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On concircular curvature tensor in a Lorentzian α-Sasakian manifold with respect to the quarter-symmetric non-metric connection

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Abstract. In the present paper, some properties of concircular curvature tensor in a Lorentzian α-Sasakian manifold with respect to the quarter-symmetric non-metric connection have been studied.

1. Introduction

In a Riemannain manifold M, a linear connection $\bar{\nabla}$ is called a quarter-symmetric connection [6] if the torsion tensor T of the connection $\bar{\nabla}$,

$$T(X,Y)=\bar{\nabla}_X Y-\bar{\nabla}_Y X-[X,Y],$$

satisfies

$$T(X,Y)=\eta(Y)\phi X-\eta(X)\phi Y,$$

where η is a 1-form and ϕ is a $(1,1)$ tensor field. If, moreover, a quarter-symmetric connection $\bar{\nabla}$ satisfies the condition

$$(\bar{\nabla}_X g)(Y,Z)=-\eta(Y)g(\phi X,Z)-\eta(Z)g(Y,\phi X),$$

where $X,Y,Z\in\chi(M)$ and $\chi(M)$ is the set of all differentiable vector fields on M, then $\bar{\nabla}$ is said to be a quarter-symmetric non-metric connection. If we change ϕX by X, then the connection reduces to a semi-symmetric non-metric connection [12]. Thus the notion of quarter-symmetric connection generalizes the notion of semi-symmetric connection.

A relation between the quarter-symmetric non-metric connection $\bar{\nabla}$ and the Levi-Civita connection ∇ in an n-dimensional Lorentzian α-Sasakian

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manifold M is given by

$$\bar{\nabla}_X Y = \nabla_X Y + \eta(Y)\phi X. \tag{1.1}$$

A transformation of an n-dimensional Riemannian manifold M, which transforms every geodesic circle of M into a geodesic circle, is called a concircular transformation [8, 11]. A concircular transformation is always a conformal transformation [8]. Here geodesic circle means a curve in M whose first curvature is constant and second curvature is identically zero. Thus the geometry of concircular transformations, i.e., the concircular geometry, is a generalization of inversive geometry in the sense that the change of metric is more general than induced by a circle preserving diffeomorphism. An interesting invariant of a concircular transformation is the concircular curvature tensor with respect to the Levi-Civita connection. It is defined by (see [11])

$$C(X,Y)Z = R(X,Y)Z - \frac{r}{n(n-1)}[g(Y,Z)X - g(X,Z)Y], \tag{1.2}$$

where $X, Y, Z \in \chi(M)$, R and r are, respectively, the curvature tensor and the scalar curvature with respect to the Levi-Civita connection. A Riemannian manifold with vanishing concircular curvature tensor is of constant curvature. Thus, the concircular curvature tensor is the measure of the failure of a Riemannian manifold to be of constant curvature. Recently, concircular curvature tensor have been studied by various authors such as Acet and Perktas [1], Barman and Ghosh [3], Haseeb [7], Taleshian and Asghari [10], and many others.

In 2005, Yildiz and Murathan [13] studied Lorentzian α-Sasakian manifolds with conformally flat and quasi-conformally flat conditions. In 2009, Yildiz et al. [15], further studied Lorentzian α-Sasakian manifolds and proved that ϕ-conformally flat, ϕ-conharmonically flat, ϕ-projectively flat and ϕ-concircularly flat Lorentzian α-Sasakian manifolds are η-Einstein manifolds. Recently, De and Majhi [5] studied ϕ-Weyl semisymmetric and ϕ-projectively semisymmetric generalized Sasakian space-forms.

The paper is organized as follows. In Section 2 we give a brief introduction of Lorentzian α-Sasakian manifolds. In Section 3 we deduce the relation between the curvature tensors of Lorentzian α-Sasakian manifolds with respect to the quarter-symmetric non-metric connection and the Levi-Civita connection. Sections 4, 5, 6, and 7 are devoted to study ξ-concircularly flat, quasi-concircularly flat, pseudoconcircularly flat, and ϕ-concircularly flat Lorentzian α-Sasakian manifolds with respect to the quarter-symmetric non-metric connection, respectively. In the last Section 8 we study ϕ-semisymmetric Lorentzian α-Sasakian manifolds with respect to the quarter-symmetric non-metric connection.

