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Asymptotics of approximation of conjugate functions by Poisson integrals

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Abstract. We obtain a decomposition of the upper bound for the deviation of Poisson integrals of conjugate periodic functions. The decomposition enables one to provide the Kolmogorov–Nikol'skii constants of an arbitrary order.

1. Introduction

Let C be the space of 2π-periodic continuous functions. The norm in this space is defined as follows:

$$\|f\|_C = \max_t |f(t)|.$$

Denote by W^r any set of 2π-periodic functions with absolutely continuous derivatives up to order $r-1$ such that $\operatorname{ess\,sup}_t \left|f^{(r)}(t)\right| \le 1$.

The set of functions that are conjugate to those from the class W^r is denoted by $\overline{W}^r$. That is,

$$\overline{W}^r = \Bigg\{ \bar{f} \colon \ \bar{f}(x) = -\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x+t)\cot\frac{t}{2}\,dt = -\frac{1}{2\pi}\int_0^{\pi} \psi_x(t)\cot\frac{t}{2}\,dt,$$
$$\psi_x(t) = f(x+t) - f(x-t), \quad f \in W^r \Bigg\}.$$

A function $f \in C$ is contained in the class Lip 1 if, for all $t_1, t_2 \in \mathbb{R}$,

$$|f(t_1) - f(t_2)| \le |t_1 - t_2|.$$

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Let us consider a boundary value problem (in the unit circle) for the equation

$$\Delta u = 0, \tag{1}$$

where Δ is the Laplace operator in polar coordinates. We can rewrite the equation (1) as follows:

$$\frac{\partial^2 u}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial u}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial x^2} = 0, \qquad 0 \le \rho < 1, \quad -\pi \le x \le \pi. \tag{2}$$

The solution of (2) that satisfies the boundary conditions

$$u(\rho, x)\big|_{\rho=1} = f(x), \qquad -\pi \le x \le \pi,$$

where f is a summable 2π-periodic function, is of the form by $\overline{W}^r$. That is

$$\begin{aligned} P_\rho(f;x) &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x+t)\left\{\frac{1}{2} + \sum_{k=1}^{\infty} \rho^k \cos kt\right\} dt \\ &= \frac{1-\rho^2}{2\pi}\int_{-\pi}^{\pi} f(x+t)\frac{dt}{1 - 2\rho\cos t + \rho^2}. \end{aligned}$$

The quantity $P_\rho(f;x)$ is called the Poisson integral of the function f. Setting $\rho = e^{-\frac{1}{\delta}}$, $\delta > 0$, we can rewrite the Poisson integral in the form

$$\begin{aligned} P_\delta(f;x) &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x+t)\left\{\frac{1}{2} + \sum_{k=1}^{\infty} e^{-\frac{k}{\delta}} \cos kt\right\} dt \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x+t)\frac{1 - e^{-\frac{2}{\delta}}}{1 - 2e^{-\frac{1}{\delta}}\cos t + e^{-\frac{2}{\delta}}}\, dt. \end{aligned}$$

The quantity

$$\begin{aligned} \bar{P}_\rho(f;x) = P_\rho(\bar{f};x) &= -\frac{1}{\pi}\int_{-\pi}^{\pi} f(x+t)\sum_{k=1}^{\infty} \rho^k \sin kt\, dt \\ &= -\frac{1}{\pi}\int_{0}^{\pi} \psi_x(t)\frac{\rho\sin t}{1 - 2\rho\cos t + \rho^2}\, dt \end{aligned}$$

is called the conjugate Poisson integral of the function f.

Let $\mathfrak{N} \subseteq C$ be a certain class of functions. According to Stepanets [10], the problem of establishing asymptotic equalities for the quantity

$$\mathcal{E}\big(\mathfrak{N}; P_\rho\big)_C = \sup_{f \in \mathfrak{N}} \|f(x) - P_\rho(f;x)\|_C$$

is called the Kolmogorov–Nikol'skii problem.

If we determine the explicit form of a function $\varphi(\rho)$ such that

$$\mathcal{E}\big(\mathfrak{N}; P_\rho\big)_C = \varphi(\rho) + o(\varphi(\rho)) \text{ as } \rho \to 1-,$$

then we say that the Kolmogorov–Nikol'skii problem for the Poisson integral P_ρ is solved on the class $\mathfrak{N}$ in the metric of the space C.

