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# Second approximation of local functions in ideal topological spaces

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Abstract. This paper gives a new dimension to discuss the local function in ideal topological spaces. We calculate error operators for various type of local functions and introduce more perfect approximation of the local functions for discussing their properties. We have also reached a topological space with the help of semi-closure.

## 1. Introduction and preliminaries

For a topological space $(X, \tau)$, the closure and the interior of $A \subseteq X$ is denoted by $Cl(A)$ and $Int(A)$, respectively, and they can be approximated by intersection of all closed sets containing $A$ and union of all open sets contained in $A$, respectively. A subset $A$ of a topological space $(X, \tau)$ is said to be semi-open ([13], [7]) (respectively, preopen [14], $\beta$-open [6], regular open ([23]), [22]) if $A \subseteq Cl(Int(A))$ (respectively, $A \subseteq Int(Cl(A)), A \subseteq Cl(Int(Cl(A))), A = Int(Cl(A))$). The collection of all semi-open (respectively, $\beta$-open) subsets of $X$ is denoted by $SO(X)$ (respectively, $\beta O(X)$). The complement of a semi-open (respectively, $\beta$-open) set is called semi-closed ([4], [5]) ($\beta$-closed [6]). The intersection of all semi-closed (respectively, $\beta$-closed) sets of $X$ containing $A \subseteq X$ is called the semi-closure [4] (respectively, $\beta$-closure [6]) of $A$ and is denoted by $sCl(A)$ (respectively, $\beta Cl(A)$). The union of all semi-open subsets of $A$ is called the semi-interior [4] of $A$ and is denoted by $sInt(A)$. Veličko [26] introduced the notion of $\theta$-open sets: a subset $A$ of a topological space $(X, \tau)$ is said to be $\theta$-open ([26], [3]) if every point $x$ of $X$ has an open neighbourhood $U_x$ such that $Cl(U_x) \subseteq A$. The complement of a $\theta$-open set is called $\theta$-closed. The collection of all $\theta$-open subsets of a topological space $(X, \tau)$ forms a topology on

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$X$ and it is coarser than $\tau$ and this collection is denoted by $\tau_\theta$. A subset $A$ of a topological space $X$ is said to be $\delta$-open [26] if, for each $x \in A$, there exists an open set $B$ such that $x \in B \subseteq Int(Cl(B)) \subseteq A$.

The study of ideal in topological spaces has been introduced by Kuratowski [12] and Vaidyanathswamy ([24], [25]). The authors in [21], [8], [2], [18], [15], [17] further considered the ideal topological spaces and studied it in detail for local functions.

An ideal [12] $\mathcal{I}$ of a topological space $(X, \tau)$ is a nonempty collection of subsets of $X$ which satisfies the following conditions:

(1) $A \in \mathcal{I}$ and $B \subseteq A$ implies $B \in \mathcal{I}$,

(2) $A \in \mathcal{I}$ and $B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$.

A topological space $(X, \tau)$ with an ideal $\mathcal{I}$ on $X$ is called an ideal topological space and is denoted by $(X, \tau, \mathcal{I})$.

Let $(X, \tau)$ be a topological space. The symbol $\mathcal{I}_{ctble}$ denotes the ideal of countable sets, and $\mathcal{I}_{cd}$ is the ideal of closed discrete sets. Further, the family of all discrete sets does not form an ideal (as it fails additivity). A set $S$ is said to be scattered if each nonempty subset of $S$ contains an isolated point. If $X$ is a $T_1$-space (each finite set is closed), then the family of all scattered sets is the ideal of scattered sets, and it is denoted by $\mathcal{I}_{sc}$. Obviously every discrete set is scattered. A set $A$ is relatively compact if $Cl(A)$ is compact. The family of all relatively compact sets forms an ideal and it is denoted by $\mathcal{I}_K$. A set $A$ is nowhere dense if $Int(Cl(A)) = \emptyset$. A countable union of nowhere dense sets is called a meager set. The family of nowhere dense sets forms the ideal $\mathcal{I}_{nwd}$, and the family of meager sets forms the ideal $\mathcal{I}_{mg}$ .

**Definition 1.1** (see [12], [10]). Let $(X, \tau, \mathcal{I})$ be an ideal topological space and let $A$ be a subset of $X$. The local function of $A$ with respect to $\mathcal{I}$ and $\tau$ is defined by

$A^*(\mathcal{I}, \tau) = \{x \in X : A \cap U \notin \mathcal{I}$ for every open set $U$ containing $x$ $\}$.

If there is no ambiguity, we will simply write $A^*$ for $A^*(\mathcal{I}, \tau)$.

**Definition 1.2** (see [1]). Let $(X, \tau, \mathcal{I})$ be an ideal topological space and let $A$ be a subset of $X$. The local closure function of $A$ with respect to $\mathcal{I}$ and $\tau$ is defined by

$\Gamma(A)(\mathcal{I}, \tau) = \{x \in X : A \cap Cl(U) \notin \mathcal{I}$ for every open set $U$ containing $x\}$.