2. Preliminaries

A differentiable manifold M of dimension n is called a Lorentzian α-Sasakian manifold, if it admits a $(1,1)$ tensor field ϕ, a contravariant vector field ξ, a covariant vector field η, and a Lorentzian metric g which satisfy the conditions (see [4])

$$\phi^2 X = X + \eta(X)\xi,$$

$$\eta(\xi) = -1,\ \phi\xi = 0,\ \eta(\phi X) = 0,\ g(X,\xi) = \eta(X), \tag{2.1}$$

$$g(\phi X, \phi Y) = g(X,Y) + \eta(X)\eta(Y),\ g(\phi X, Y) = g(X, \phi Y) \tag{2.2}$$

for all vector fields X, Y on M.

Also Lorentzian α-Sasakian manifolds satisfy the equations (see [13]–[15])

$$\nabla_X \xi = -\alpha\phi X,$$

$$\Phi(X,Y) = (\nabla_X \eta)Y = -\alpha g(\phi X, Y), \tag{2.3}$$

where ∇ denotes the operator of covariant differentiation with respect to the Lorentzian metric g, and $\alpha \in R$.

Further, on a Lorentzian α-Sasakian manifold M, the following relations hold (see [2, 13, 15]):

$$g(R(X,Y)Z,\xi) = \eta(R(X,Y)Z) = \alpha^2[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)],$$

$$R(\xi,X)Y = \alpha^2[g(X,Y)\xi - \eta(Y)X],$$

$$R(X,Y)\xi = \alpha^2[\eta(Y)X - \eta(X)Y],$$

$$R(\xi,X)\xi = \alpha^2[X + \eta(X)\xi],$$

$$S(X,\xi) = (n-1)\alpha^2\eta(X),\ S(\xi,\xi) = -(n-1)\alpha^2,$$

$$Q\xi = (n-1)\alpha^2\xi,$$

$$(\nabla_X\phi)Y = \alpha[g(X,Y)\xi - \eta(Y)X], \tag{2.4}$$

$$S(\phi Y, \phi Z) = S(Y,Z) + (n-1)\alpha^2 g(Y,Z) \tag{2.5}$$

for any vector fields X, Y and Z on M.

Example 2.1. We consider the 3-dimensional manifold $M^3 = \{(x,y,z) \in \mathbb{R}^3 : z > 0\}$, where (x,y,z) are standard coordinates of $\mathbb{R}^3$. Let e_1, e_2 and e_3 be the vector fields on M^3 given by

$$e_1 = z\frac{\partial}{\partial x},\ e_2 = z\frac{\partial}{\partial y},\ e_3 = z\frac{\partial}{\partial z} = \xi,$$

which are linearly independent at each point of M^3, and hence form a basis of T_pM^3. Define a Lorentzian metric g on M^3 as

$$g(e_1,e_1) = 1,\ g(e_2,e_2) = 1,\ g(e_3,e_3) = -1,$$

$$g(e_1,e_2) = g(e_1,e_3) = g(e_2,e_3) = 0.$$

Let η be the 1-form on M^3 defined as $\eta(X) = g(X, e_3) = g(X, \xi)$ for all $X \in \chi(M)$, and let ϕ be the $(1, 1)$ tensor field on M^3 defined as

$$\phi e_1 = -e_1,\ \phi e_2 = -e_2,\ \phi e_3 = 0.$$

By applying linearity of ϕ and g, we have

$$\eta(\xi) = g(\xi, \xi) = -1,\ \phi^2 X = X + \eta(X)\xi,\ \eta(\phi X) = 0,$$
$$g(X, \xi) = \eta(X),\ g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y)$$

for all $X, Y \in \chi(M)$.

