ACTA ET COMMENTATIONES UNIVERSITATIS TARTUENSIS DE MATHEMATICA
Volume 22, Number 2, December 2018
Available online at http://acutm.math.ut.ee

On two integrability methods

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Abstract. Let p be a non-negative, non-decreasing function on $[0,\infty)$ and let $k \geq 1$. In this paper, we introduce the concept of the Riesz integrability (shortly, $|\bar{N},p|_k$ integrability) of improper integrals. We prove an equivalence theorem of $|\bar{N},p|_k$ and $|\bar{N},q|k$ integrability of improper integrals.

1. Introduction

Throughout this paper we assume that f is a real valued function which is continuous on $[0,\infty)$ and $s(x)=\int_0^x f(t)dt$. By $\sigma(x)$, we denote the Cesàro mean of $s(x)$. The improper integral $\int_0^\infty f(t)dt$ is said to be integrable $|C,1|_k$, $k\geq 1$, in the sense of Flett [4], if the improper integral

$$\int_0^\infty x^{k-1}|\sigma'(x)|^k dx$$

is convergent.

Let p be a real valued non-decreasing function on $[0,\infty)$ with $p(0)=0$, $p(x)\neq 0$ for $x>0$, and let

$$P(x)=\int_0^x p(t)dt. \tag{1.1}$$

Then

$$\sigma_p(x)=\frac{1}{P(x)}\int_0^x p(t)s(t)dt$$

defines the Riesz mean of the function s.

Received September 28, 2017.
2010 *Mathematics Subject Classification.* 26D15; 40F05; 40D25; 40A10; 35A23.
Key words and phrases. Cesàro and Riesz method; improper integrals; divergent integrals; inequalities for integrals; Hölder's inequality.
http://dx.doi.org/10.12697/ACUTM.2018.22.21

We say that the integral $\int_0^\infty f(t)dt$ is integrable $|\bar{N},p|_k$, $k \geq 1$, if the improper integral

$$\int_0^\infty \left(\frac{P(x)}{p(x)}\right)^{k-1} |\sigma_p'(x)|^k dx$$

is convergent. If we take $p(x) = 1$ for all values of x, then $|\bar{N},p|_k$ integrability reduces to $|C,1|_k$ integrability of improper integrals.

Since (see [11], p. 392)

$$\sigma_p'(x) = \frac{p(x)}{P(x)} v_p(x), \quad v_p(x) = \frac{1}{P(x)} \int_0^x P(u)f(u)du,$$

an improper integral $\int_0^\infty f(t)dt$ is integrable $|\bar{N},p|_k$ if and only if the improper integral

$$\int_0^\infty \frac{p(x)}{P(x)} |v_p(x)|^k dx \tag{1.2}$$

is convergent.

Several authors have presented theorems on the absolute summability of infinite integrals by functional Nörlund methods (see [5, 6, 7, 8, 9]). Özgen [10, 11] proved some theorems dealing with Riesz and Cesàro integrability of improper integrals.

We note that for infinite series, an analogous definition was introduced by Bor [1]. Using this definition, Bor and Thorpe [2] proved the following theorem about the equivalence of $|\bar{N},p_n|_k$ and $|\bar{N},q_n|_k$ summability methods.

Theorem 1.1. *Let $k \geq 1$, and let (p_n) and (q_n) be positive sequences with $P_n = \sum_{k=1}^n p_k$ and $Q_n = \sum_{k=1}^n q_k$. The $|\bar{N},p_n|_k$ summability of series $\sum a_n$ is equivalent to the $|\bar{N},q_n|_k$ summability provided that*

$$\frac{p_n}{P_n} = O\left(\frac{q_n}{Q_n}\right), \quad \frac{q_n}{Q_n} = O\left(\frac{p_n}{P_n}\right), \quad \text{as } n \to \infty.$$

2. Main result

The aim of this paper is to prove the following analogue of the theorem of Bor and Thorpe for $|\bar{N},p|_k$ and $|\bar{N},q|_k$ integrability of improper integrals.

Theorem 2.1. *Let $k \geq 1$, and let p and q be real valued non-decreasing functions on $[0,\infty)$ with $P(x)$ defined by (1.1) and $Q(x) = \int_0^x q(t)\,dt$, $q(x) \neq 0$ for $x > 0$, $q(0) = 0$. If*

$$p(x)Q(x) = O(q(x)P(x)), \tag{2.1}$$

$$q(x)P(x) = O(p(x)Q(x)), \tag{2.2}$$

as $x \to \infty$, then the integrability $|\bar{N},p|_k$ of $\int_0^\infty f(t)\,dt$ is equivalent to the integrability $|\bar{N},q|_k$.