If there is no ambiguity, we will simply write $\Gamma(A)$ for $\Gamma(A)(\mathcal{I}, \tau)$.

An approximation of the local function has been done by Al-Omari and Noiri [1] in 2013 with the help of closure operator of the topological space. In this paper, we have introduced another approximation of the local function with the help of semi-closure operator of the topological space and we have shown that almost all of the properties of local function (see [9], [10], [12], [24]) and Al-Omari and Noiri's local function [1] are satisfied. We have also shown that our approximation is much closer than the approximation of Al-Omari and Noiri [1].

## 2. Semi-closure local functions

**Definition 2.1.** Let $(X, \tau, \mathcal{I})$ be an ideal topological space. For a subset $A$ of $X$, we define the following operator:

$$\gamma(A)(\mathcal{I}, \tau) = \{x \in X : \; A \cap sCl(U) \notin \mathcal{I} \text{ for every } U \in \tau(x)\},$$

where $\tau(x) = \{U \in \tau : \; x \in U\}$. In case there is no confusion, $\gamma(A)(\mathcal{I}, \tau)$ is briefly denoted by $\gamma(A)$ and is called the semi-closure local function of $A$ with respect to $\mathcal{I}$ and $\tau$.

**Lemma 2.2.** *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. Then $A^*(\mathcal{I}, \tau) \subseteq \gamma(A)(\mathcal{I}, \tau) \subseteq \Gamma(A)(\mathcal{I}, \tau)$ for every subset $A$ of $X$.*

*Proof.* If $A \cap U \notin \mathcal{I}$ for every $U \in \tau(x)$, then $A \cap sCl(U) \notin \mathcal{I}$, since $A \cap U \subseteq A \cap sCl(U)$. Again from similar reason, $A \cap Cl(U) \notin \mathcal{I}$. Thus $A^*(\mathcal{I}, \tau) \subseteq \gamma(A)(\mathcal{I}, \tau) \subseteq \Gamma(A)(\mathcal{I}, \tau)$. □

Following examples illustrate the above lemma.

**Example 2.3.** Let $\mathbb{R}$ be the set of reals and let $\mathbb{Q}$ be the set of all rational numbers. Consider $X = \mathbb{R}$, $\tau = \{\emptyset, \mathbb{Q}, \mathbb{R}\}$, and $\mathcal{I} = \wp(\mathbb{Q})$. Let $i \in \mathbb{R} \setminus \mathbb{Q}$. Then $(\mathbb{Q} \cup \{i\})^* = \mathbb{R} \setminus \mathbb{Q}$ and $\gamma(\mathbb{Q} \cup \{i\}) = \mathbb{R}$.

**Example 2.4.** Let $X = \{a, b, c, d\}, \tau = \{\emptyset, X, \{a, c\}, \{d\}, \{a, c, d\}\}, \mathcal{I} = \{\emptyset, \{c\}\}$, and $A = \{b, c, d\}$. Then $\gamma(A) = \{b, d\}$ and $\Gamma(A) = X$.

**Theorem 2.5.** *Let $(X, \tau)$ be a topological space, let $\mathcal{I}$ and $\mathcal{J}$ be two ideals on $X$, and let $A$ and $B$ be two subsets of $X$. Then the following properties hold.*

(1) *If $A \subseteq B$, then $\gamma(A) \subseteq \gamma(B)$.*
(2) *If $\mathcal{I} \subseteq \mathcal{J}$, then $\gamma(A)(\mathcal{I}) \supseteq \gamma(A)(\mathcal{J})$.*
(3) *$\gamma(A) = sCl(\gamma(A)) \subseteq Cl_\theta(A)$ and $\gamma(A)$ is semi-closed.*
(4) *If $A \in \mathcal{I}$, then $\gamma(A) = \emptyset$.*

*Proof.* (1) Suppose that $x \in \gamma(A)$. Then $A \cap sCl(U) \notin \mathcal{I}$ for all $U \in \tau(x)$. Then $B \cap sCl(U) \notin \mathcal{I}$, otherwise $A \cap sCl(U) \in \mathcal{I}$, a contradiction. Thus $\gamma(A) \subseteq \gamma(B)$.

(2) Let $x \in \gamma(A)(\mathcal{J})$. Then $A \cap sCl(U) \notin \mathcal{J}$ and hence $A \cap sCl(U) \notin \mathcal{I}$.

(3) It is obvious that $\gamma(A) \subseteq sCl(\gamma(A))$. For reverse inclusion, let $x \in sCl(\gamma(A))$. Then $U_x \cap \gamma(A) \neq \emptyset$, for every $U_x \in \tau(x)$. Let $y \in U_x$ and $y \in \gamma(A)$. Then $U_x \in \tau(y)$ and $sCl(U_x) \cap A \notin \mathcal{I}$ and hence $x \in \gamma(A)$ for every $U_x \in \tau(x)$. Thus $\gamma(A) = sCl(\gamma(A))$.