Let ∇ be the Levi-Civita connection with respect to the Lorentzian metric g. Then we have

$$[e_1, e_2] = 0,\ [e_2, e_1] = 0,\ [e_1, e_3] = -e_1,$$
$$[e_3, e_1] = e_1,\ [e_2, e_3] = -e_2,\ [e_3, e_2] = e_2.$$

The Riemannian connection ∇ of the Lorentzian metric g is given by

$$\begin{aligned} 2g(\nabla_X Y, Z) &= Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) - g(X, [Y, Z]) \\ &\quad + g(Y, [Z, X]) + g(Z, [X, Y]), \end{aligned}$$

which is known as Koszul's formula (see [4]). Using Koszul's formula, we can easily calculate

$$\begin{aligned} &\nabla_{e_1} e_1 = -e_3,\ \nabla_{e_1} e_2 = 0,\ \nabla_{e_1} e_3 = -e_1,\ \nabla_{e_2} e_1 = 0,\ \nabla_{e_2} e_2 = -e_3, \\ &\nabla_{e_2} e_3 = -e_2,\ \nabla_{e_3} e_1 = 0,\ \nabla_{e_3} e_2 = 0,\ \nabla_{e_3} e_3 = 0. \end{aligned} \tag{2.6}$$

Now let

$$X = \sum_{i=1}^{3} X^i e_i = X^1 e_1 + X^2 e_2 + X^3 e_3,$$
$$Y = \sum_{j=1}^{3} Y^j e_j = Y^1 e_1 + Y^2 e_2 + Y^3 e_3,$$
$$Z = \sum_{k=1}^{3} Z^k e_k = Z^1 e_1 + Z^2 e_2 + Z^3 e_3$$

for all $X, Y, Z \in \chi(M)$. It is known that

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z. \tag{2.7}$$

From the equations (2.6) and (2.7), it can be easily verified that

$$\begin{aligned} &R(e_1, e_2)e_1 = -e_2,\ R(e_1, e_3)e_1 = -e_3,\ R(e_2, e_3)e_1 = 0, \\ &R(e_1, e_2)e_2 = e_1,\ R(e_1, e_3)e_2 = 0,\ R(e_2, e_3)e_2 = -e_3, \\ &R(e_1, e_2)e_3 = 0,\ R(e_1, e_3)e_3 = -e_1,\ R(e_2, e_3)e_3 = -e_2. \end{aligned} \tag{2.8}$$

From (2.8), it follows that

$$R(X, Y)Z = g(Y, Z)X - g(X, Z)Y.$$

Thus, for $e_3 = \xi$, the manifold M^3 is a Lorentzian almost contact metric manifold of constant curvature 1 and is locally isometric to the unit sphere $S^3(1)$.

Definition 2.2 (see [15])**.** A Lorentzian α-Sasakian manifold M is said to be an η-Einstein manifold if its Ricci tensor S is of the form

$$S(X, Y) = ag(X, Y) + b\eta(X)\eta(Y),$$

where a and b are scalar functions on M.

A Lorentzian α-Sasakian manifold M is said to be a generalized η-Einstein manifold if its Ricci tensor S is of the form

$$S(X, Y) = ag(X, Y) + b\eta(X)\eta(Y) + c\Omega(X, Y),$$

where a, b, c are scalar functions on M and $\Omega(X, Y) = g(\phi X, Y)$. If $c = 0$, then it reduces to an η-Einstein manifold.

3. Curvature tensor of Lorentzian α-Sasakian manifolds with respect to the quarter-symmetric non-metric connection

The curvature tensor $\bar{R}$ of a Lorentzian α-Sasakian manifold with respect to the quarter-symmetric non-metric connection $\bar{\nabla}$ is defined by

$$\bar{R}(X, Y)Z = \bar{\nabla}_X\bar{\nabla}_Y Z - \bar{\nabla}_Y\bar{\nabla}_X Z - \bar{\nabla}_{[X,Y]}Z. \tag{3.1}$$

From the equations (1.1), (2.1), (2.3), (2.4), and (3.1), we get

$$\begin{aligned}\bar{R}(X, Y)Z = R(X, Y)Z &+ \alpha[g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y] \\ &+ \alpha[\eta(X)Y - \eta(Y)X]\eta(Z),\end{aligned} \tag{3.2}$$

where

$$R(X, Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$$

is the Riemannian curvature tensor of the connection ∇. Contracting X in (3.2), we get

$$\bar{S}(Y, Z) = S(Y, Z) + \alpha g(\phi Y, Z)\psi - \alpha g(Y, Z) - \alpha n\eta(Y)\eta(Z), \tag{3.3}$$

where S and $\bar{S}$ are the Ricci tensors with respect to the connections ∇ and $\bar{\nabla}$, respectively, on M and ψ=trace ϕ. Contracting again Y and Z in (3.3), we get

$$\bar{r} = r + \alpha\psi^2, \tag{3.4}$$

where r and $\bar{r}$ are, respectively, the scalar curvatures with respect to the connections ∇ and $\bar{\nabla}$ on M.

