded in z^0. Simultaneously, it is known [5] that, for an entire function F, there exists a positive continuous function $L(z)$ such that $F(z)$ is a function of bounded L-index in the direction $\mathbf{b}$ if and only if the multiplicities of zeros of the function $g_{z^0}(t) \not\equiv 0$ are uniformly bounded in z^0.

Note that the concept of bounded L-index in a direction has a few advantages in the comparison with traditional approaches to study the properties of entire solutions of differential equations. In particular, if an entire solution has bounded index, then it immediately yields its growth estimates, a uniform in a sense distribution of its zeros, a certain regular behavior of the solution, etc (see the bibliography in [6]).

2. Main definitions and notations

An entire function $F(z)$, $z \in \mathbb{C}^n$, is called *a function of bounded L-index in a direction* $\mathbf{b} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$ (see [4, 6, 7]) if there exists $m_0 \in \mathbb{Z}_+$ such that, for every $m \in \mathbb{Z}_+$ and every $z \in \mathbb{C}^n$,

$$\frac{1}{m!L^m(z)} \left| \frac{\partial^m F(z)}{\partial \mathbf{b}^m} \right| \le \max \left\{ \frac{1}{k!L^k(z)} \left| \frac{\partial^k F(z)}{\partial \mathbf{b}^k} \right| : 0 \le k \le m_0 \right\}, \quad (1)$$

where $\frac{\partial^0 F(z)}{\partial \mathbf{b}^0} := F(z)$, $\frac{\partial F(z)}{\partial \mathbf{b}} := \sum\limits_{j=1}^{n} \frac{\partial F(z)}{\partial z_j} b_j = \langle \mathbf{grad} F, \overline{\mathbf{b}} \rangle$, $\frac{\partial^k F(z)}{\partial \mathbf{b}^k} := \frac{\partial}{\partial \mathbf{b}}\big(\frac{\partial^{k-1} F(z)}{\partial \mathbf{b}^{k-1}}\big)$, $k \ge 2$.

The least such integer $m_0 = m_0(\mathbf{b})$ is called the *L-index in the direction $\mathbf{b} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$ of the entire function* $F(z)$, and is denoted by $N_{\mathbf{b}}(F, L) = m_0$. If such an m_0 does not exist, then F is called *a function of unbounded L-index in the direction* $\mathbf{b}$, and we write $N_{\mathbf{b}}(F, L) = \infty$.

If $L(z) \equiv 1$, then $F(z)$ is called *a function of bounded index in the direction* $\mathbf{b}$, and $N_{\mathbf{b}}(F) = N_{\mathbf{b}}(F, 1)$.

In the case $n = 1$, we obtain the definition of an entire function of one variable of bounded l-index (see [9, 12]); in the case $n = 1$ and $L(z) \equiv 1$, it is reduced to the definition of bounded index, suggested by Lepson [10].

For $\eta > 0$, $z \in \mathbb{C}^n$, $\mathbf{b} = (b_1, \ldots, b_n) \in \mathbb{C}^n \setminus \{\mathbf{0}\}$, and a positive continuous function $L : \mathbb{C}^n \to \mathbb{R}_+$, we define

$$\lambda_1^{\mathbf{b}}(z, t_0, \eta) = \inf \left\{ \frac{L(z + t\mathbf{b})}{L(z + t_0\mathbf{b})} : |t - t_0| \le \frac{\eta}{L(z + t_0\mathbf{b})} \right\},$$

$$\lambda_1^{\mathbf{b}}(z, \eta) = \inf\{\lambda_1^{\mathbf{b}}(z, t_0, \eta) : t_0 \in \mathbb{C}\}, \quad \lambda_1^{\mathbf{b}}(\eta) = \inf\{\lambda_1^{\mathbf{b}}(z, \eta) : z \in \mathbb{C}^n\},$$

$$\lambda_2^{\mathbf{b}}(z,t_0,\eta)=\sup\left\{\frac{L(z+t\mathbf{b})}{L(z+t_0\mathbf{b})}:|t-t_0|\le\frac{\eta}{L(z+t_0\mathbf{b})}\right\},$$

$$\lambda_2^{\mathbf{b}}(z,\eta)=\sup\{\lambda_2^{\mathbf{b}}(z,t_0,\eta):t_0\in\mathbb{C}\},\quad \lambda_2^{\mathbf{b}}(\eta)=\sup\{\lambda_2^{\mathbf{b}}(z,\eta):z\in\mathbb{C}^n\}.$$

By $Q_{\mathbf{b}}^n$ we denote the class of functions L, which satisfy the condition

$$(\forall\eta\ge 0):\ 0<\lambda_1^{\mathbf{b}}(\eta)\le\lambda_2^{\mathbf{b}}(\eta)<+\infty. \tag{2}$$

For a positive continuous function $l(z)$ with $z\in\mathbb{C}$, and $z_0\in\mathbb{C}$, $\eta>0$, we denote $\lambda_1(z_0,\eta)\equiv\lambda_1^{\mathbf{b}}(0,z_0,\eta)$ and $\lambda_2(z_0,\eta)\equiv\lambda_2^{\mathbf{b}}(0,z_0,\eta)$ in the case $z=0$, $\mathbf{b}=1$, $n=1$, $L\equiv l$, and $\lambda_1(\eta)=\inf\{\lambda_1(z_0,\eta):z_0\in\mathbb{C}\}$, $\lambda_2(\eta)=\sup\{\lambda_2(z_0,\eta):z_0\in\mathbb{C}\}$, and $Q=Q_1^1$.

