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Simplifying coefficients in a family of nonlinear ordinary differential equations

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Abstract. By virtue of the Faá di Bruno formula, properties of the Stirling numbers and the Bell polynomials of the second kind, the binomial inversion formula, and other techniques in combinatorial analysis, the author finds a simple, meaningful, and significant expression for coefficients in a family of nonlinear ordinary differential equations.

1. Motivation and main results

In [6, Theorem 2.1], it was established inductively and recursively that the function

$$F(t) = \frac{1}{e^{\ln(1+t)} - 1} \tag{1}$$

satisfies the family of differential functions

$$n!F^{n+1}(t) = (-1)^n \sum_{k=0}^{n} a_k(n)(1+t)^k F^{(k)}(t) \tag{2}$$

for $n \in \mathbb{N}$, where $a_0(n) = n!$, $a_n(n) = 1$, and

$$a_k(n) = \sum_{n_1=k-1}^{n-1} \sum_{n_2=k-2}^{n_1-1} \cdots \sum_{n_k=0}^{n_{k-1}-1} \frac{(n+k)!}{\prod_{\ell=1}^{k}(n_\ell + k - \ell + 2)(n_\ell + k - \ell + 1)} \tag{3}$$

for $0 < k < n$. Hereafter, the expression (3) was employed in the whole paper [6].

In this paper, since

(1) the original proof of [6, Theorem 2.1] is long and tedious,

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(2) the expression (3) is too complex to be remembered, understood, and computed easily,

we will supply a simple and standard proof for [6, Theorem 2.1] and, more importantly, find a simple, meaningful, and significant form for the quantities $a_k(n)$.

Our main results can be stated as the following theorem.

Theorem 1. *For $n \in \{0\} \cup \mathbb{N}$, the function $F(t)$ defined by* (1) *satisfies*

$$F^{(n)}(t) = (-1)^n \frac{n!}{(1+t)^n} \sum_{m=0}^{n} \binom{n}{m} F^{m+1}(t) \tag{4}$$

and

$$F^{n+1}(t) = (-1)^n \sum_{m=0}^{n} \binom{n}{m} \frac{1}{m!} (1+t)^m F^{(m)}(t). \tag{5}$$

2. Proof of Theorem 1

By virtue of the Faà di Bruno formula

$$\frac{\mathrm{d}^n}{\mathrm{d}\, t^n} f \circ h(t) = \sum_{k=0}^{n} f^{(k)}(h(t))\, \mathrm{B}_{n,k}\big(h'(t), h''(t), \dots, h^{(n-k+1)}(t)\big)$$

for $n \ge 0$, where the Bell polynomials of the second kind $\mathrm{B}_{n,k}(x_1, \dots, x_{n-k+1})$ for $n \ge k \ge 0$ are defined [2, p. 134, Theorem A] and [2, p. 139, Theorem C] by

$$\mathrm{B}_{n,k}(x_1, x_2, \dots, x_{n-k+1}) = \sum_{\substack{1 \le i \le n, \ell_i \in \{0\} \cup \mathbb{N} \\ \sum_{i=1}^{n} i\ell_i = n \\ \sum_{i=1}^{n} \ell_i = k}} \frac{n!}{\prod_{i=1}^{n-k+1} \ell_i!} \prod_{i=1}^{n-k+1} \left(\frac{x_i}{i!}\right)^{\ell_i},$$

we obtain

$$F^{(n)}(t) = \sum_{k=0}^{n} \frac{\mathrm{d}^k}{\mathrm{d}\, u^k}\left(\frac{1}{e^u - 1}\right) \mathrm{B}_{n,k}\left(\frac{1}{1+t}, -\frac{1}{(1+t)^2}, \dots, \frac{(-1)^{n-k}(n-k)!}{(1+t)^{n-k+1}}\right)$$

where $u = u(t) = \ln(1+t)$. In view of two identities

$$\mathrm{B}_{n,k}\big(abx_1, ab^2x_2, \dots, ab^{n-k+1}x_{n-k+1}\big) = a^k b^n\, \mathrm{B}_{n,k}(x_1, x_2, \dots, x_{n-k+1})$$

and

$$\mathrm{B}_{n,k}(0!, 1!, 2!, \dots, (n-k)!) = (-1)^{n-k} s(n,k)$$

in [2, p. 135], where a and b are any complex numbers and $s(n,k)$ stands for the Stirling number of the first kind, we have

$$F^{(n)}(t) = \sum_{k=0}^{n} \frac{\mathrm{d}^k}{\mathrm{d}\, u^k}\left(\frac{1}{e^u - 1}\right)\left(\frac{1}{1+t}\right)^n (-1)^{n+k}\, \mathrm{B}_{n,k}(0!, 1!, \dots, (n-k)!)$$

$$= \frac{1}{(1+t)^n} \sum_{k=0}^{n} \frac{\mathrm{d}^k}{\mathrm{d}\,u^k}\left(\frac{1}{e^u-1}\right) s(n,k).$$