Lemma 3.1. *Let M be an n-dimensional Lorentzian α-Sasakian manifold with respect to the quarter-symmetric non-metric connection. Then we have*

$$\bar{R}(X, Y)\xi = (\alpha^2 + \alpha)[\eta(Y)X - \eta(X)Y], \tag{3.5}$$

$$\begin{aligned}\bar{R}(\xi, X)Y &= -\bar{R}(X,\xi)Y \\ &= \alpha^2 g(X,Y)\xi - (\alpha^2+\alpha)\eta(Y)X - \alpha\eta(X)\eta(Y)\xi, \\ \bar{R}(\xi, X)\xi &= (\alpha^2+\alpha)[X+\eta(X)\xi], \\ \bar{S}(X,\xi) &= (n-1)(\alpha^2+\alpha)\eta(X),\ \bar{S}(\xi,\xi) = -(n-1)(\alpha^2+\alpha), \\ \bar{Q}\xi &= (n-1)(\alpha^2+\alpha)\xi\end{aligned} \tag{3.6}$$

for all $X, Y \in \chi(M)$.

4. ξ-Concircularly flat Lorentzian α-Sasakian manifolds with respect to the quarter-symmetric non-metric connection

Analogous to the equation (1.2), the concircular curvature tensor $\bar{C}$ with respect to the quarter-symmetric non-metric connection is given by

$$\bar{C}(X,Y)Z = \bar{R}(X,Y)Z - \frac{\bar{r}}{n(n-1)}[g(Y,Z)X - g(X,Z)Y], \tag{4.1}$$

where $\bar{R}$ and $\bar{r}$ are, respectively, the Riemannian curvature tensor and the scalar curvature with respect to the connection $\bar{\nabla}$.

Definition 4.1. A Lorentzian α-Sasakian manifold with respect to the quarter-symmetric non-metric connection is said to be ξ-concircularly flat if

$$\bar{C}(X,Y)\xi = 0 \tag{4.2}$$

for all X, Y on M.

Taking $Z = \xi$ in (4.1) and then using (2.1), (3.4), and (3.5), we have

$$\bar{C}(X,Y)\xi = \left[(\alpha^2+\alpha) - \frac{r+\alpha\psi^2}{n(n-1)}\right][\eta(Y)X - \eta(X)Y]. \tag{4.3}$$

From the equations (4.2) and (4.3), it follows that

$$\left[(\alpha^2+\alpha) - \frac{r+\alpha\psi^2}{n(n-1)}\right][\eta(Y)X - \eta(X)Y] = 0. \tag{4.4}$$

Taking $Y = \xi$ in (4.4) and using (2.1), we have

$$\left[(\alpha^2+\alpha) - \frac{r+\alpha\psi^2}{n(n-1)}\right][X + \eta(X)\xi] = 0. \tag{4.5}$$

Now taking inner product of (4.5) with U, we find

$$\left[(\alpha^2+\alpha) - \frac{r+\alpha\psi^2}{n(n-1)}\right][g(X,U) + \eta(X)\eta(U)] = 0. \tag{4.6}$$

By replacing X by QX in (4.6) and using the fact that $S(X,U) = g(QX,U)$, we obtain

$$\left[(\alpha^2+\alpha) - \frac{r+\alpha\psi^2}{n(n-1)}\right][S(X,U) + \alpha^2(n-1)\eta(X)\eta(U)] = 0.$$

This implies that either the scalar curvature of M is $n(n-1)(\alpha^2+\alpha)-\alpha\psi^2$ or

$$S(X,U) = -\alpha^2(n-1)\eta(X)\eta(U).$$

Hence, we can state the following theorem.