Again from above $sCl(U_x) \cap A \notin \mathcal{I}$ for every $U_x \in \tau(x)$, then $sCl(U_x) \cap A \neq \emptyset$ for every $U_x \in \tau(x)$. Thus $Cl(U_x) \cap A \neq \emptyset$ for every $U_x \in \tau(x)$. Therefore $x \in Cl_\theta(A)$.

(4) Suppose that $x \in \gamma(A)$. Then for any $U_x \in \tau(x)$, $A \cap sCl(U) \notin \mathcal{I}$. Given that $A \in \mathcal{I}$, $A \cap sCl(U) \in \mathcal{I}$ for every $U_x \in \tau(x)$. This is a contradiction. Hence $\gamma(A) = \emptyset$. □

**Lemma 2.6.** *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. If $U \in \tau_\theta$, then $U \cap \gamma(A) = U \cap \gamma(U \cap A) \subseteq \gamma(U \cap A)$ for any subset $A$ of $X$.*

*Proof.* Let $x \in U \cap \gamma(A)$. Since $U \in \tau_\theta$, there exists $W \in \tau$ such that $x \in W \subseteq Cl(W) \subseteq U$ and hence $x \in W \subseteq sCl(W) \subseteq Cl(W) \subseteq U$ . Let $V$ be any open set containing $x$. Then $V \cap W \in \tau(x)$ and $sCl(V \cap W) \cap A \notin \mathcal{I}$, since $x \in \gamma(A)$. Therefore $sCl(V) \cap (U \cap A) \notin \mathcal{I}$. This shows that $x \in \gamma(U \cap A)$. Moreover, $U \cap \gamma(A) \subseteq U \cap \gamma(U \cap A)$. Again from Theorem 2.5, $\gamma(U \cap A) \subseteq \gamma(A)$ and $U \cap \gamma(A \cap U) \subseteq U \cap \gamma(A)$. Thus we have, $U \cap \gamma(A) = U \cap \gamma(U \cap A) \subseteq \gamma(U \cap A)$ for any subset $A$ of $X$. □

**Theorem 2.7.** *Let $(X, \tau, \mathcal{I})$ be an ideal topological space and let $A$, $B$ be subsets of $X$. Then the following properties hold.*

(1) $\gamma(\emptyset) = \emptyset$.
(2) $\gamma(A \cup B) = \gamma(A) \cup \gamma(B)$.

*Proof.* (1) The proof is obvious.

(2) It is obvious that from Theorem 2.5 that $\gamma(A) \cup \gamma(B) \subseteq \gamma(A \cup B)$. To prove $\gamma(A) \cup \gamma(B) \supseteq \gamma(A \cup B)$, suppose $x \notin \gamma(A) \cup \gamma(B)$. Then $x$ belongs neither to $\gamma(A)$ nor to $\gamma(B)$. Therefore, there exist $U_x$, $V_x \in \tau(x)$ such that $sCl(U_x) \cap A \in \mathcal{I}$ and $sCl(V_x) \cap B \in \mathcal{I}$. Then $[sCl(U_x) \cap A] \cup [sCl(V_x) \cap B] \in \mathcal{I}$.

Now

$$\begin{aligned} sCl(U_x \cap V_x) \cap (A \cup B) &\subseteq [sCl(U_x) \cap sCl(V_x)] \cap (A \cup B) \\ &= [[sCl(U_x) \cap sCl(V_x)] \cap A]] \cup [[sCl(U_x) \cap sCl(V_x)] \cap B] \\ &\subseteq [sCl(U_x) \cap A] \cup [sCl(V_x) \cap B] \in \mathcal{I}. \end{aligned}$$

Thus $x \notin \gamma(A \cup B)$ as $U_x \cap V_x \in \tau(x)$.

Therefore $\gamma(A \cup B) = \gamma(A) \cup \gamma(B)$. □

**Theorem 2.8.** *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. Then $\gamma Cl : \wp(X) \to \wp(X)$, defined by $\gamma Cl(A) = A \cup \gamma(A)$, is a Kuratowski closure operator.*

Thus we obtain a topological space from the Kuratowski closure operator $\gamma Cl$.