Let $L^*(z)$ be a positive continuous function in $\mathbb{C}^n$. We write $L\asymp L^*$ if for some θ_1,θ_2 with $0<\theta_1\le\theta_2<+\infty$, and for all $z\in\mathbb{C}^n$, the inequalities $\theta_1L(z)\le L^*(z)\le\theta_2L(z)$ hold.

For a given $z^0\in\mathbb{C}^n$, we denote $g_{z^0}(t):=F(z^0+t\mathbf{b})$. If one has $g_{z^0}(t)\ne 0$ for all $t\in\mathbb{C}$, then $G_r^{\mathbf{b}}(F,z^0):=\emptyset$; if $g_{z^0}(t)\equiv 0$, then $G_r^{\mathbf{b}}(F,z^0):=\{z^0+t\mathbf{b}\colon t\in\mathbb{C}\}$. If $g_{z^0}(t)\not\equiv 0$ and a_k^0 are zeros of the function $g_{z^0}(t)$, then

$$G_r^{\mathbf{b}}(F,z^0):=\bigcup_k\left\{z^0+t\mathbf{b}\colon |t-a_k^0|\le\frac{r}{L(z^0+a_k^0\mathbf{b})}\right\},\quad r>0.$$

Let

$$G_r^{\mathbf{b}}(F)=\bigcup_{z^0\in\mathbb{C}^n}G_r^{\mathbf{b}}(F,z^0). \tag{3}$$

Remark that if $L(z)\equiv 1$, then $G_r^{\mathbf{b}}(F)\subset\{z\in\mathbb{C}^n: dist(z,Z_F)<r|\mathbf{b}|\}$, where Z_F is the zero set of the function F. By $n\big(r,z^0,t_0,1/F\big)=\sum_{|a_k^0-t_0|\le r}1$ we denote the counting function of the zero sequence (a_k^0).

Exploring properties of entire functions of bounded L-index in a direction, we obtained the following assertion.

Theorem 1 (see [4], [6])**.** *An entire function $F(z)$ is of bounded L-index in the direction $\mathbf{b}$ if and only if there exists a number $M>0$ such that, for all $z^0\in\mathbb{C}^n$, the function $g_{z^0}(t)$ is of bounded l_{z^0}-index with $N(g_{z^0},l_{z^0})\le M<+\infty$ as a function of variable $t\in\mathbb{C}$, and $N_{\mathbf{b}}(F,L)=\max\{N(g_{z^0},l_{z^0}):z^0\in\mathbb{C}^n\}$.*

In view of Theorem 1, a natural question (see [3]) arises.

Problem 1. Does there exist an entire function $F(z)$, $z\in\mathbb{C}^n$, with $N(g_{z^0},l_{z^0})<+\infty$ for every $z^0\in\mathbb{C}^n$, but $N_{\mathbf{b}}(F,L)=+\infty$?

We gave an affirmative answer (see [3]) to the above-mentioned question: we proved that

$$F(z_1,z_2)=\cos\sqrt{z_1z_2}=\sum_{p=0}^{\infty}\frac{(-1)^p(z_1z_2)^p}{(2p)!}$$

is of unbounded index in the direction $\mathbf{b} = (1, 1)$. Recently, this result was generalized for each direction $\mathbf{b} \in \mathbb{C}^2 \setminus \{0\}$ in [6].

At the Lviv city seminar on the theory of analytic functions (fall, 2007), A. A. Kondratyuk asked the following question.

Problem 2. Does there exist a continuous function $L : \mathbb{C}^2 \to \mathbb{R}_+$ providing the boundedness of L-index in the direction $\mathbf{b}$ of the function $F(z_1, z_2) = \cos\sqrt{z_1 z_2}$?

Using methods from [3], we give an answer to Kondratyuk's question in Theorem 5.

3. Auxiliary propositions

The notation $L \asymp L^*$ means that for some $\theta_1, \theta_2 \in \mathbb{R}_+$ with $0 < \theta_1 \le \theta_2 < +\infty$ and for all $z \in \mathbb{C}^n$, the inequalities $\theta_1 L(z) \le L^*(z) \le \theta_2 L(z)$ hold.

Theorem 2 (see [4,6])**.** *Let $L \in Q^n_{\mathbf{b}}$, $L \asymp L^*$. An entire function $F(z)$ is of bounded L^*-index in the direction $\mathbf{b}$ if and only if F is of bounded L-index in the direction $\mathbf{b}$.*

We consider the partial differential equation (PDE)

$$g_0(z)\frac{\partial^p w}{\partial \mathbf{b}^p} + g_1(z)\frac{\partial^{p-1} w}{\partial \mathbf{b}^{p-1}} + \ldots + g_p(z)w = h(z). \tag{4}$$

Theorem 3 (see [4,6])**.** *Let $L \in Q^n_{\mathbf{b}}$, let entire functions $g_0(z), \ldots, g_p(z)$, $h(z)$ be functions of bounded L-index in a direction $\mathbf{b}$, and suppose that, for every $r > 0$, there exists $T = T(r) > 0$ such that, for each $z \in \mathbb{C}^n \setminus G^{\mathbf{b}}_r(g_0)$ and $j = 1, \ldots, p$,*

$$|g_j(z)| \le T L^j(z)|g_0(z)|. \tag{5}$$

Then an entire function $F(z)$ satisfying (4) *has bounded L-index in the direction $\mathbf{b}$.*

Recently, an improved version of Theorem 3 was obtained in [8] with restrictions for the values of r.