Theorem 4.2. *For a ξ-concircularly flat Lorentzian α-Sasakian manifold with respect to the quarter-symmetric non-metric connection, either the scalar curvature is $n(n-1)(\alpha^2+\alpha)-\alpha\psi^2$ or the manifold is a special type of η-Einstein manifold with respect to the Levi-Civita connection.*

5. Quasi-concircularly flat Lorentzian α-Sasakian manifolds with respect to the quarter-symmetric non-metric connection

Definition 5.1. A Lorentzian α-Sasakian manifold with respect to the quarter-symmetric non-metric connection is said to be quasi-concircularly flat if

$$g[\bar{C}(X,Y)Z,\phi W] = 0 \tag{5.1}$$

for all X, Y, Z, W on M.

Let M be an n-dimensional quasi-concircularly flat Lorentzian α-Sasakian manifold with respect to the quarter-symmetric non-metric connection. Therefore, from (4.1) and (5.1), it follows that

$$g[\bar{R}(X,Y)Z,\phi W] = \frac{\bar{r}}{n(n-1)}[g(Y,Z)g(X,\phi W) - g(X,Z)g(Y,\phi W)],$$

which by using (3.2) takes the form

$$\begin{aligned} g[R(X,Y)Z,\phi W] = &-\alpha[g(\phi Y,Z)g(\phi X,\phi W) \\ &- g(\phi X,Z)g(\phi Y,\phi W)] \\ &- \alpha[\eta(X)g(Y,\phi W) - \eta(Y)g(X,\phi W)]\eta(Z) \\ &+ \frac{\bar{r}}{n(n-1)}[g(Y,Z)g(X,\phi W) - g(X,Z)g(Y,\phi W)]. \end{aligned} \tag{5.2}$$

Let $\{e_1, e_2, ..., e_{n-1}, \xi\}$ be a local orthonormal basis of vector fields in M. Then $\{\phi e_1, \phi e_2, ..., \phi e_{n-1}, \xi\}$ is also a local orthonormal basis. If we put

$X = \phi e_i$ and $W = e_i$ in (5.2) and sum up with respect to i, then we have

$$\begin{aligned}\sum_{i=1}^{n-1} g[R(\phi e_i, Y)Z, \phi e_i] = -\alpha \sum_{i=1}^{n-1}[g(\phi Y, Z)g(\phi^2 e_i, \phi e_i)\\ - g(\phi^2 e_i, Z)g(\phi Y, \phi e_i)]\\ - \alpha \sum_{i=1}^{n-1}[\eta(\phi e_i)g(Y, \phi e_i) - \eta(Y)g(\phi e_i, \phi e_i)]\eta(Z)\\ + \frac{\bar{r}}{n(n-1)} \sum_{i=1}^{n-1}[g(Y, Z)g(\phi e_i, \phi e_i) - g(\phi e_i, Z)g(Y, \phi e_i)].\end{aligned} \tag{5.3}$$

It can be easily verified that

$$\sum_{i=1}^{n-1} g[R(\phi e_i, Y)Z, \phi e_i] = S(Y, Z) - \alpha^2[g(Y, Z) + \eta(Y)\eta(Z)],$$

$$\sum_{i=1}^{n-1} g(\phi^2 e_i, \phi e_i) = \psi, \quad \sum_{i=1}^{n-1} g(\phi e_i, \phi e_i) = (n-1), \tag{5.4}$$

$$\sum_{i=1}^{n-1} \eta(\phi e_i)g(Y, \phi e_i) = 0,$$

$$\sum_{i=1}^{n-1} g(\phi^2 e_i, Z)g(\phi Y, \phi e_i) = g(Y, Z) + \eta(Y)\eta(Z),$$

$$\sum_{i=1}^{n-1} g(\phi e_i, Z)g(Y, \phi e_i) = g(Y, Z) + \eta(Y)\eta(Z).$$

Thus, by virtue of (3.4), the equation (5.3) takes the form

$$\begin{aligned}S(Y, Z) = \left[\alpha^2 + \alpha + \frac{(r + \alpha\psi^2)(n-2)}{n(n-1)}\right] g(Y, Z)\\ + \left[\alpha^2 + n\alpha - \frac{r + \alpha\psi^2}{n(n-1)}\right] \eta(Y)\eta(Z) - \alpha\psi g(\phi Y, Z),\end{aligned}$$

where $\psi = \text{trace } \phi$. Thus we can state the following theorem.

