Behaviour of multivariate tail dependence coefficients

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Abstract. In applications tail dependence is an important property of a copula. Bivariate tail dependence is investigated in many papers, but multivariate tail dependence has not been studied widely. We define multivariate upper and lower tail dependence coefficients as limits of the probability that values of one marginal will be large if at least one of other marginals will be as large also. Further we derive some relations between introduced tail dependence and bivariate tail dependence coefficients. Applications have shown that the multivariate t-copula has been successfully used in practice because of it's tail dependence property. Therefore, t-copula can be used as an alternative method for risk assessment under Solvency II for insurance models. We have paid attention to the properties of the introduced multivariate tail dependence coefficient for t-copula and examine it in the simulation experiment.

1. Introduction

In applications an important property of a copula is tail dependence. Tail dependence coefficients characterize the degree of dependence between marginals in the tail area. It becomes very important in risk measuring. Bivariate tail dependence is investigated in many papers because of its importance in applications. It gives an answer to the question: will large values of one variable increase the probability that another variable will be as large either. Bivariate tail dependence refers to the degree of dependence in the corner of the lower-left quadrant or upper-right quadrant of a bivariate distribution. An answer to this question is characterized mathematically by the limit of the conditional probability $P(Y > x | X > x)$ when x tends to infinity. This limit is the characteristic of upper tail dependence. In applications

very important are multivariate copulas constructed from multivariate distributions. The Gaussian copula does not have the upper tail dependence as the described above limit equals to zero when the Pearson correlation coefficient is smaller than one, Cherubini et al. [5]. This property makes Gaussian copula not suitable for many applications. The same time t-copula takes into account tail dependence when using it as a data model, Cherubini et al. [5]. In applications usually calculation of tail dependence coefficients is realised using copulas. At the same time there are only few skewed copulas in use. Archimedian copulas are axial symmetric by construction, the copulas built via elliptical multivariate distributions are also symmetric (Gaussian and t-copula, for instance). It seems natural to join into a multivariate distribution skewed marginals by a skewed copula density. Skew elliptical families give a good possibility for that. In Demarta and McNeil [7] a skew t-copula is introduced which is based on the multivariate generalized hyperbolic distribution. In Kollo and Pettere [18] the skew t-copula was defined on the basis of multivariate skew t-distribution following Azzalini and Capitanio [1]. In this paper we study tail dependence for n-dimensional t-copula and leave investigation of skew-elliptical copulas for future.

In problems of financial mathematics multivariate tail dependence is even more important than bivariate tail dependence. The main reason is that in financial problems a variable of interest usually depends on behaviour of several other random variables. And it is not possible to describe multivariate dependence as a set of bivariate tail dependence coefficients. More important are the questions: how will behave a random variable of interest if some other variables are larger than a given value; or more generally, how will behave a given variable if at least one of other variables is larger than a certain given value.

2. Tail dependence coefficients

Bivariate lower and upper tail dependence coefficients are defined and expressed via copulas (see [10, 14, 20]). Let X and Y be random variables. Then the lower tail dependence coefficient is

$$\lambda_L(X, Y) = \lim_{u \to 0} P(F_Y(Y) \leq u | F_X(X) \leq u) = \lim_{u \to 0} \frac{C(u, u)}{u}$$

and the upper tail dependence coefficient is

$$\lambda_U(X, Y) = \lim_{u \to 1} P(F_Y(Y) > u | F_X(X) > u) = \lim_{u \to 1} \frac{1 - 2u + C(u, u)}{1 - u}.$$

Here F_X and F_Y are distribution functions of the marginals X and Y, and C is the copula of X and Y. For notions and results on copula theory an interested reader is referred to Nelsen [20] or Cherubini et al. [5].

Because bivariate tail dependence coefficients can be calculated using copulas most often they have been used for Archimedean copulas. For symmetric copulas upper and lower tail dependence coefficients are equal, therefore we shall not use indexes L and U for t-copula later on. In Demarta and McNeil [7] a formula for tail dependence coefficient for t-copula is given in the bivariate case:

$$\lambda = 2T(-\sqrt{\nu+1}\sqrt{1-r} \,/\, \sqrt{1+r}, \nu+1), \qquad (2.1)$$

where $T(\cdot, \nu+1)$ is the distribution function of the univariate t-distribution with $\nu+1$ degrees of freedom.