From [4, Theorem 2.1], [9, Theorems 3.1 and 3.2], and [10, Lemma 2.1], it follows that

$$\frac{\mathrm{d}^k}{\mathrm{d}\,t^k}\left(\frac{1}{e^t-1}\right) = (-1)^k \sum_{m=1}^{k+1} (m-1)! S(k+1,m)\left(\frac{1}{e^t-1}\right)^m, \quad k \ge 0,$$

where $S(n,k)$ stands for the Stirling number of the second kind. Therefore, we acquire

$$\begin{aligned} F^{(n)}(t) &= \frac{1}{(1+t)^n} \sum_{k=0}^{n} (-1)^k \left[\sum_{m=1}^{k+1} (m-1)! S(k+1,m)\left(\frac{1}{e^u-1}\right)^m\right] s(n,k) \\ &= \frac{1}{(1+t)^n} \sum_{m=1}^{n+1} \left[\sum_{k=m-1}^{n} (-1)^k s(n,k) S(k+1,m)\right] (m-1)! \left(\frac{1}{e^u-1}\right)^m \\ &= \frac{1}{(1+t)^n} \sum_{m=0}^{n} \left[\sum_{k=m}^{n} (-1)^k s(n,k) S(k+1,m+1)\right] m! \left(\frac{1}{e^u-1}\right)^{m+1} \\ &= \frac{1}{(1+t)^n} \sum_{m=0}^{n} \left[\sum_{k=m}^{n} (-1)^k s(n,k) S(k+1,m+1)\right] m! F^{m+1}(t). \end{aligned}$$

Since

$$S(n+1,k) = kS(n,k) + S(n,k-1), \quad 0 \le k-1 \le n,$$

see [8, p. 114, Eq. (9.1)], we have

$$\begin{aligned} &\sum_{k=m}^{n} (-1)^k s(n,k) S(k+1,m+1) \\ &\qquad = (m+1) \sum_{k=m}^{n} (-1)^k s(n,k) S(k,m+1) + \sum_{k=m}^{n} (-1)^k s(n,k) S(k,m). \end{aligned}$$

Further utilizing the formula

$$\sum_{k=m}^{n} (-1)^k s(n,k) S(k,m) = (-1)^n L(n,m)$$

in [1, pp. 304–305, Remark 8.6] and simplifying yield

$$\sum_{k=m}^{n} (-1)^k s(n,k) S(k+1,m+1) = (-1)^n [(m+1) L(n,m+1) + L(n,m)],$$

where $L(n,k)$ stands for the Lah number which can be defined [5] by

$$L(n,k)=\frac{(n-1)!}{(k-1)!}\binom{n}{k}$$

with the conventions $L(0,0)=1$, $L(n,0)=0$ if $n\ge1$, and $L(n,k)=0$ if $k>n$. Consequently, it follows that

$$F^{(n)}(t)=\frac{(-1)^n}{(1+t)^n}\sum_{m=0}^{n}[(m+1)L(n,m+1)+L(n,m)]m!F^{m+1}(t).$$

Making use of

$$L(n,k+1)=\frac{n-k}{k(k+1)}L(n,k)$$

reveals

$$[(m+1)L(n,m+1)+L(n,m)]m!=(m-1)!nL(n,m)=n!\binom{n}{m}.$$

Accordingly, we obtain the equality (4).

The formula (5.48) in [3, p. 192] reads that

$$g(k)=\sum_{\ell=0}^{k}\binom{k}{\ell}(-1)^{\ell}f(\ell)\iff f(k)=\sum_{\ell=0}^{k}\binom{k}{\ell}(-1)^{\ell}g(\ell).$$

Applying this binomial inversion formula to (4) results in

$$(-1)^nF^{n+1}(t)=\sum_{m=0}^{n}\binom{n}{m}(-1)^m(-1)^m\frac{(1+t)^m}{m!}F^{(m)}(t)$$

which can be rewritten as (5). The proof of Theorem 1 is complete.

3. Remarks

Finally, we list several remarks on our main results and closely related things.

Remark 1. It is clear that the function $F(t)$ defined by (1) is $\frac{1}{t}$.

Remark 2. Comparing (2) with (5) reveals that

$$a_k(n)=\frac{n!}{k!}\binom{n}{k},\quad n\ge k\ge0,$$

which is simpler, more meaningful, and more significant than the expression (3).

Remark 3. This paper is a slightly modified version of the preprint [7].

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