Definition 1. A formal series $\sum\limits_{n=0}^{\infty} g_n(\rho)$ is called a complete asymptotic decomposition of a function $f(\rho)$ as $\rho \to 1-$, if for every natural number m, the equation

$$f(\rho) = \sum_{n=0}^{m} g_n(\rho) + o\left(g_m(\rho)\right)$$

holds as $\rho \to 1-$, and for all $n \in N$,

$$|g_{n+1}(\rho)| = o\left(|g_n(\rho)|\right).$$

In what follows, we denote this fact by

$$f(\rho) \cong \sum_{n=0}^{\infty} g_n(\rho).$$

Approximation properties of the method of approximation by Poisson integrals on classes of differentiable functions have been well studied. The Kolmogorov–Nikol'skii problem for the Poisson integral on the classes W^1 was solved by Natanson in [8]:

$$\mathcal{E}\big(W^1; P_\rho\big)_C = \frac{2}{\pi}(1-\rho)|\ln(1-\rho)| + O(1-\rho) \qquad \text{as } \rho \to 1-.$$

In [11], Timan obtained the exact values of the approximative characteristics $\mathcal{E}\big(W^r; P_\rho\big)_C$.

In the paper [6], Malei determined the complete asymptotic decomposition of the upper bounds of deviations of Poisson integrals from functions of the class W^1:

$$\mathcal{E}\big(W^1; P_\rho\big)_C = \frac{2}{\pi}\sum_{k=1}^{\infty}\left\{\alpha_k(1-\rho)^k \ln\frac{1}{1-\rho} + \beta_k(1-\rho)^k\right\},$$

$$\alpha_k = \frac{1}{k}, \quad \beta_k = \frac{1}{k}\left\{\ln 2 + \frac{1}{k} - \sum_{i=1}^{k-1}\frac{1}{i2^i}\right\}, \qquad k = 1, 2, \ldots.$$

Later, this decomposition was reproved by Stark [9].

The complete asymptotic decomposition of the quantity $\mathcal{E}\big(W^r; P_\delta\big)_C$ in powers of $\frac{1}{\delta}$ as $\delta \to \infty$ was obtained by Baskakov [2] for $r = 1, 2, 3$. Later, the Kolmogorov–Nikol'skii problem for the Poisson integral on classes of differentiable functions was solved in the works [5, 12–15]. On the other hand, approximation properties of the method of approximation by Poisson integrals on classes of conjugate functions have not been sufficiently studied.

The first estimates for $\mathcal{E}\big(\overline{W}^1; P_\rho\big)_C$ were obtained by Nagy [7]. In particular, he established the equalities

$$\mathcal{E}\big(\overline{W}^1; P_\rho\big)_C = \frac{4}{\pi}\int_\rho^1 \frac{\arctan t}{t}\,dt, \qquad 0 \le \rho < 1,$$
$$\mathcal{E}\big(\overline{W}^1; P_\rho\big)_C = (1-\rho) + O\big((1-\rho)^2\big) \qquad \text{as } \rho \to 1- .$$

Later, the general expressions that allow one to get asymptotic decompositions of the quantity $\mathcal{E}\big(\overline{W}^r; P_\delta\big)_C$ in powers of $\frac{1}{\delta}$ as $\delta \to \infty$, were determined by Baskakov [1].

In the present paper, we establish a complete asymptotic decomposition of the quantity

$$\mathcal{E}\big(\mathrm{Lip}\ 1; \overline{P}_\rho\big)_C = \sup_{f\in \mathrm{Lip}\ 1} \big\|\overline{f}(\cdot) - \overline{P}_\rho(f;\cdot)\big\|_C.$$

This decomposition allows one to provide the Kolmogorov–Nikol'skii constants of an arbitrary order.

2. The main result

The following theorem is the main result of the paper.

Theorem 1. *Let $f \in$ Lip 1. Then the following complete asymptotic decomposition holds as $\rho \to 1-$:*

$$\mathcal{E}\big(\mathrm{Lip}\ 1; \overline{P}_\rho\big)_C = \sum_{k=1}^{\infty} \frac{(1-\rho)^k}{k2^{k-1}} + \sum_{m=1}^{\infty} \frac{1}{m} \sum_{s=0}^{\infty} \frac{(1-\rho)^{2m+s}}{2^{2m+s}} \binom{2m+s-1}{s} \alpha_m, \tag{3}$$

where

$$\alpha_m = \frac{4}{\pi}\sum_{k=1}^{m} \frac{(-1)^{m+k+1}}{2m-2k+1} + 1. \tag{4}$$