Bortot [2] has studied tail dependence of the bivariate skew normal and skew t-distributions and derived inequalities for the tail dependence coefficients of these copulas. In more details tail behavior of the skew t-copula has been examined by simulation in Kollo et al. [19]. Because of interest to multivariate distributions and corresponding to them copulas in applications a question about tail dependence for multivariate distributions has raised. But this has not been studied widely. We have found several studies on multivariate tail dependence but they do not give answers to the questions formulated in the end of Introduction. For example, Frahm [12] has defined lower and upper tail dependence coefficients for a d-dimensional random vector with distribution function F in the following way:

$$\varepsilon_L = \lim_{t \to 0} P(F_{max} \le t | F_{min} \le t), \quad \varepsilon_U = \lim_{t \to 0} P(F_{min} > t | F_{max} > t), \quad (2.2)$$

where $F_1, F_2, \ldots, F_d$ are marginal distributions,

$$F_{min} = \min\{F_1(X_1), F_2(X_2), \ldots, F_d(X_d)\},$$

and

$$F_{max} = \max\{F_1(X_1), F_2(X_2), \ldots, F_d(X_d)\}.$$

He has expressed the tail dependence coefficients (2.2) through copulas

$$\varepsilon_L = \lim_{u \to 0^+} \frac{P(F_1(X_1) \le u, \ldots, F_d(X_d) \le u)}{1 - P(F_1(X_1) > u, \ldots, F_d(X_d) > u)}$$
$$= \lim_{u \to 0^+} \frac{C(u, \ldots, u)}{1 - \overline{C}(1-u, \ldots, 1-u)}$$

and

$$\varepsilon_U = \lim_{u \to 1^-} \frac{P(F_1(X_1) > u, \ldots, F_d(X_d) > u)}{1 - P(F_1(X_1) \le u, \ldots, F_d(X_d) \le u)}$$
$$= \lim_{u \to 1^-} \frac{\overline{C}(1-u, \ldots, 1-u)}{1 - C(u, \ldots, u)},$$

where C is copula of random vector $\mathbf{X}$ and $\overline{C}$ is the survival copula corresponding to C.

Additionally he has found explicit expressions of tail dependence coefficients for t-distribution and a certain class of elliptically distributed distributions. Later Chan and Li [3] defined extreme dependence index for vector $\mathbf{X}$ in the following way:

$$
\begin{aligned}
\gamma = \lim_{u \to 1} P\{X_j(X_j) > u, \ \ j \in \{1, 2, \ldots, n\}| \\
F_i(X_i) > u \ \text{ for some } \ i \in \{1, 2, \ldots, n\}\}.
\end{aligned} \tag{2.3}
$$

The authors have found explicit expressions of the index for the multivariate t-distributions and have investigated their monotonicity properties. Kluppelberg et al. [17] have proposed a semi-parametric model for (asymptotically dependent) tail dependence functions for an elliptical copula. Under this model assumption they have created a novel estimator for the tail dependence function, which has been examined both theoretically and empirically.

De Luca and Rivieccio [6] have studied multivariate tail dependence coefficient (2.3) in the case of Clayton copula, but estimated tail dependence coefficient only in three dimensional case. Bernardino [8] has estimated tail dependence coefficients (2.3) for transformed multivariate Archimedian copulas. Several important facts about tail dependence and Archimedian copulas can be found in Juri and Wuthrich [16], Charpentier and Segers [4], Joe et al. [15], and Hua and Joe [13]. Flores [11] introduced monotonic copulas. Embrecht et al. [9] have studied properties of tail dependence coefficients matrix in multivariate case.

But all these definitions do not give an answer to the question about tail behaviour of one variable if at least one other variable becomes larger than some critical value. Therefore the aim of this paper is to introduce a different measure of tail dependence for n-dimensional copulas. We define multivariate upper and lower tail dependence coefficients as limits of the probability that values of one marginal will be large if at least one from other marginals will be as large too. Such approach gives more flexibility for describing tail behaviour of multivariate distributions.