By $\overline{G}$, we denote the closure of a domain G. The following assertion gives sufficient conditions for a positive continuous function to belong to $Q^n_{\mathbf{b}}$ (some other similar propositions are given in [1]). In a conversation with the authors (2015), S. Yu. Favorov posed a problem to describe functions in $Q^n_{\mathbf{b}}$ by their differential characteristics. The next lemma is a proposition of such type.

Proposition 1. *Let $L : \mathbb{C}^n \to \mathbb{C}$ and let $\frac{\partial L}{\partial \mathbf{b}}$ be continuous functions in a domain G. If there exist numbers $P > 0$ and $c > 0$ such that, for all $z \in \overline{G}$,*

$$\frac{1}{c + |L(z)|}\left|\frac{\partial L(z)}{\partial \mathbf{b}}\right| \le P, \tag{6}$$

then the inequalities

$$0 < \inf_{z\in\overline{G}} \inf_{\substack{t_0\in\mathbb{C},\\ z+t_0\mathbf{b}\in\overline{G}}} \inf_{|t-t_0|\le\frac{\eta}{L_1(z+t_0\mathbf{b})}} \frac{L_1(z+t\mathbf{b})}{L_1(z+t_0\mathbf{b})}$$

$$\le \sup_{z\in\overline{G}} \sup_{\substack{t_0\in\mathbb{C},\\ z+t_0\mathbf{b}\in\overline{G}}} \sup_{|t-t_0|\le\frac{\eta}{L_1(z+t_0\mathbf{b})}} \frac{L_1(z+t\mathbf{b})}{L_1(z+t_0\mathbf{b})} < \infty$$

hold for every $\eta \ge 0$*, where* $L_1(z) = c + |L(z)|$*. If, in addition,* $G = \mathbb{C}^n$*, then* $L_1 \in Q^n_{\mathbf{b}}$*.*

Proof. Clearly, the function $L_1(z)$ is positive and continuous. For given $z \in \mathbb{C}^n$ and $t_0 \in \mathbb{C}$, we define an analytic curve

$$\varphi(\tau) = z + t_0\mathbf{b} + \tau e^{i\arg(t-t_0)}\mathbf{b}, \quad \tau \in [0, |t-t_0|].$$

For every continuously differentiable function g of real variable τ, the inequality $\frac{d}{dt}|g(\tau)| \le |g'(\tau)|$ holds except for the points where $g'(\tau) = 0$. Using restrictions of this lemma, we deduce the upper estimate of $\lambda_2^{\mathbf{b}}(z, t_0, \eta)$ for the function L_1 :

$$\lambda_2^{\mathbf{b}}(z,t_0,\eta) = \sup\left\{\frac{c+|L(z+t\mathbf{b})|}{c+|L(z+t_0\mathbf{b})|} : |t-t_0| \le \frac{\eta}{c+|L(z+t_0\mathbf{b})|}\right\}$$
$$= \sup_{|t-t_0|\le\eta/(c+|L(z+t_0\mathbf{b})|)} \{\exp\{\ln(c+|L(z+t\mathbf{b})|) - \ln(c+|L(z+t_0\mathbf{b})|)\}\}$$
$$= \sup\left\{\exp\left\{\int_0^{|t-t_0|} \frac{d(c+|L(z+\varphi(\tau)\mathbf{b})|)}{c+|L(z+\varphi(\tau)\mathbf{b})|}\right\} : |t-t_0| \le \frac{\eta}{c+|L(z+t_0\mathbf{b})|}\right\} q$$
$$\le \sup_{|t-t_0|\le\frac{\eta}{c+|L(z+t_0\mathbf{b})|}} \left\{\exp\left\{\int_0^{|t-t_0|} \frac{|\varphi'(\tau)|}{c+|L(z+\varphi(\tau)\mathbf{b})|} \left|\frac{\partial L(z+\varphi(\tau)\mathbf{b})}{\partial \mathbf{b}}\right| |d\tau|\right\}\right\}$$
$$\le \sup_{|t-t_0|\le\frac{\eta}{c+|L(z+t_0\mathbf{b})|}} \left\{\exp\left\{\frac{P|\mathbf{b}|\eta}{c+|L(z+t_0\mathbf{b})|}\right\}\right\} \le \exp\left(\frac{P|\mathbf{b}|\eta}{c}\right).$$

Hence, for all $\eta \ge 0$,

$$\lambda_2(\eta) = \sup_{z\in\mathbb{C}^n} \sup_{t_0\in\mathbb{C}} \lambda_2^{\mathbf{b}}(z,t_0,\eta) \le \exp\left(\frac{P|\mathbf{b}|\eta}{c}\right) < \infty.$$

Using the inequality $\frac{d}{dt}|g(t)| \ge -|g'(t)|$, it can be proved that $\lambda_1(\eta) \ge \exp\left(-\frac{P|\mathbf{b}|\eta}{c}\right) > 0$. for every $\eta \ge 0$. Therefore, $L_1 \in Q^n_{\mathbf{b}}$. □

Note that the assertion of Proposition 1 is new also in the case $n = 1$.