Proof. First we show that the integrability $|\bar{N},p|_k$ of $\int_0^\infty f(t)\,dt$ implies the integrability $|\bar{N},q|_k$. Let $\sigma_q(x)$ be the Riesz mean of the function s. Since $\int_0^\infty f(t)dt$ is integrable $|\bar{N},p|_k$, the integral (1.2) converges and thus

$$\int_0^m \frac{p(x)}{P(x)}|v_p(x)|^k dx = O(1) \text{ as } m\to\infty.$$

Using the equality $s(x)-\sigma_p(x)=v_p(x)$, we can write

$$\begin{aligned}\sigma_q'(x) &= \frac{q(x)}{Q^2(x)}\int_0^x Q(t)f(t)\,dt\\ &= \frac{q(x)}{Q^2(x)}\int_0^x Q(t)v_p'(t)\,dt + \frac{q(x)}{Q^2(x)}\int_0^x Q(t)\frac{p(t)}{P(t)}v_p(t)\,dt.\end{aligned}$$

Integrating by parts, we have

$$\begin{aligned}\sigma_q'(x) &= \frac{q(x)}{Q^2(x)}\left[Q(x)v_p(x) - \int_0^x v_p(t)q(t)dt\right] + \frac{q(x)}{Q^2(x)}\int_0^x Q(t)\frac{p(t)}{P(t)}v_p(t)dt\\ &= \frac{q(x)}{Q(x)}v_p(x) + \frac{q(x)}{Q^2(x)}\int_0^x \frac{v_p(t)}{P(t)}Q(t)p(t)dt - \frac{q(x)}{Q^2(x)}\int_0^x q(t)v_p(t)dt\\ &= \sigma_{q,1}(x)+\sigma_{q,2}(x)+\sigma_{q,3}(x).\end{aligned}$$

To prove the integrability $|\bar{N},q|_k$ of $\int_0^\infty f(t)\,dt$, by Minkowski's inequality it is sufficient to show that, for $r=1,2,3$,

$$\int_0^m \left(\frac{Q(x)}{q(x)}\right)^{k-1}|\sigma_{q,r}(x)|^k dx = O(1)\ as\ m\to\infty.$$

By (2.2), we have

$$\begin{aligned}\int_0^m \left(\frac{Q(x)}{q(x)}\right)^{k-1}|\sigma_{q,1}(x)|^k dx &= \int_0^m \frac{q(x)}{Q(x)}|v_p(x)|^k dx\\ &= O(1)\int_0^m \frac{p(x)}{P(x)}|v_p(x)|^k dx = O(1) \text{ as } m\to\infty.\end{aligned}$$

Now, applying Hölder's inequality with $k>1$, by (2.1) and (2.2) we get

$$\begin{aligned}\int_0^m\left(\frac{Q(x)}{q(x)}\right)^{k-1}|\sigma_{q,2}(x)|^k dx &= \int_0^m \frac{q(x)}{Q^{k+1}(x)}\left(\int_0^x \frac{Q(t)p(t)}{P(t)}\frac{q(t)}{q(t)}|v_p(t)|dt\right)^k dx\\ &= O(1)\int_0^m \frac{q(x)}{Q^2(x)}dx\left(\int_0^x\left(\frac{Q(t)}{q(t)}\right)^k q(t)\left(\frac{p(t)}{P(t)}\right)^k|v_p(t)|^k dt\right)\\ &\quad\times\left(\frac{1}{Q(x)}\int_0^x q(t)dt\right)^{k-1}\\ &= O(1)\int_0^m\left(\frac{Q(t)}{q(t)}\right)^{k-1}Q(t)\left(\frac{p(t)}{P(t)}\right)^k|v_p(t)|^k dt\int_t^m \frac{q(x)}{Q^2(x)}dx\end{aligned}$$

$$= O(1)\int_0^m \left(\frac{Q(t)}{q(t)}\right)^{k-1} \left(\frac{p(t)}{P(t)}\right)^k |v_p(t)|^k dt$$
$$= O(1)\int_0^m \frac{p(t)}{P(t)}|v_p(t)|^k dt = O(1) \text{ as } m \to \infty.$$

Finally, again by Hölder's inequality with $k > 1$, using (2.2), we have

$$\int_0^m \left(\frac{Q(x)}{q(x)}\right)^{k-1} |\sigma_{q,3}(x)|^k dx = \int_0^m \frac{q(x)}{Q^{k+1}(x)} dx \left(\int_0^x q(t)|v_p(t)|dt\right)^k$$
$$= O(1)\int_0^m \frac{q(x)}{Q^2(x)} dx \left(\int_0^x q(t)|v_p(t)|^k dt\right)\left(\frac{1}{Q(x)}\int_0^x q(t)dt\right)^{k-1}$$
$$= O(1)\int_0^m q(t)|v_p(t)|^k dt \int_t^m \frac{q(x)}{Q^2(x)} dx = O(1)\int_0^m \frac{q(t)}{Q(t)}|v_p(t)|^k dt$$
$$= O(1)\int_0^m \frac{p(t)}{P(t)}|v_p(t)|^k dt = O(1) \text{ as } m \to \infty.$$

Interchanging in our proof the roles of the functions p and q, we find that the integrability $|\bar{N}, q|_k$ of $\int_0^\infty f(t)\,dt$ implies the integrability $|\bar{N}, p|_k$. □

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