**Lemma 2.9.** *Let $(X, \tau, \mathcal{I})$ be an ideal topological space and let $A$ and $B$ be subsets of $X$. Then $\gamma(A) \setminus \gamma(B) = \gamma(A \setminus B) \setminus \gamma(B)$.*

*Proof.* Clearly,

$$\gamma(A) = \gamma[(A \setminus B) \cup (A \cap B)] = \gamma(A \setminus B) \cup \gamma(A \cap B) \subseteq \gamma(A \setminus B) \cup \gamma(B).$$

Then $\gamma(A) \setminus \gamma(B) \subseteq \gamma(A \setminus B) \setminus \gamma(B)$. Again from Theorem 2.5, $\gamma(A \setminus B) \subseteq \gamma(A)$ and hence $\gamma(A \setminus B) \setminus \gamma(B) \subseteq \gamma(A) \setminus \gamma(B)$. Consequently, $\gamma(A) \setminus \gamma(B) = \gamma(A \setminus B) \setminus \gamma(B)$. □

**Corollary 2.10.** *Let $(X, \tau, \mathcal{I})$ be an ideal topological space and let $A$ and $B$ be any subsets of $X$ with $B \in \mathcal{I}$. Then $\gamma(A \cup B) = \gamma(A) = \gamma(A \setminus B)$.*

*Proof.* Given that $\gamma(B) = \emptyset$. From Lemma 2.9, $\gamma(A) = \gamma(A \setminus B)$, and by Theorem 2.7, $\gamma(A \cup B) = \gamma(A) \cup \gamma(B) = \gamma(A)$. □

**Theorem 2.11** (see [20])**.** *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. Then each of the following conditions implies that the local function and the local closure function coincide:*
(a) *$\tau$ has a clopen base $\mathcal{B}$,*
(b) *$\tau$ has a $T_3$-topology on $X$,*
(c) *$\mathcal{I} = \mathcal{I}_{cd}$,*
(d) *$\mathcal{I} = \mathcal{I}_K$,*
(e) *$\mathcal{I}_{nwd} \subseteq \mathcal{I}$,*
(f) *$\mathcal{I} = \mathcal{I}_{mg}$.*

**Corollary 2.12.** *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. Then each of the conditions* (a)–(f) *in Theorem 2.11 implies that the local function, local closure function and semi-closure local function coincide.*

**Remark 2.13.** We have shown that in Example 2.3 and Example 2.4, the two operators are different. But there are some situations in which they are the same and this follows from the next section.

## 3. Semi-closure compatibility of ideal topological spaces

**Definition 3.1** (see [19])**.** Let $(X, \tau, \mathcal{I})$ be an ideal topological space. We say that $\tau$ is compatible with the ideal $\mathcal{I}$, denoted $\tau \sim \mathcal{I}$, if the following holds for every $A \subseteq X$: if for every $x \in A$ there exists $U \in \tau(x)$ such that $U \cap A \in \mathcal{I}$, then $A \in \mathcal{I}$.

**Definition 3.2** (see [11])**.** Let $(X, \tau, \mathcal{I})$ be an ideal topological space. We say that the topology $\tau$ is semi-compatible with the ideal $\mathcal{I}$, denoted $\tau \sim\sim \mathcal{I}$, if the following holds for every $A \subseteq X$: if for every $x \in A$ there exists a $U \in SO(X, x)$ such that $U \cap A \in \mathcal{I}$, then $A \in \mathcal{I}$, where $SO(X, x) = \{U \in SO(X) : x \in U\}$.

**Definition 3.3** (see [1])**.** Let $(X, \tau, \mathcal{I})$ be an ideal topological space. We say that $\tau$ is closure compatible with the ideal $\mathcal{I}$, denoted $\tau \sim_\Gamma \mathcal{I}$, if the following holds for every $A \subseteq X$: if for every $x \in A$ there exists $U \in \tau(x)$ such that $Cl(U) \cap A \in \mathcal{I}$, then $A \in \mathcal{I}$.

**Definition 3.4.** Let $(X, \tau, \mathcal{I})$ be an ideal topological space. We say that $\tau$ is semi-closure compatible with the ideal $\mathcal{I}$, denoted $\tau \sim_\gamma \mathcal{I}$, if the following holds for every $A \subseteq X$: if for every $x \in A$ there exists $U \in \tau(x)$ such that $sCl(U) \cap A \in \mathcal{I}$, then $A \in \mathcal{I}$.

**Remark 3.5.** If $\tau$ is semi-compatible with $\mathcal{I}$, then $\tau$ is compatible with $\mathcal{I}$.

*Proof.* Proof is obvious from the fact that $U \cap A \subseteq Cl(U) \cap A$. $\square$

**Remark 3.6.** If $\tau$ is semi-closure compatible with $\mathcal{I}$, then $\tau$ is closure compatible with $\mathcal{I}$.

*Proof.* Proof is obvious from the fact that $sCl(U) \cap A \subseteq Cl(U) \cap A$. $\square$

**Theorem 3.7.** *For an ideal topological space $(X, \tau, \mathcal{I})$, the following properties are equivalent.*

(1) $\tau \sim_\gamma \mathcal{I}$.
(2) *If a subset $A$ of $X$ has a cover of open sets each of whose semi-closure intersection with $A$ is in $\mathcal{I}$, then $A \in \mathcal{I}$.*
(3) *For every $A \subseteq X$, $A\gamma(A) = \emptyset$ implies that $A \in \mathcal{I}$.*
(4) *For every $A \subseteq X$, $A \setminus \gamma(A) \in \mathcal{I}$.*
(5) *For every $A \subseteq X$, if $A$ contains no nonempty subset $B$ with $B \subseteq \gamma(B)$, then $A \in \mathcal{I}$.*

*Proof.* (1) $\Longrightarrow$ (2) and (2) $\Longrightarrow$ (3) are obvious.