Lemma 1. *Let* $\varepsilon > 0$, $\mathbf{b} = (b_1, b_2) \in \mathbb{C}^2 \setminus \{0\}$, *and*

$$L_{\varepsilon,\mathbf{b}}(z_1, z_2) := \begin{cases} \frac{|b_1 z_2 + b_2 z_1|}{\sqrt{|z_1 z_2|}} + 1, & |z_1 z_2| > \varepsilon^2, \\ \frac{|b_1 z_2 + b_2 z_1|}{\varepsilon} + 1, & |z_1 z_2| \le \varepsilon^2. \end{cases}$$

Then $L_{\varepsilon,\mathbf{b}} \in Q^2_{\mathbf{b}}$.

Proof. At first, we show that $L_{\varepsilon,\mathbf{b}} \in Q^n_{\mathbf{b}}$. Let $b_1 \neq 0$, $b_2 \neq 0$, $|z_1 z_2| > \varepsilon^2$. Then

$$\lambda_2^{\mathbf{b}}(z, t^0, \eta)$$
$$= \sup\Bigg\{\Big(\frac{|b_1(z_2 + tb_2) + b_2(z_1 + tb_1)|}{\sqrt{|(z_1 + tb_1)(z_2 + tb_2)|}} + 1\Big)\Big/\Big(\frac{|b_1(z_2 + t_0 b_2) + b_2(z_1 + t_0 b_1)|}{\sqrt{|(z_1 + t_0 b_1)(z_2 + t_0 b_2)|}} + 1\Big),$$
$$|t - t_0| \le \frac{\eta}{\frac{|b_1(z_2 + t_0 b_2) + b_2(z_1 + t_0 b_1)|}{\sqrt{|(z_1 + t_0 b_1)(z_2 + t_0 b_2)|}} + 1}\Bigg\}$$
$$\le \sup_{|t - t_0| \le \eta}\Bigg\{\Big(\frac{|b_1(z_2 + tb_2) + b_2(z_1 + tb_1)|}{\sqrt{|(z_1 + tb_1)(z_2 + tb_2)|}} + 1\Big)\Big/\Big(\frac{|b_1(z_2 + t_0 b_2) + b_2(z_1 + t_0 b_1)|}{\sqrt{|(z_1 + t_0 b_1)(z_2 + t_0 b_2)|}} + 1\Big)\Bigg\}.$$

Since $||z_j + b_j t| - |z_j + b_j t_0|| \le |z_j + b_j t_0 - (z_j + b_j t_0)| = |b_j| \cdot |t - t_0| \le |b_j|\eta$, it follows that $|z_j + b_j t_0| - |b_j|\eta \le |z_j + b_j t| \le |z_j + b_j t_0| + |b_j|\eta$. Hence,

$$\lambda_2^{\mathbf{b}}(z, t^0, \eta)$$
$$\le \sup_{|t - t_0| \le \eta}\Bigg\{\frac{|b_1(z_2 + tb_2) + b_2(z_1 + tb_1)| + \sqrt{|(z_2 + tb_2)(z_1 + tb_1)|}}{|b_1(z_2 + t_0 b_2) + b_2(z_1 + t_0 b_1)| + \sqrt{|(z_1 + t_0 b_1)(z_2 + t_0 b_2)|}}$$
$$\times \frac{\sqrt{|(z_1 + t_0 b_1)(z_2 + t_0 b_2)|}}{\sqrt{|(z_2 + tb_2)(z_1 + tb_1)|}}\Bigg\}$$
$$\le \sup_{t_0 \in \mathbb{C}}\Bigg\{\Big(|b_1(z_2 + t_0 b_2)| + |b_2(z_1 + t_0 b_1)| + 2|b_1 b_2|\eta$$
$$+ \sqrt{(|z_1 + t_0 b_1| + |b_1|\eta)(|z_2 + t_0 b_2| + |b_2|\eta)}\Big)\Big/\Big(|b_1(z_2 + t_0 b_2) + b_2(z_1 + t_0 b_1)|$$
$$+ \sqrt{|(z_1 + t_0 b_1)(z_2 + t_0 b_2)|}\Big)\Bigg\} \times \sup_{t_0 \in \mathbb{C}}\left\{\frac{\sqrt{|(z_1 + t_0 b_1)(z_2 + t_0 b_2)|}}{\sqrt{(|z_2 + t_0 b_2| - |b_2|\eta)(|z_1 + t_0 b_1| - |b_1|\eta)|}}\right\}$$
$$\le C_2(\eta, b_1, b_2) < +\infty.$$

Similarly, $\lambda_1^{\mathbf{b}}(z, t^0, \eta) \ge C_1(\eta, b_1, b_2) > 0$.