Proof. It is obvious that

$$\begin{aligned}
\overline{P}_\rho(f;x) - \overline{f}(x) &= -\frac{1}{\pi}\int_0^\pi \psi_x(t)\frac{\rho \sin t}{1-2\rho\cos t+\rho^2}\,dt + \frac{1}{2\pi}\int_0^\pi \psi_x(t)\cot\frac{t}{2}\,dt \\
&= \frac{1}{2\pi}\int_0^\pi \psi_x(t)\Big(\cot\frac{t}{2} - \frac{2\rho\sin t}{1-2\rho\cos t+\rho^2}\Big)\,dt \\
&= \frac{1}{2\pi}\int_0^\pi \psi_x(t)\cot\frac{t}{2}\Big(1 - \frac{2\rho\sin t\cdot \tan\frac{t}{2}}{1-2\rho\cos t+\rho^2}\Big)\,dt \\
&= \frac{1}{2\pi}\int_0^\pi \psi_x(t)\cot\frac{t}{2}\cdot \frac{1-2\rho\cos t+\rho^2 - 2\rho\sin t\cdot\tan\frac{t}{2}}{1-2\rho\cos t+\rho^2}\,dt.
\end{aligned}$$

Using double-angle formulas and the main trigonometric identity, we obtain

$$\overline{P}_\rho(f;x)-\overline{f}(x)=\frac{1}{2\pi}\int_0^{\pi}\psi_x(t)\cot\frac{t}{2}\cdot\frac{(1-\rho)^2}{1-2\rho\cos t+\rho^2}\,dt.$$

In view of the fact that $f(x)$ is 2π-periodic, we have

$$\begin{aligned}\overline{P}_\rho(f;x)-\overline{f}(x)&=\frac{1}{2\pi}\int_0^{\frac{\pi}{2}}\psi_x(t)\cot\frac{t}{2}\cdot\frac{(1-\rho)^2}{1-2\rho\cos t+\rho^2}\,dt\\&\quad-\frac{1}{2\pi}\int_{\frac{\pi}{2}}^{\pi}\psi_{x+\pi}(\pi-t)\cot\frac{t}{2}\cdot\frac{(1-\rho)^2}{1-2\rho\cos t+\rho^2}\,dt.\end{aligned}\tag{5}$$

Since $f\in \mathrm{Lip}\ 1$, by virtue of the subadditivity of the absolute value, we have

$$\begin{aligned}\mathcal{E}\big(\mathrm{Lip}\ 1;\overline{P}_\rho\big)_C&\le\frac{1}{\pi}\int_0^{\frac{\pi}{2}}t\,\cot\frac{t}{2}\cdot\frac{(1-\rho)^2}{1-2\rho\cos t+\rho^2}\,dt\\&\quad+\frac{1}{\pi}\int_{\frac{\pi}{2}}^{\pi}(\pi-t)\cot\frac{t}{2}\cdot\frac{(1-\rho)^2}{1-2\rho\cos t+\rho^2}\,dt.\end{aligned}\tag{6}$$

Let us prove that, in (6), the inequality is, in fact, an equality. To this end, note that the class Lip 1 is invariant to the shift of an argument, i.e., if $f\in\mathrm{Lip}\ 1$, then, for any $h\in\mathbb{R}$, the function $f_1(x)=f(x+h)$ is also in the class Lip 1. Hence, from (5) we get

$$\begin{aligned}\mathcal{E}\big(\mathrm{Lip}\ 1;\overline{P}_\rho\big)_C&=\sup_{f\in\mathrm{Lip}\ 1}\big\|\overline{f}(x)-\overline{P}_\rho(f;x)\big\|_C=\sup_{f\in\mathrm{Lip}\ 1}\big|\overline{f}(0)-\overline{P}_\rho(f;0)\big|\\&=\sup_{f\in\mathrm{Lip}\ 1}\bigg|\frac{1}{2\pi}\int_0^{\frac{\pi}{2}}\psi_0(t)\cot\frac{t}{2}\cdot\frac{(1-\rho)^2}{1-2\rho\cos t+\rho^2}\,dt\\&\quad-\frac{1}{2\pi}\int_{\frac{\pi}{2}}^{\pi}\psi_\pi(\pi-t)\cot\frac{t}{2}\cdot\frac{(1-\rho)^2}{1-2\rho\cos t+\rho^2}\,dt\bigg|.\end{aligned}$$

Let us denote by $g(x)$ any odd 2π-periodic function from the class Lip 1 which can be defined on the segment $[0;\pi]$ as follows:

$$g(x)=\begin{cases}x, & 0\le x\le\frac{\pi}{2},\\ \pi-x, & \frac{\pi}{2}\le x\le\pi.\end{cases}$$