3. Definitions of multivariate tail dependence coefficients

First we introduce lower and upper tail dependence coefficients in multivariate case in the following intuitive way:

$$
\lambda_U(X_i) = \lim_{u \to 1} \lambda_U(u, X_i),
$$

where $\lambda_U(u, X_i)$ is a sum of all possible conditional probabilities that given variable will be larger than given percentile if at least one other variable is larger too, $i \in \{1, 2, \ldots, n\}$, and

$$
\lambda_L(X_i) = \lim_{u \to 0} \lambda_L(u, X_i),
$$

where $\lambda_L(u, X_i)$ is a sum of all possible conditional probabilities when given variable will be smaller than given percentile if at least one other variable is smaller, too.

Such coefficient shows how sensitive is given variable to increases or decreases of some other variables.

Before we introduce the main notion we define necessary additional coefficients. Let $\mathbf{X} = \{X_1, X_2, \ldots, X_n\}$ be n-dimensional multivariate vector with marginals distribution functions F_i.

Definition 1. The one-variable upper tail dependence coefficient is defined by

$$\lambda_U^1(X_i) = \sum_{m=1, m\neq i}^{n} \lim_{u\to 1} P\left(F_i(x) > u | F_m(x) > u, \bigcap_{k=1, k\neq m\neq i}^{n} F_k(x) < u \right)$$

or

$$\lambda_U^1(X_i) = \sum_{m=1, m\neq i}^{n} \lim_{u\to 1} \frac{P\left(F_i(x) > u, F_m(x) > u, \bigcap_{k=1, k\neq m\neq i}^{n} F_k(x) < u \right)}{P\left(F_m(x) > u, \bigcap_{k=1, k\neq m\neq i_j}^{n} F_k(x) < u \right)},$$
$$i, k, m \in \{1, 2, \ldots, n\}.$$
(3.1)

This coefficient expresses the probability that one variable is greater than a constant, if one other variable is greater than the constant and all other variables are less than this constant.

Definition 2. The l-variables upper tail dependence coefficient is

$$\lambda_U^l(X_i) = \frac{1}{l!} \sum_{\substack{m_l=1, \\ m_l\neq i,\ldots,m_1\neq i}}^{n} \cdots \sum_{\substack{m_2=1, \\ m_2\neq i, m_1\neq i}}^{n} \sum_{\substack{m_1=1, \\ m_1\neq i}}^{n} \lim_{u\to 1} P\Bigg(F_i(x) > u |$$
$$F_{m_1}(x) > u, \ldots, F_{m_l}(x) > u, \bigcap_{\substack{k=1, k\neq i \\ k\neq m_1, \ldots, k\neq m_l}}^{n} F_k(x) < u \Bigg)$$

or

$$\lambda_U^l(X_i) = \frac{1}{l!} \sum_{\substack{m_l=1, \\ m_l \neq i, \ldots, m_1 \neq i}}^{n} \cdots \sum_{\substack{m_2=1, \\ m_2 \neq i, m_1 \neq i}}^{n} \sum_{\substack{m_1=1, \\ m_1 \neq i}}^{n} \lim_{u \to 1} \left[P \left(F_i(x) > u \right| \right.$$

$$\left. F_{m_1}(x) > u, \ldots, F_{m_l}(x) > u, \bigcap_{\substack{k=1, k \neq i \\ k \neq m_1, \ldots, k \neq m_l}}^{n} F_k(x) < u \right) \quad (3.2)$$

$$\left/ P \left(F_{m_1}(x) > u, \ldots, F_{m_l}(x) > u, \bigcap_{\substack{k=1, k \neq i \\ k \neq m_1, \ldots, k \neq m_l}}^{n} F_k(x) < u \right) \right],$$

$$2 \leq l \leq n - 2, \quad m_j, k, i \in \{1, 2, \ldots, n\}, \quad j \in \{1, 2, \ldots, l - 1\}.$$

These coefficients are describing tail behaviour of a given variable X_i if directly l variables, $l \in \{2, 3, \ldots, n-2\}$, are larger than the given percentile.