Let $b_1 \neq 0$, $b_2 \neq 0$, $|z_1 z_2| \le \varepsilon^2$, and $L_1(z_1, z_2) = \frac{b_1 z_2 + b_2 z_1}{\varepsilon}$. Then

$$\frac{1}{1 + |L_1(z_1, z_2)|}\left|\frac{\partial L_1(z_1, z_2)}{\partial \mathbf{b}}\right| = \frac{2|b_1 b_2|}{\frac{|b_1 z_2 + b_2 z_1|}{\varepsilon} + 1} \le 2|b_1 b_2|.$$

Hence, $L_{\varepsilon,\mathbf{b}} \in Q^n_{\mathbf{b}}$ by Proposition 1. The cases $b_1 = 0$, $b_2 \neq 0$ or $b_1 \neq 0$, and $b_2 = 0$, can be considered analogously. □

4. Main results

We consider the following partial differential equations related to a direction $\mathbf{b} = (b_1, b_2)$:

$$\begin{aligned} z_1 z_2 (b_1 z_2 + b_2 z_1) \frac{\partial^2 F}{\partial \mathbf{b}^2} + \frac{(b_1 z_2 - b_2 z_1)^2}{2} \frac{\partial F}{\partial \mathbf{b}} \\ + \frac{(b_1 z_2 + b_2 z_1)^3}{4} F(z_1, z_2) = 0, \end{aligned} \tag{7}$$

for $b_1 \neq 0,\ b_2 \neq 0$, and

$$4z_1 \frac{\partial^2 F}{\partial \mathbf{b}^2} + 2b_1 \frac{\partial F}{\partial \mathbf{b}} + b_1^2 z_2 F(z_1, z_2) = 0, \tag{8}$$

for $b_1 \neq 0,\ b_2 = 0$. For $b_1 = 0,\ b_2 \neq 0$, one can consider a PDE which matches to (8) up to permutations of the variables z_1 and z_2.

Theorem 4. *Let $\mathbf{b} = (b_1, b_2) \in \mathbb{C}^2 \setminus \{0\}$. If an entire bivariate function $F(z_1, z_2)$ satisfies either equation (7) or (8), then, for each $\varepsilon > 0$, the function F has bounded $L_{\varepsilon,\mathbf{b}}$-index in the direction $\mathbf{b}$, where $L_{\varepsilon,\mathbf{b}}$ is the function from Lemma 1.*

Proof. We will verify the conditions of Theorem 3 for the partial differential equation (7). The functions

$$g_0(z_1, z_2) = z_1 z_2 (b_1 z_2 + b_2 z_1), \quad g_1(z_1, z_2) = (b_1 z_2 - b_2 z_1)^2/2$$

and

$$g_2(z_1, z_2) = (b_1 z_2 + b_2 z_1)^3/4$$

are polynomials. Therefore, they have bounded L-index in the direction $\mathbf{b}$ for every positive continuous function $L(z_1, z_2)$.

We assume that $b_1 \neq 0$, $b_2 \neq 0$. For $z_1 = z_1^0 + b_1 t$, $z_2 = z_2^0 + b_2 t$, the function

$$g_0(z^0 + t\mathbf{b}) = (z_1^0 + b_1 t)(z_2^0 + b_2 t)(b_1 z_2^0 + b_2 z_1^0 + 2b_1 b_2 t)$$

has three zeros

$$t_1 = -\frac{z_1^0}{b_1}, \quad t_2 = -\frac{z_2^0}{b_2}, \quad t_3 = -\frac{b_1 z_2^0 + b_2 z_1^0}{2b_1 b_2}.$$

The condition $(z_1, z_2) \in \mathbb{C}^2 \setminus G_r^{\mathbf{b}}(g_0)$ means that

$$|t - t_j| > \frac{r}{L_{\varepsilon,\mathbf{b}}(z^0 + t_j \mathbf{b})} \quad \text{for any } t_j,\ j \in \{1, 2, 3\}.$$

Hence,

$$|z_k^0 + b_k t - (z_k^0 + b_k t_j)| > r|b_k|/L_{\varepsilon,\mathbf{b}}(z^0 + t_j \mathbf{b}), \quad k \in \{1, 2\}.$$

In particular,

$$|z_1| = |z_1^0 + b_1 t| = |z_1^0 + b_1 t - (z_1^0 + b_1 t_1)|$$
$$> \frac{r|b_1|}{L_{\varepsilon,\mathbf{b}}\left(0, z_2^0 - \frac{b_2}{b_1} z_1^0\right)} = \frac{\varepsilon r|b_1|}{\varepsilon + |b_1(z_2^0 - \frac{b_2}{b_1} z_1^0)|} \tag{9}$$
$$= \frac{\varepsilon r|b_1|}{\varepsilon + |b_1 z_2^0 + b_1 b_2 t - b_2 z_1^0 - b_1 b_2 t|} = \frac{r|b_1|}{1+|b_1 z_2 - b_2 z_1|/\varepsilon},$$

$$|z_2| = |z_2^0 + b_2 t| = |z_2^0 + b_2 t - (z_2^0 + b_2 t_2)|$$
$$> \frac{r|b_2|}{L_{\varepsilon,\mathbf{b}}(z_1^0 - \frac{b_1}{b_2} z_2^0, 0)} = \frac{\varepsilon r|b_2|}{\varepsilon + |b_2(z_1^0 - \frac{b_1}{b_2} z_2^0)|} \tag{10}$$
$$= \frac{\varepsilon r|b_2|}{\varepsilon + |b_2 z_1^0 + b_2 b_1 t - b_1 z_2^0 - b_1 b_2 t|} = \frac{\varepsilon r|b_2|}{\varepsilon+|b_1 z_2 - b_2 z_1|/\varepsilon},$$

$$|t - t_3| = |t + \frac{b_1 z_2^0 + b_2 z_1^0}{2 b_1 b_2}| < \frac{r}{L_{\varepsilon,\mathbf{b}}(z^0 + t_3 \mathbf{b})} = r \Leftrightarrow$$
$$|2 b_1 b_2 t + b_1 z_2^0 + b_2 z_1^0| > 2|b_1 b_2| r \Leftrightarrow$$
$$|b_1(z_2^0 + b_2 t) + b_2(z_1^0 + b_1 t)| > 2|b_1 b_2| r \Leftrightarrow |b_1 z_2 + b_2 z_1| > 2|b_1 b_2| r. \tag{11}$$