(3) $\Longrightarrow$ (4). From given condition, for any $A \subseteq X$, we have $A \setminus \gamma(A) \subseteq A$ and

$$(A \setminus \gamma(A)) \cap \gamma(A \setminus \gamma(A)) \subseteq (A \setminus \gamma(A)) \cap \gamma(A) = \emptyset.$$

Then $A \setminus \gamma(A) \in \mathcal{I}$ (from (3)).

(4) $\Longrightarrow$ (5). Given that $A \setminus \gamma(A) \in \mathcal{I}$ for every $A \subseteq X$. Put $J = A \setminus \gamma(A)$, then $A = J \cup (A \cap \gamma(A))$ and $\gamma(A) = \gamma(J) \cup \gamma(A \cap \gamma(A)) = \gamma(A \cap \gamma(A))$ as $\gamma(J) = \emptyset$. This implies that $A \cap \gamma(A) = A \cap \gamma(A \cap \gamma(A)) \subseteq \gamma(A \cap \gamma(A))$ and $A \cap \gamma(A) \subseteq A$. By the given condition, $A \cap \gamma(A) = \emptyset$ and hence $A = A \setminus \gamma(A) \in \mathcal{I}$.

(5) $\Longrightarrow$ (1). Let $A \subseteq X$ and assume that, for every $x \in A$, there exists $U \in \tau(x)$ such that $sCl(U) \cap A \in \mathcal{I}$. Then $A \cap \gamma(A) = \emptyset$. Suppose that $A$ contains $B$ such that $B \subseteq \gamma(B)$. Then $B = B \cap \gamma(B) \subseteq A \cap \gamma(B) \subseteq A \cap \gamma(A) = \emptyset$. Therefore, $A$ contains no nonempty subset $B$ with $B \subseteq \gamma(B)$. Hence $A \in \mathcal{I}$. $\square$

**Theorem 3.8.** *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. If $\tau \sim_\gamma \mathcal{I}$, then following equivalent properties hold.*

(1) *For every $A \subseteq X$, $A \cap \gamma(A) = \emptyset$ implies that $\gamma(A) = \emptyset$.*
(2) *For every $A \subseteq X$, $\gamma(A \setminus \gamma(A)) = \emptyset$.*
(3) *For every $A \subseteq X$, $\gamma(A \cap \gamma(A)) = \gamma(A)$.*

*Proof.* At first we show that (1) holds when $\tau \sim_\gamma \mathcal{I}$. If $A \cap \gamma(A) = \emptyset$ for any subset $A$ of $X$, then, from Theorem 3.7 (3), $A \in \mathcal{I}$. Since $A \in \mathcal{I}$, $\gamma(A) = \emptyset$.

(1) $\Longrightarrow$ (2). Let $B = A \setminus \gamma(A)$, then

$$\begin{aligned}B \cap \gamma(B) &= (A \setminus \gamma(A)) \cap \gamma(A \setminus \gamma(A)) = (A \cap (X \setminus \gamma(A))) \cap \gamma(A \cap (X \setminus \gamma(A)))\\ &\subseteq [A \cap (X \setminus (X \setminus \gamma(A))] \cap [\gamma(A) \cap (\gamma(X \setminus \gamma(A)))] = \emptyset.\end{aligned}$$

This implies that $\gamma(B) = \emptyset$, and hence $\gamma(A \setminus \gamma(A)) = \emptyset$.

(2) $\Longrightarrow$ (3). Assume, for every $A \subseteq X$, that $A = (A \setminus \gamma(A)) \cup (A \cap \gamma(A))$. Then

$$\begin{aligned}\gamma(A) &= \gamma[(A \setminus \gamma(A)) \cup (A \cap \gamma(A))]\\ &= \gamma((A \setminus \gamma(A))) \cup \gamma((A \cap \gamma(A))) = \gamma(A \cap \gamma(A))\end{aligned}$$

as $\gamma(A \setminus \gamma(A)) = \emptyset$.

(3) $\Longrightarrow$ (1). Suppose $A \cap \gamma(A) = \emptyset$ and $\gamma(A \cap \gamma(A)) = \gamma(A)$, for every $A \subseteq X$. This implies that $\emptyset = \gamma(\emptyset) = \gamma(A)$. $\square$

**Theorem 3.9.** *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. Then the following properties are equivalent.*

(1) *$sCl(\tau) \cap \mathcal{I} = \emptyset$, where $sCl(\tau) = \{sCl(V) : V \in \tau\}$.*
(2) *If $I \in \mathcal{I}$, then $Int_\theta(I) = \emptyset$.*
(3) *For every clopen subset $G$, $G \subseteq \gamma(G)$.*
(4) *$X = \gamma(X)$.*

*Proof.* (1) $\Longrightarrow$ (2). Let $I \in \mathcal{I}$. Suppose $x \in Int_\theta(I)$. Then there exists $U \in \tau$ such that $x \in U \subseteq sCl(U) \subseteq Cl(U) \subseteq I$. Since $I \in \mathcal{I}$, and hence $\emptyset \neq \{x\} \subseteq sCl(U) \in sCl(\tau) \cap \mathcal{I}$. This is contrary to $sCl(\tau) \cap \mathcal{I} = \emptyset$. Therefore, $Int_\theta(I) = \emptyset$.