Theorem 5.2. *A quasi-concircularly flat Lorentzian α-Sasakian manifold with respect to the quarter-symmetric non-metric connection is a generalized η-Einstein manifold with respect to the Levi-Civita connection.*

6. Pseudoconcircularly flat Lorentzian α-Sasakian manifolds with respect to the quarter-symmetric non-metric connection

Definition 6.1. A Lorentzian α-Sasakian manifold with respect to the quarter-symmetric non-metric connection is said to be pseudoconcircularly flat if

$$g[\bar{C}(\phi X, Y)Z, \phi W] = 0 \tag{6.1}$$

for all X, Y, Z, W on M.

Let M be an n-dimensional pseudoconcircularly flat Lorentzian α-Sasakian manifold with respect to the quarter-symmetric non-metric connection. Therefore, from (4.1) and (6.1), it follows that

$$g[\bar{R}(\phi X, Y)Z, \phi W] = \frac{\bar{r}}{n(n-1)}[g(Y, Z)g(\phi X, \phi W) - g(\phi X, Z)g(Y, \phi W)],$$

which in view of (3.2) takes the form

$$\begin{aligned} g[R(\phi X, Y)Z, \phi W] &= \alpha[g(X, Z)g(\phi Y, \phi W) + \eta(X)\eta(Z)g(\phi Y, \phi W) \\ &\quad - g(\phi Y, Z)g(X, \phi W) + \eta(Y)\eta(Z)g(\phi X, \phi W)] \\ &\quad + \frac{\bar{r}}{n(n-1)}[g(Y, Z)g(\phi X, \phi W) - g(\phi X, Z)g(Y, \phi W)]. \end{aligned} \tag{6.2}$$

In view of (2.2), (6.2) becomes

$$\begin{aligned} g[R(\phi X, Y)Z, \phi W] &= \alpha[g(X, Z)g(Y, W) + g(X, Z)\eta(Y)\eta(W) \\ &\quad + g(Y, W)\eta(X)\eta(Z) + 2\eta(X)\eta(Z)\eta(Y)\eta(W) \\ &\quad - g(\phi Y, Z)g(X, \phi W) + g(X, W)\eta(Y)\eta(Z)] \\ &\quad + \frac{\bar{r}}{n(n-1)}[g(Y, Z)g(X, W) + g(Y, Z)\eta(X)\eta(W) \\ &\quad - g(\phi X, Z)g(Y, \phi W)]. \end{aligned} \tag{6.3}$$

Let $\{e_1, e_2, ..., e_{n-1}, \xi\}$ be a local orthonormal basis of vector fields in M. Then $\{\phi e_1, \phi e_2, ..., \phi e_{n-1}, \xi\}$ is also a local orthonormal basis. If we put $X = W = e_i$ in (6.3) and sum up with respect to i, then we have

$$\begin{aligned} \sum_{i=1}^{n-1} g[R(\phi e_i, Y)Z, \phi e_i] &= \alpha \sum_{i=1}^{n-1} [g(e_i, Z)g(Y, e_i) + g(e_i, Z)\eta(Y)\eta(e_i) \\ &\quad + g(Y, e_i)\eta(e_i)\eta(Z) + 2\eta(e_i)\eta(e_i)\eta(Z)\eta(Y) - g(e_i, \phi e_i)g(\phi Y, Z) \\ &\quad + g(e_i, e_i)\eta(Y)\eta(Z)] + \frac{\bar{r}}{n(n-1)} \sum_{i=1}^{n-1} [g(Y, Z)g(e_i, e_i) \\ &\quad + g(Y, Z)\eta(e_i)\eta(e_i) - g(\phi e_i, Z)g(Y, \phi e_i)]. \end{aligned} \tag{6.4}$$