From (11), for $|z_1 z_2| > \varepsilon^2$ and $(z_1, z_2) \in \mathbb{C}^2 \setminus G_r^{\mathbf{b}}(g_0)$, we obtain

$$|g_2(z_1, z_2)| = \frac{1}{4}|b_1 z_2 + b_2 z_1|^3 = \frac{1}{4}\frac{|b_1 z_2 + b_2 z_1|^2}{|z_1 z_2|}|z_1 z_2(b_1 z_2 + b_2 z_1)|$$
$$= \frac{1}{4}\left(\frac{|b_1 z_2 + b_2 z_1|}{\sqrt{|z_1 z_2|}}\right)^2 |g_0(z)| \le \frac{1}{4} L^2_{\varepsilon,\mathbf{b}}(z_1, z_2)|g_0(z_1, z_2)|,$$

$$\frac{|g_1(z_1, z_2)|}{|g_0(z_1, z_2)| L_{\varepsilon,\mathbf{b}}(z_1, z_2)} = \frac{|b_1 z_2 - b_2 z_1|^2}{\sqrt{|z_1 z_2|}|b_1 z_2 + b_2 z_1|(|b_1 z_2 + b_2 z_1| + \sqrt{|z_1 z_2|})}$$
$$= \frac{|b_1 \frac{z_2}{z_1} - b_2|^2}{\sqrt{|z_1 z_2|}|b_1 \frac{z_2}{z_1} + b_2|(|b_1 \frac{z_2}{z_1} + b_2| + \sqrt{|\frac{z_2}{z_1}|})} \le C_1(r, \varepsilon),$$

where $C_1(r, \varepsilon)$ is some positive constant independent of z_1 and z_2. The last inequality is valid because $|z_1 z_2| > \varepsilon^2$, the degree of the numerator of the fraction $\frac{z_2}{z_1}$ is not greater than the degree of the denominator and, in view of (11), the modulus of the denominator is greater than some positive constant.

For $|z_1 z_2| \le \varepsilon^2$ and $(z_1, z_2) \in \mathbb{C}^2 \setminus G_r^{\mathbf{b}}(g_0)$, we consider the following three cases:

a) $|z_1| \le \varepsilon, |z_2| \le \varepsilon$. In view of (9) and (10), the following inequalities are valid:

$$|g_2(z_1, z_2)| = \frac{1}{4}|b_1 z_2 + b_2 z_1|^3 = \frac{1}{4}\frac{|b_1 z_2 + b_2 z_1|^2}{\varepsilon^2}\varepsilon^2|b_1 z_2 + b_2 z_1|$$

$$\leq \frac{1}{4} L^2_{\varepsilon,\mathbf{b}}(z_1,z_2)|g_0(z_1,z_2)| \frac{\varepsilon^2}{|z_1 z_2|} < \frac{1}{4} L^2_{\varepsilon,\mathbf{b}}(z_1,z_2)|g_0(z_1,z_2)| \frac{(\varepsilon + |b_1 z_2 - b_2 z_1|)^2}{r^2|b_1 b_2|}$$
$$\leq \frac{\varepsilon^2(1+|b_1|+|b_2|)^2}{4r^2|b_1 b_2|} L^2_{\varepsilon,\mathbf{b}}(z_1,z_2)|g_0(z_1,z_2)|,$$

$$\frac{|g_1(z_1,z_2)|}{|g_0(z_1,z_2)|L_{\varepsilon,\mathbf{b}}(z_1,z_2)} = \frac{|b_1 z_2 - b_2 z_1|^2}{2|z_1 z_2(b_1 z_2 + b_2 z_1)|\left(\frac{|b_1 z_2 - b_2 z_1|}{\varepsilon}+1\right)}$$
$$\leq \frac{|b_1 z_2 - b_2 z_1|^2\left(\frac{|b_1 z_2 - b_2 z_1|}{\varepsilon}+1\right)}{2r^2|b_1 b_2(b_1 z_2 + b_2 z_1)|} \leq \frac{\varepsilon^2(|b_1|+|b_2|)^2(|b_1|+|b_2|+1)}{4r^3|b_1 b_2|}.$$

b) $|z_1| > \varepsilon$, $|z_2| \leq \varepsilon$. In view of (10) and (11), the following inequalities hold:

$$|g_2(z_1,z_2)| = \frac{1}{4}|b_1 z_2 + b_2 z_1|^3 = \frac{1}{4}\frac{|b_1 z_2 + b_2 z_1|^2}{\varepsilon^2}\varepsilon^2|b_1 z_2 + b_2 z_1|$$
$$\leq \frac{1}{4} L^2_{\varepsilon,\mathbf{b}}(z_1,z_2)|g_0(z_1,z_2)| \frac{\varepsilon^2}{|z_1 z_2|} < \frac{1}{4} L^2_{\varepsilon,\mathbf{b}}(z_1,z_2)|g_0(z_1,z_2)| \frac{\varepsilon(\varepsilon + |b_1 z_2 - b_2 z_1|)}{|z_1 b_2| r}$$
$$\leq \frac{\varepsilon(1+|b_1|+|b_2|)}{4r|b_2|} L^2_{\varepsilon,\mathbf{b}}(z_1,z_2)|g_0(z_1,z_2)|,$$