(2) $\Longrightarrow$ (3). Obvious from [1].

(3) $\Longrightarrow$ (4). Obvious as $X$ is clopen.

(4) $\Longrightarrow$ (1). Given that $X = \gamma(X) = \{x \in X : sCl(U) \cap X = sCl(U) \notin \mathcal{I}, U \in \tau(x)\}$. Hence $sCl(\tau) \cap \mathcal{I} = \emptyset$. $\square$

**Theorem 3.10.** *Let $(X, \tau, \mathcal{I})$ be an ideal topological space and let $\tau$ be semi-closure compatible with $\mathcal{I}$. Then for every $G \in \tau_\theta$ and every subset $A$ of $X$,*

$$sCl(\gamma(G \cap A)) = \gamma(G \cap A) \subseteq \gamma(G \cap \gamma(A)) \subseteq Cl_\theta(G \cap \gamma(A)).$$

*Proof.* We know, from Theorem 3.8 (3), that $\gamma(G \cap A) = \gamma((G \cap A) \cap \gamma(G \cap A)) \subseteq \gamma(G \cap \gamma(A)) \subseteq Cl_\theta(G \cap \gamma(A))$ (from Theorem 2.5). $\square$

## 4. $\Psi_\gamma$-operator

**Definition 4.1.** Let $(X, \tau, \mathcal{I})$ be an ideal topological space. An operator $\Psi_\gamma : \wp(X) \to SO(X)$ is defined as follows: for every $A \subseteq X$,

$$\Psi_\gamma(A) = \{x \in X : \text{ there exists } U \in \tau(x) \text{ such that } sCl(U) \setminus A \in \mathcal{I}\}.$$

Observe that $\Psi_\gamma(A) = X \setminus \gamma(X \setminus A)$.

**Theorem 4.2.** *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. Then the following properties hold.*

(1) *If $A \subseteq B$, then $\Psi_\gamma(A)$ is semi-open.*
(2) *If $A \subseteq B$, then $\Psi_\gamma(A) \subseteq \Psi_\gamma(B)$.*
(3) *If $A,\ B \subseteq X$, then $\Psi_\gamma(A \cap B) = \Psi_\gamma(A) \cap \Psi_\gamma(B)$.*
(4) *If $A \subseteq X$, then $\Psi_\gamma(A) = \Psi_\gamma(\Psi_\gamma(A))$ if and only if $\gamma(X \setminus A) = \gamma(\gamma(X \setminus A))$.*
(5) *If $A \in \mathcal{I}$, then $\Psi_\gamma(A) = X \setminus \gamma(A)$.*
(6) *If $A \subseteq X,\ I \in \mathcal{I}$, then $\Psi_\gamma(A \setminus I) = \Psi_\gamma(A)$.*
(7) *If $A \subseteq X,\ I \in \mathcal{I}$, then $\Psi_\gamma(A \cup I) = \Psi_\gamma(A)$.*
(8) *If $(A \setminus B) \cup (B \setminus A) \in \mathcal{I}$, then $\Psi_\gamma(A) = \Psi_\gamma(B)$.*

*Proof.* (1) This follows from Theorem 2.5(3).

(2) This follows from Theorem 2.5(1).

(3) One has $\Psi_\gamma(A \cap B) = X \setminus \gamma(X \setminus (A \cap B)) = X \setminus \gamma[(X \setminus A) \cup (X \setminus B)] = X \setminus [\gamma(X \setminus A) \cup (X \setminus B)] = [X \setminus \gamma(X \setminus A) \cap [X \setminus \gamma(X \setminus B)] = \Psi_\gamma(A) \cap \Psi_\gamma(B)$.

(4) We have $\Psi_\gamma(A) = \Psi_\gamma(\Psi_\gamma(A))$ if and only if $X \setminus \gamma(X \setminus A) = \Psi_\gamma[X \setminus \gamma(X \setminus A)]$ if and only if $X \setminus \gamma(X \setminus A) = X \setminus \gamma(\gamma(X \setminus A))$ if and only if $\gamma(X \setminus A) = \gamma(\gamma(X \setminus A))$.

(5) Obvious from Corollary 2.10.

(6) Obvious from (5).

(7) Obvious from (5).