It can be easily verified that

$$\sum_{i=1}^{n-1} g[R(\phi e_i, Y)Z, \phi e_i] = S(Y,Z) - \alpha^2[g(Y,Z) + \eta(Y)\eta(Z)], \tag{6.5}$$

$$\sum_{i=1}^{n-1} g(e_i, Z)g(Y, e_i) = \sum_{i=1}^{n-1} g(\phi e_i, Z)g(Y, \phi e_i) = g(Y,Z) + \eta(Y)\eta(Z), \tag{6.6}$$

$$\sum_{i=1}^{n-1} g(e_i, e_i) = (n-1). \tag{6.7}$$

By virtue of (3.4) and (6.5)–(6.7), the equation (6.4) takes the form

$$\begin{aligned} S(Y,Z) = {} & [\alpha^2 + \alpha + \frac{(r + \alpha\psi^2)(n-2)}{n(n-1)}]g(Y,Z) \\ & + [\alpha^2 + \alpha n - \frac{r + \alpha\psi^2}{n(n-1)}]\eta(Y)\eta(Z) - \alpha\psi g(\phi Y, Z), \end{aligned}$$

where $\psi = \text{trace } \phi$. Thus we can state the following theorem.

Theorem 6.2. *A pseudoconcircularly flat Lorentzian α-Sasakian manifold with respect to the quarter-symmetric non-metric connection is a generalized η-Einstein manifold with respect to the Levi-Civita connection.*

7. ϕ-Concircularly flat Lorentzian α-Sasakian manifolds with respect to the quarter-symmetric non-metric connection

Definition 7.1. A Lorentzian α-Sasakian manifold with respect to the quarter-symmetric non-metric connection is said to be ϕ-concircularly flat if (see [9])

$$\phi^2 \bar{C}(\phi X, \phi Y)\phi Z = 0 \tag{7.1}$$

for all X, Y, Z on M.

Let M be an n-dimensional ϕ-concircularly flat Lorentzian α-Sasakian manifold with respect to the quarter-symmetric non-metric connection. Then from (7.1) it follows that

$$g(\bar{C}(\phi X, \phi Y)\phi Z, \phi W) = 0. \tag{7.2}$$

In view of (4.1), (7.2) becomes

$$\begin{aligned} g[\bar{R}(\phi X, \phi Y)\phi Z, \phi W] = {} & \frac{\bar{r}}{n(n-1)}[g(\phi Y, \phi Z)g(\phi X, \phi W) \\ & - g(\phi X, \phi Z)g(\phi Y, \phi W)], \end{aligned}$$

which in view of (3.2) takes the form

$$g\left[R(\phi X,\phi Y)\phi Z,\phi W\right]=\alpha[g(X,\phi Z)g(Y,\phi W)-g(Y,\phi Z)g(X,\phi W)] + \frac{\bar{r}}{n(n-1)}\left[g(\phi Y,\phi Z)g(\phi X,\phi W)-g(\phi X,\phi Z)g(\phi Y,\phi W)\right]. \tag{7.3}$$

Let $\{e_1, e_2, ..., e_{n-1}, \xi\}$ be a local orthonormal basis of vector fields in M. Then $\{\phi e_1, \phi e_2, ..., \phi e_{n-1}, \xi\}$ is also a local orthonormal basis. If we put $X = W = e_i$ in (7.4) and sum up with respect to i, then we have

$$\sum_{i=1}^{n-1} g[R(\phi e_i,\phi Y)\phi Z,\phi e_i]=\alpha\sum_{i=1}^{n-1}[g(e_i,\phi Z)g(Y,\phi e_i) -g(Y,\phi Z)g(e_i,\phi e_i)] + \frac{\bar{r}}{n(n-1)}\sum_{i=1}^{n-1}\left[g(\phi Y,\phi Z)g(\phi e_i,\phi e_i)-g(\phi e_i,\phi Z)g(\phi Y,\phi e_i)\right]. \tag{7.4}$$

It can be easily verified that

$$\sum_{i=1}^{n-1} g[R(\phi e_i,\phi Y)\phi Z,\phi e_i]=S(\phi Y,\phi Z)-\alpha^2 g(\phi Y,\phi Z), \tag{7.5}$$

$$\sum_{i=1}^{n-1} g(\phi e_i,\phi Z)g(\phi Y,\phi e_i)=g(\phi Y,\phi Z). \tag{7.6}$$

By virtue of (5.4), (6.6), (7.5), and (7.6), the equation (7.4) gives

$$S(\phi Y,\phi Z)=\left[\frac{\bar{r}(n-2)}{n(n-1)}+\alpha^2\right]g(\phi Y,\phi Z)+\alpha g(Y,Z) + \alpha\eta(Y)\eta(Z)-\alpha g(Y,\phi Z)\psi$$

which in view of (2.2), (2.5), and (3.4) turns to

$$S(Y,Z)=\left[\frac{(r+\alpha\psi^2)(n-2)}{n(n-1)}-(n-2)\alpha^2+\alpha\right]g(Y,Z) + \left[\frac{(r+\alpha\psi^2)(n-2)}{n(n-1)}+\alpha^2+\alpha\right]\eta(Y)\eta(Z)-\alpha g(Y,\phi Z)\psi,$$

where $\psi = \text{trace}\ \phi$. Thus we can state the following theorem.