$$\frac{|g_1(z_1,z_2)|}{|g_0(z_1,z_2)|L_{\varepsilon,\mathbf{b}}(z_1,z_2)} = \frac{|b_1 z_2 - b_2 z_1|^2}{2|z_1 z_2(b_1 z_2 + b_2 z_1)|\left(\frac{|b_1 z_2 - b_2 z_1|}{\varepsilon}+1\right)}$$
$$\leq \frac{|b_1 z_2 - b_2 z_1|^2}{2r|z_1 b_2(b_1 z_2 + b_2 z_1)|} \leq \frac{|z_1|(|b_1|+|b_2|)^2}{2r|b_2| \cdot |b_1 z_2 + b_2 z_1|} \leq C_2(r,\varepsilon),$$

where $C_2(r,\varepsilon)$ is a positive constant independent of z_1 and z_2.

c) $|z_1| \leq \varepsilon$, $|z_2| > \varepsilon$. Using (9) and (11), this case can be considered as the previous case.

By Theorem 3, the function $F(z_1,z_2)$ has bounded $L_{\varepsilon,\mathbf{b}}$-index in the direction $\mathbf{b} = (b_1,b_2)$, where $b_1 \neq 0$, $b_2 \neq 0$.

Let $\mathbf{b} = (b_1, 0)$, where $b_1 \neq 0$. Now we will verify the conditions of Theorem 3 for the partial differential equation (8). The functions $g_0(z_1,z_2) = z_1$, $g_1(z_1,z_2) = \frac{b_1}{2}$, $g_2(z_1,z_2) = \frac{b_1^2 z_2}{4}$ are polynomials or constants. Hence, they have bounded L-index in any direction and for every positive continuous function L.

Let $r > 0$. For $z_1 = z_1^0 + tb_1$, the function $g_0(z_1^0 + tb_1, z_2) = z_1^0 + tb_1$ has one zero $t_0 = -\frac{z_1^0}{b_1}$. The condition $(z_1,z_2) \in \mathbb{C}^2 \setminus G_r^{\mathbf{b}}(g_0)$ means that

$$|t - t_0| > \frac{r}{L_{\varepsilon,\mathbf{b}}(z_1^0 + t_0 b_1, z_2)} = \frac{r}{L_{\varepsilon,\mathbf{b}}(0,z_2)} = \frac{r}{\frac{|b_1 z_2|}{\varepsilon,\mathbf{b}}+1}.$$

Hence,

$$|z_1| = |z_1^0 + b_1 t - (z_1^0 + b_1 t_0)| = |b_1(r - t_0)| > \frac{|b_1| r}{\frac{|b_1 z_2|}{\varepsilon} + 1}. \tag{12}$$

For $|z_1 z_2| > \varepsilon^2$ and $(z_1, z_2) \in \mathbb{C}^2 \setminus G_r^{\mathbf{b}}(g_0)$, we have

$$\frac{|g_1(z)|}{|g_0(z)| L_{\varepsilon,\mathbf{b}}(z)} = \frac{|b_1|}{2(|b_1\sqrt{z_1 z_2}| + |z_1|)} < \frac{|b_1|}{2(|b_1 \varepsilon| + |z_1|)} < \frac{1}{2\varepsilon},$$

$$\frac{|g_2(z)|}{|g_0(z)| L^2_{\varepsilon,\mathbf{b}}(z)} = \frac{|b_1^2 z_2|}{4|z_1|(|b_1|\sqrt{\frac{|z_2|}{|z_1|}} + 1)^2} = \frac{|b_1^2 z_2|}{4(|b_1^2 z_2| + 2|b_1\sqrt{z_2 z_1}| + |z_1|)} < \frac{1}{4}.$$

In view of (12), for $|z_1 z_2| \le \varepsilon^2$ and $(z_1, z_2) \in \mathbb{C}^2 \setminus G_r^{\mathbf{b}}(g_0)$, the following estimates are valid:

$$\frac{|g_1(z)|}{|g_0(z)| L_{\varepsilon,\mathbf{b}}(z)} = \frac{|b_1|}{2|z_1|(\frac{|b_1 z_2|}{\varepsilon} + 1)} < \frac{1}{2r},$$

$$\frac{|g_2(z)|}{|g_0(z)| L^2_{\varepsilon,\mathbf{b}}(z)} = \frac{|b_1^2 z_2|}{4|z_1|(\frac{|b_1 z_2|}{\varepsilon} + 1)^2} < \frac{|b_1 z_2|}{4r(\frac{|b_1 z_2|}{\varepsilon} + 1)} < \frac{\varepsilon}{4r}.$$

By Theorem 3, the function $F(z_1, z_2)$ has bounded $L_{\varepsilon,\mathbf{b}}$-index in the direction $\mathbf{b} = (b_1, 0)$. □

Theorem 5. *The entire bivariate function* $F(z_1, z_2) = \cos\sqrt{z_1 z_2}$ *has bounded* $L_{\varepsilon,\mathbf{b}}$*-index in the direction* $\mathbf{b}$.