(8) Given that $(A \setminus B) \cup (B \setminus A) \in \mathcal{I}$. Then from hereditary property, $(A \setminus B) \in \mathcal{I}$ and $(B \setminus A) \in \mathcal{I}$. Note that $B = [A \setminus (A \setminus (A \setminus B))] \cup (B \setminus A)$. Then $\Psi_\gamma(A) = \Psi_\gamma(A \setminus (A \setminus B)) = \Psi_\gamma[(A \setminus (A \setminus B)) \cup (B \setminus A)] = \Psi_\gamma(B)$. $\square$

**Corollary 4.3.** *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. Then $U \subseteq \Psi_\gamma(U)$, for every $\theta$-open set $U \subseteq X$.*

*Proof.* We have $\Psi_\gamma(U) = X \setminus \gamma(X \setminus A)$. As we know, $\gamma(X \setminus U) \subseteq Cl_\theta(X \setminus U) = (X \setminus U)$ ($(X \setminus U)$ is $\theta$-closed). Then $U = X \setminus (X \setminus U) \subseteq X \setminus \gamma(X \setminus U) = \Psi_\gamma(U)$. $\square$

However, Modak and Bandyopadhyay [16] have shown that the above relation is true for the open set when the local function is defined in Kuratowski's sense.

Now we shall give an example of a set $A$ which is not $\theta$-open set but satisfies $A \subseteq \Psi_\gamma(A)$.

**Example 4.4.** Let $X = \{a, b, c, d\}$, $\tau = \{\emptyset,\ X, \{a, c\},\ \{d\},\ \{a, c, d\}\}$, and let $\mathcal{I} = \{\emptyset,\ \{b\},\ \{c\},\ \{b, c\}\}$. Then

$$SO(X) = \{\emptyset,\ X,\ \{d\},\ \{a, c\},\ \{b, d\},\ \{a, b, c\},\ \{a, c, d\}\}$$

and semi-closed sets are $\{\emptyset,\ X, \{a, b, c\},\ \{b, d\},\ \{a, c\},\ \{d\},\ \{b\}\}$. Let $A = \{a\}$. Then $\Psi_\gamma(\{a\}) = X \setminus \gamma(\{b, c, d\}) = X \setminus \{b, d\} = \{a, c\}$. Therefore, $\{a\} \subseteq \Psi_\gamma(\{a\})$, but $A$ is not $\theta$-open.

**Theorem 4.5.** *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. For $A \subseteq X$, the following properties hold.*

(1) $\Psi_\gamma(A) = \bigcup\{U \in \tau :\ sCl(U) \setminus A \in \mathcal{I}\}$.
(2) $\Psi_\gamma(A) \supseteq \bigcup\{U \in \tau :\ [(sCl(U) \setminus A) \cup (A \setminus sCl(U))] \in \mathcal{I}\}$.

*Proof.* (1) This follows from the definition of $\Psi_\gamma$-operator.
(2) One has

$$\bigcup\{U \in \tau :\ [(sCl(U) \setminus A) \cup (A \setminus sCl(U))] \in \mathcal{I}\}$$
$$\subseteq \bigcup\{U \in \tau :\ sCl(U) \setminus A \in \mathcal{I}\} = \Psi_\gamma(A)$$

for every $A \subseteq X$. □

**Theorem 4.6.** *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. If $\sum = \{A \subseteq X :\ A \subseteq \Psi_\gamma(A)\}$, then $\sum$ is a topology on $X$.*

*Proof.* We have $\Psi_\gamma(X) = X \setminus \gamma(X \setminus X) = X$, since $\emptyset \in \mathcal{I}$ implies $\gamma(\emptyset) = \emptyset$. Again $\Psi_\gamma(\emptyset) = X \setminus \gamma(X \setminus \emptyset) = X \setminus X = \emptyset$. Since $A \cap B \subseteq \Psi_\gamma(A)$ and $A \cap B \subseteq \Psi_\gamma(B)$, $A \cap B \subseteq \Psi_\gamma(A) \cap \Psi_\gamma(B)$. Suppose $\{A_\alpha :\ \alpha \in \Lambda\} \subseteq \sum$, then $A_\alpha \subseteq \Psi_\gamma(A_\alpha) \subseteq \Psi_\gamma(\cup A_\alpha)$ for every $\alpha$, and hence $\cup A_\alpha \subseteq \Psi_\gamma(\cup A_\alpha)$. This shows that $\sum$ is a topology. □

## 5. Error operators

**Definition 5.1.** Let $(X, \tau, \mathcal{I})$ be an ideal topological space. We define two operators $E_1,\ E_2 : \wp(X) \to \wp(X)$ as follows: for every $A \subseteq X$,

$$E_1(A) = \{x \in X :\ [(Cl(U) \cap A) \setminus (sCl(U) \cap A)] \notin \mathcal{I},\ U \in \tau(x)\},$$
$$E_2(A) = \{x \in X :\ [(Cl(U) \cap A) \setminus (U \cap A)] \notin \mathcal{I},\ U \in \tau(x)\}.$$

These two operators are called error operators.