Theorem 7.2. *A ϕ-concircularly flat Lorentzian α-Sasakian manifold with respect to the quarter-symmetric non-metric connection is a generalized η-Einstein manifold with respect to the Levi-Civita connection.*

8. ϕ-Concircularly semisymmetric Lorentzian α-Sasakian manifolds with respect to the quarter-symmetric non-metric connection

Definition 8.1. A Lorentzian α-Sasakian manifold with respect to the quarter-symmetric non-metric connection is said to be ϕ-concircularly semisymmetric if (see [5])

$$\bar{C}(X,Y)\cdot\phi = 0$$

for all X, Y on M.

Let M be an n-dimensional ϕ-concircularly semisymmetric Lorentzian α-Sasakian manifold with respect to the quarter-symmetric non-metric connection. Therefore $\bar{C}(X,Y)\cdot\phi = 0$ turns into

$$(\bar{C}(X,Y)\cdot\phi)Z = \bar{C}(X,Y)\phi Z - \phi\bar{C}(X,Y)Z = 0,$$

which, by $X = \xi$, takes the form

$$(\bar{C}(\xi,Y)\cdot\phi)Z = \bar{C}(\xi,Y)\phi Z - \phi\bar{C}(\xi,Y)Z = 0. \tag{8.1}$$

Taking $X = \xi$ in (4.1) and using (2.1), we have

$$\bar{C}(\xi,Y)Z = \bar{R}(\xi,Y)Z - \frac{\bar{r}}{n(n-1)}[g(Y,Z)\xi - \eta(Z)Y].$$

In view of (3.6), it becomes

$$\begin{aligned}\bar{C}(\xi,Y)Z = &\left[\alpha^2 - \frac{\bar{r}}{n(n-1)}\right] g(Y,Z)\xi \\ &- \left[(\alpha^2+\alpha) - \frac{\bar{r}}{n(n-1)}\right]\eta(Z)Y - \alpha\eta(Y)\eta(Z)\xi\end{aligned} \tag{8.2}$$

from which we get

$$\phi\bar{C}(\xi,Y)Z = -[(\alpha^2+\alpha) - \frac{\bar{r}}{n(n-1)}]\eta(Z)\phi Y. \tag{8.3}$$

Replacing Z by ϕZ in (8.2) and using (2.1), we get

$$\bar{C}(\xi,Y)\phi Z = \left[\alpha^2 - \frac{\bar{r}}{n(n-1)}\right] g(Y,\phi Z)\xi. \tag{8.4}$$

Therefore, combining (8.1), (8.3), and (8.4), we have

$$\left[\alpha^2 - \frac{\bar{r}}{n(n-1)}\right] g(Y,\phi Z)\xi + \left[(\alpha^2+\alpha) - \frac{\bar{r}}{n(n-1)}\right]\eta(Z)\phi Y = 0. \tag{8.5}$$

Taking inner product of (8.5) with ξ and using (2.1), we find

$$\left[\alpha^2 - \frac{\bar{r}}{n(n-1)}\right] g(Y, \phi Z) = 0.$$

Since $g(X, \phi Z) \neq 0$, we get

$$\bar{r} = n(n-1)\alpha^2 \qquad \Longrightarrow \qquad r = n(n-1)\alpha^2 - \alpha\psi^2.$$

Thus we can state the following theorem.

Theorem 8.2. *For an n-dimensional ϕ-concircularly semisymmetric Lorentzian α-Sasakian manifold with respect to the quarter-symmetric non-metric connection, the scalar curvature with respect to the Levi-Civita connection is $n(n-1)\alpha^2 - \alpha\psi^2$.*

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