Proof. We prove that the partial differential equation (4) has entire solution $F(z_1, z_2)$. Let $b_1 \ne 0$, $b_2 \ne 0$. Formally, the first and second order directional derivatives of F are

$$\frac{\partial F}{\partial \mathbf{b}} = -(b_1 z_2 + b_2 z_1)\frac{\sin\sqrt{z_1 z_2}}{2\sqrt{z_1 z_2}},$$

$$\frac{\partial^2 F}{\partial \mathbf{b}^2} = -\frac{1}{2^2}\left(b_1\sqrt{\frac{z_2}{z_1}} + b_2\sqrt{\frac{z_1}{z_2}}\right)^2 \cos\sqrt{z_1 z_2} - \frac{1}{2}\left(\Big(-\frac{b_1}{2}\frac{\sqrt{z_2}}{z_1^{3/2}} + \frac{b_2}{2}\frac{1}{\sqrt{z_1 z_2}}\Big)b_1 + \Big(\frac{b_1}{2}\frac{1}{\sqrt{z_1 z_2}} - \frac{b_2}{2}\frac{\sqrt{z_1}}{z_2^{3/2}}\Big)b_2\right)\sin\sqrt{z_1 z_2} = -\frac{(b_1 z_2 + b_2 z_1)^2}{4 z_1 z_2}F(z_1, z_2)$$
$$+\frac{(b_1 z_2 - b_2 z_1)^2}{4(z_1 z_2)^{3/2}}\sin\sqrt{z_1 z_2} = -\frac{(b_1 z_2 + b_2 z_1)^2}{4 z_1 z_2}F(z_1, z_2) - \frac{(b_1 z_2 - b_2 z_1)^2}{4(z_1 z_2)^{3/2}}$$
$$\times\frac{2\sqrt{z_1 z_2}}{b_1 z_2 + b_2 z_1}\cdot\frac{\partial F}{\partial \mathbf{b}} = -\frac{(b_1 z_2 + b_2 z_1)^2}{4 z_1 z_2}F(z_1, z_2) - \frac{(b_1 z_2 - b_2 z_1)^2}{2 z_1 z_2 (b_1 z_2 + b_2 z_1)}\frac{\partial F}{\partial \mathbf{b}}.$$

Hence, we obtain

$$z_1 z_2 (b_1 z_2 + b_2 z_1)\frac{\partial^2 F}{\partial \mathbf{b}^2} + \frac{(b_1 z_2 - b_2 z_1)^2}{2}\frac{\partial F}{\partial \mathbf{b}} + \frac{(b_1 z_2 + b_2 z_1)^3}{4}F(z_1, z_2) = 0,$$

i.e., F satisfies equation (7).

If $b_1 \neq 0$, $b_2 = 0$, then we calculate the directional derivatives of F and similarly show that the function satisfies (8). By Theorem 4, the function F has bounded $L_{\varepsilon,\mathbf{b}}$-index in the direction $\mathbf{b}$. □

Theorem 5 implies the following assertion.

Theorem 6. *Let*

$$L_\varepsilon(z_1, z_2) = \begin{cases} \sqrt{\frac{|z_2|}{|z_1|}} + \sqrt{\frac{|z_1|}{|z_2|}} + 1, & |z_1 z_2| > \varepsilon^2, \\ \frac{|z_2|+|z_1|}{\varepsilon} + 1, & |z_1 z_2| \leq \varepsilon^2. \end{cases}.$$

Then, for each $\mathbf{b} \in \mathbb{C}^n \setminus \{0\}$, *one has* $L_\varepsilon \in Q^n_{\mathbf{b}}$ *and the entire bivariate function* $F(z_1, z_2) = \cos\sqrt{z_1 z_2}$ *has bounded* L_ε*-index in the direction* $\mathbf{b}$.

Proof. The function $L_{\varepsilon,\mathbf{b}}(z_1, z_2)$ does not exceed $c_3 \cdot L_\varepsilon(z_1, z_2)$: indeed,

$$\begin{cases} \frac{|b_1 z_2 + b_2 z_1|}{\sqrt{|z_1 z_2|}} + 1 \leq c_3 \left(\sqrt{\frac{|z_2|}{|z_1|}} + \sqrt{\frac{|z_1|}{|z_2|}} + 1\right), & |z_1 z_2| > \varepsilon^2, \\ \frac{|b_1 z_2 + b_2 z_1|}{\varepsilon} + 1 \leq c_3 \left(\frac{|z_2|+|z_1|}{\varepsilon} + 1\right), & |z_1 z_2| \leq \varepsilon^2, \end{cases} \tag{13}$$

where $c_3 = \max\{|b_1|, |b_2|, 1\}$.

In fact, in the proof of Theorem 2 (see the proofs of Theorem 3 in [4] and Proposition 2.1 in [6, p. 25]), we proved that if $G(z)$ has bounded L_1-index in the direction $\mathbf{b} \in \mathbb{C}^n \setminus \{0\}$ and, for all $z \in \mathbb{C}^n$ one has $L_1(z) \leq \theta \cdot L_2(z)$, when L_1, $L_2 \in Q^n_{\mathbf{b}}$, then $G(z)$ has bounded L_2-index in the direction $\mathbf{b}$. From (13), it follows that $F(z_1, z_2)$ is a function of bounded L_ε-index in each direction $\mathbf{b}$. □

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