Then $\psi_x(t)=g(x+t)-g(x-t)$. Hence, for $t\in[0,\frac{\pi}{2}]$, we can write

$$\psi_0(t)=g(t)-g(-t)=2g(t)=2t,$$

and, for $t\in[\frac{\pi}{2},\pi]$, we have

$$\psi_\pi(\pi-t)=g(2\pi-t)-g(t)=g(-t)-g(t)=-2g(t)=-2(\pi-t).$$

Therefore,

$$\big|\overline{g}(0)-\overline{P}_\rho(g;0)\big|=\frac{1}{\pi}\int_0^{\frac{\pi}{2}}t\,\cot\frac{t}{2}\cdot\frac{(1-\rho)^2}{1-2\rho\cos t+\rho^2}\,dt$$

$$+\frac{1}{\pi}\int_{\frac{\pi}{2}}^{\pi}(\pi-t)\cot\frac{t}{2}\cdot\frac{(1-\rho)^2}{1-2\rho\cos t+\rho^2}\,dt.$$

This completes the proof that, in (6), the inequality can not be strict, i.e., the following equality holds:

$$\mathcal{E}\big(\mathrm{Lip}\ 1;\overline{P}_{\rho}\big)_C=\frac{1}{\pi}\int_0^{\frac{\pi}{2}}t\ \cot\frac{t}{2}\cdot\frac{(1-\rho)^2}{1-2\rho\cos t+\rho^2}\,dt$$
$$+\int_{\frac{\pi}{2}}^{\pi}\cot\frac{t}{2}\cdot\frac{(1-\rho)^2}{1-2\rho\cos t+\rho^2}\,dt-\frac{1}{\pi}\int_{\frac{\pi}{2}}^{\pi}t\ \cot\frac{t}{2}\cdot\frac{(1-\rho)^2}{1-2\rho\cos t+\rho^2}\,dt \quad (7)$$
$$=I_1+I_2-I_3.$$

To calculate the integrals I_1, I_2, and I_3, we first consider the integral

$$\int\cot\frac{t}{2}\cdot\frac{1-\rho^2}{1-2\rho\cos t+\rho^2}\,dt.$$

Integrating by parts and using the equality (see [4, formula (2.556.1)])

$$\int\frac{1-a^2}{1-2a\cos t+a^2}\,dt=2\arctan\Big(\frac{1+a}{1-a}\tan\frac{t}{2}\Big)+C,\quad 0<a<1,\quad |t|<\pi,$$

we obtain

$$\int\cot\frac{t}{2}\cdot\frac{1-\rho^2}{1-2\rho\cos t+\rho^2}\,dt=2\arctan\Big(\frac{1+\rho}{1-\rho}\tan\frac{t}{2}\Big)\cdot\cot\frac{t}{2}$$
$$-2\int\arctan\Big(\frac{1+\rho}{1-\rho}\tan\frac{t}{2}\Big)\,d\left(\cot\frac{t}{2}\right).$$

Using the equality (see [4, formula (2.854)])

$$\int\frac{1}{t^2}\arctan\frac{t}{a}\,dt=-\frac{1}{t}\arctan\frac{t}{a}-\frac{1}{2a}\ln\frac{a^2+t^2}{t^2}+C,$$

we have

$$\int\cot\frac{t}{2}\cdot\frac{1-\rho^2}{1-2\rho\cos t+\rho^2}\,dt=-\frac{1+\rho}{1-\rho}\ln\frac{\tan^2\frac{t}{2}+\left(\frac{1-\rho}{1+\rho}\right)^2}{\tan^2\frac{t}{2}}+C. \quad (8)$$

From (8), it follows that

$$I_2=\int_{\frac{\pi}{2}}^{\pi}\cot\frac{t}{2}\cdot\frac{(1-\rho)^2}{1-2\rho\cos t+\rho^2}\,dt=\ln\left(1+\left(\frac{1-\rho}{1+\rho}\right)^2\right).$$

Integrating by parts yields

$$I_1=\frac{1}{\pi}\int_0^{\frac{\pi}{2}}t\ \cot\frac{t}{2}\cdot\frac{(1-\rho)^2}{1-2\rho\cos t+\rho^2}\,dt$$

$$= -\frac{1}{2}\ln\left(1+\left(\frac{1-\rho}{1+\rho}\right)^2\right)+\frac{1}{\pi}\int_0^{\frac{\pi}{2}}\ln\frac{\tan^2\frac{t}{2}+\left(\frac{1-\rho}{1+\rho}\right)^2}{\tan^2\frac{t}{2}},$$

$$I_3=\frac{1}{\pi}\int_{\frac{\pi}{2}}^{\pi} t\cot\frac{t}{2}\cdot\frac{(1-\rho)^2}{1-\rho\cos t+\rho^2}\,dt$$

$$=\frac{1}{2}\ln\left(1+\left(\frac{1-\rho}{1+\rho}\right)^2\right)+\frac{1}{\pi}\int_{\frac{\pi}{2}}^{\pi}\ln\frac{\tan^2\frac{t}{2}+\left(\frac{1-\rho}{1+\rho}\right)^2}{\tan^2\frac{t}{2}}.$$