**Theorem 5.2.** *For an ideal topological space $(X, \tau, \mathcal{I})$, the following properties hold.*

(1) $E_1(A) \subseteq \Gamma(A)$, *for any subset $A$ of $X$.*
(2) $E_2(A) \subseteq \Gamma(A)$, *for any subset $A$ of $X$.*
(3) $E_1(A) \subseteq E_2(A)$, *for any subset $A$ of $X$.*
(4) $E_1(A) \setminus E_2(A) = \emptyset$, *for any subset $A$ of $X$.*
(5) $E_2(A) \setminus E_1(A) \neq \emptyset$, *for any subset $A$ of $X$.*

*Proof.* (1) Proof is obvious from the fact that $[(Cl(U) \cap A) \setminus (sCl(U) \cap A)] \notin \mathcal{I}$ implies $Cl(U) \cap A \notin \mathcal{I}$.

(2) Proof is obvious from the fact that $[(Cl(U) \cap A) \setminus (U \cap A)] \notin \mathcal{I}$ implies $Cl(U) \cap A \notin \mathcal{I}$.

(3) Suppose $x \in E_1(A)$. Then for all $U \in \tau(x)$, $[(Cl(U) \cap A) \setminus (U \cap A)] \notin \mathcal{I}$ otherwise $[(Cl(U) \cap A) \setminus (sCl(U) \cap A)] \in \mathcal{I}$. Thus $x \in E_2(A)$. □

We have from the above that the error $E_1(A)$ is smaller than the error $E_2(A)$, for a subset $A$ of $X$. Thus $\gamma(A)$ is a better approximation of $\Gamma(A)$ than the approximation of $A^*$ by $\Gamma(A)$.

**Definition 5.3.** Let $(X, \tau, \mathcal{I})$ be an ideal topological space. Two operators $E_{\gamma *}$, $E_{\Gamma\gamma} : \wp(X) \to \wp(X)$ are defined as follows: for every $A \subseteq X$, $E_{\gamma *}(A) = \gamma(A) \setminus A^*$ and $E_{\Gamma\gamma}(A) = \Gamma(A) \setminus \gamma(A)$.

**Theorem 5.4.** *For an ideal topological space $(X, \tau, \mathcal{I})$, the following properties hold.*

1. $E_{\gamma *}(A) \subseteq \gamma(A)$, *for any subset $A$ of $X$.*
2. $E_{\Gamma\gamma}(A) \subseteq \Gamma(A)$, *for any subset $A$ of $X$.*
3. $E_{\gamma *}(A) \subseteq E_{\Gamma\gamma}(A)$, *for any subset $A$ of $X$.*
4. $E_{\Gamma\gamma}(A) \setminus E_{\gamma *}(A) = \emptyset$, *for any subset $A$ of $X$.*
5. $E_{\Gamma\gamma}(A) \setminus E_{\gamma *}(A) \neq \emptyset$, *for any subset $A$ of $X$.*

*Proof.* (3) Let $x \in E_{\gamma *}(A)$. Then for all $U_x \in \tau(x)$, $sCl(U_x) \cap A \notin \mathcal{I}$ and there exists $V_x \in \tau(x)$ such that $V_x \cap A \in \mathcal{I}$. Then $(Cl(U_x) \cap A) \notin \mathcal{I}$. Thus $[(Cl(U_x) \cap A) \setminus (V_x \cap A)] \notin \mathcal{I}$. (If not, that is $[(Cl(U_x) \cap A) \setminus (V_x \cap A)] \in \mathcal{I}$ and hence, $[(Cl(U_x) \cap A) \setminus (V_x \cap A)] = I_1 \in \mathcal{I}$. Thus $Cl(U_x) \cap A = I_1 \cup (I_1 \cap (V_x \cap A)) \in \mathcal{I}$, a contradiction.) □

## 6. Conclusion

One can define a local function in an ideal topological space replacing the open sets by

$$\{x \in X : U_x \cap A \notin \Gamma, U_x \in \tau(x)\}. \tag{6.1}$$

But then the following occurs.

(i) If the open sets $U_x$ of (6.1) are replaced by the base members $B_x$ (such as regular open sets) then this local function is finer than the original local function (see [12], [24], [9], [10]) and this new type of local function does not give a topology, however the collection of regular open sets is a smaller collection than the topology.

(ii) If the open sets $U_x$ of (6.1) are replaced by $sCl(U_x)$ (respectively, $Cl(U_x)$), then this local function is a generalization of the original local function (see [12], [24], [9], [10]), however $U_x \subseteq sCl(U_x)$ (respectively, $Cl(U_x)$), and this induces again a topology.

(iii) If the open sets $U_x$ of (6.1) are replaced by the semi-open sets, then the new local function [11] is weaker than the original local function and it does not induce a topology, although the collection of all semi-open sets is finer than the topology (see [11]).

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