Substituting the obtained values for I_1, I_2, and I_3 in (7), we get

$$\mathcal{E}\big(\mathrm{Lip}\ 1;\overline{P}_\rho\big)_C=\frac{1}{\pi}\int_0^{\pi}\ln\frac{\tan^2\frac{t}{2}+\left(\frac{1-\rho}{1+\rho}\right)^2}{\tan^2\frac{t}{2}}\,dt$$
$$-\frac{2}{\pi}\int_{\frac{\pi}{2}}^{\pi}\ln\frac{\tan^2\frac{t}{2}+\left(\frac{1-\rho}{1+\rho}\right)^2}{\tan^2\frac{t}{2}}\,dt=U_1+U_2. \tag{9}$$

Using the formulas (4.227.3) and (4.227.17) from [4], i.e.,

$$\int_0^{\frac{\pi}{2}}\ln(a\tan t)\,dt=\frac{\pi}{2}\ln a,\qquad a>0,$$

$$\int_0^{\frac{\pi}{2}}\ln(a^2+b^2\tan^2 t)\,dt=\pi\ln(a+b),\qquad a>0,\quad b>0,$$

we obtain

$$U_1=\frac{1}{\pi}\int_0^{\pi}\ln\frac{\tan^2\frac{t}{2}+\left(\frac{1-\rho}{1+\rho}\right)^2}{\tan^2\frac{t}{2}}\,dt$$
$$=\frac{1}{\pi}\int_0^{\pi}\ln\left(\tan^2\frac{t}{2}+\left(\frac{1-\rho}{1+\rho}\right)^2\right)-\frac{1}{\pi}\int_0^{\pi}\ln\left(\tan^2\frac{t}{2}\right)dt=2\ln\frac{2}{1+\rho}.$$

Let us find the Taylor series of $\phi(\rho)=2\ln\frac{2}{1+\rho}$ in powers of $1-\rho$:

$$U_1=\phi(\rho)=\sum_{k=1}^{\infty}\frac{(1-\rho)^k}{k\cdot 2^{k-1}}. \tag{10}$$

We continue the estimate making a substitution in U_2 and finding the expansion in series of the logarithm function. Thus,

$$U_2=-\frac{2}{\pi}\int_{\frac{\pi}{2}}^{\pi}\ln\frac{\tan^2\frac{t}{2}+\left(\frac{1-\rho}{1+\rho}\right)^2}{\tan^2\frac{t}{2}}\,dt=-\frac{4}{\pi}\int_{-\frac{\pi}{4}}^{0}\ln\left(1+\left(\frac{1-\rho}{1+\rho}\right)^2\tan^2 t\right)dt$$

$$= -\frac{4}{\pi}\int_{-\frac{\pi}{4}}^{0}\sum_{m=1}^{\infty}\frac{(-1)^{m+1}}{m}\left(\frac{1-\rho}{1+\rho}\right)^{2m}\tan^{2m}t\,dt.$$

Since the series under the last integral is uniformly convergent, we can use term-by-term integration. We have

$$U_2 = -\frac{4}{\pi}\sum_{m=1}^{\infty}\frac{(-1)^{m+1}}{m}\left(\frac{1-\rho}{1+\rho}\right)^{2m}\int_{-\frac{\pi}{4}}^{0}\tan^{2m}t\,dt. \qquad (11)$$

To calculate the integral on the right-hand side of (11), we use the equality (see [3, formula (5.10.2)])

$$\int \tan^{2n} t\ dt = \sum_{k=1}^{n}(-1)^{k-1}\frac{\tan^{2n-2k+1} t}{2n-2k+1} + (-1)^n t.$$

We get

$$U_2 = \sum_{m=1}^{\infty}\frac{1}{m}\left(\frac{1-\rho}{1+\rho}\right)^{2m}\alpha_m,$$

where α_m is defined by (4).

Finding the expansion in series of $\phi(\rho) = \left(\frac{1}{1+\rho}\right)^{2m}$ in powers of $1-\rho$, we obtain

$$U_2 = \sum_{m=1}^{\infty}\frac{1}{m}\sum_{s=0}^{\infty}\frac{(1-\rho)^{2m+s}}{2^{2m+s}}\binom{2m+s-1}{s}\alpha_m. \qquad (12)$$

Combining (10), (12), and (9), we get (3).

The theorem has been proved. □

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