Definition 3. The multivariate upper overall tail dependence coefficient for X_i is

$$\lambda_U^T(X_i) = \lim_{u \to 1} P \left(F_i(x) > u \right| \bigcap_{m=1, m \neq i}^{n} F_m(x) > u \right)$$

or

$$\lambda_U^T(X_i) = \lim_{u \to 1} \frac{P \left(F_i(x) > u, \bigcap_{m=1, m \neq i}^{n} F_m(x) > u \right)}{P \left(\bigcap_{m=1, m \neq i}^{n} F_m(x) > u \right)}, \quad (3.3)$$

$$i, m \in \{1, 2, \ldots, n\}.$$

This coefficient describes behaviour of a given variable if all other variables become larger than the given percentile. Finally we can define the multivariate upper tail dependence coefficient.

Definition 4. The multivariate upper tail dependence coefficient of X_i is

$$\lambda_U(X_i) = \lambda_U^1(X_i) + \sum_{l=2}^{n-1} \lambda_U^l(X_i) + \lambda_U^T(X_i), \quad i \in \{1, 2, \ldots, n\}. \quad (3.4)$$

Proposition 1. *Tail dependence coefficients are invariant under ordering of marginals.*

Proof. It is obvious because of invariance of intersection of sets. $\square$

Remark 1. Because of Proposition 1 it is possible to use only one ordering of marginals and the coefficient before sum in formula (3.2) is not needed.

In the similar way it is possible to define lower tail dependence coefficient. The formulas (3.1)–(3.4) have, for $n = 4$, the form

$$\lambda_U(X_i) = \lambda_U^1(X_i) + \lambda_U^2(X_i) + \lambda_U^T(X_i), \quad i \in \{1, 2, 3, 4\}, \qquad (3.5)$$

where

$$\lambda_U^1(X_i) = \sum_{\substack{m=1, \\ m \neq i}}^{4} \lim_{u \to 1} \frac{P\left(F_i(x) > u, F_m(x) > u, \bigcap_{k=1, k \neq m \neq i}^{n} F_k(x) < u\right)}{P\left(F_m(x) > u, \bigcap_{k=1, k \neq m \neq i}^{4} F_k(x) < u\right)}, \quad (3.6)$$
$$i, k, m \in \{1, 2, \ldots, n\},$$

$$\lambda_U^2(X_i) =$$
$$= \frac{1}{2!} \sum_{\substack{m_2=1, \\ m_2 \neq m_1 \neq i}}^{n} \sum_{\substack{m_1=1, \\ m_1 \neq i}}^{n} \lim_{u \to 1} \frac{P(F_i(x) > u, F_{m_1}(x) > u, F_{m_2}(x) > u, F_k(x) < u)}{P(F_{m_1}(x) > u, F_{m_2}(x) > u, F_k(x) < u)},$$
$$l = 2, \ m_j, k, i \in \{1, 2, 3, 4\}, \ j \in \{1, 2\}, \ k \neq m_j,$$

$$(3.7)$$

$$\lambda_U^T(X_i) = \lim_{u \to 1} \frac{P\left(F_i(x) > u, \bigcap_{m=1, m \neq i}^{4} F_m(x) > u\right)}{P\left(\bigcap_{m=1, m \neq i}^{4} F_m(x) < u\right)}, \quad (3.8)$$
$$i, m \in \{1, 2, \ldots, n\}.$$

Take $i = 1$, then because of invariance of probability of intersection, formulas (3.6)–(3.8) have the following form:

$$\lambda_U^1(X_1) = \lim_{u \to 1} \frac{P(F_1(x) > u, F_2(x) > u, F_3(x) < u, F_4(x) < u)}{P(F_2(x) > u, F_3(x) < u, F_4(x) < u)}$$
$$+ \lim_{u \to 1} \frac{P(F_1(x) > u, F_3(x) > u, F_2(x) < u, F_4(x) < u)}{P(F_3(x) > u, F_2(x) < u, F_4(x) < u)} \quad (3.9)$$
$$+ \lim_{u \to 1} \frac{P(F_1(x) > u, F_4(x) > u, F_2(x) < u, F_3(x) < u)}{P(F_4(x) > u, F_2(x) < u, F_3(x) < u)},$$

$$\lambda_U^2(X_1) = \lim_{u \to 1} \frac{P(F_1(x) > u, F_2(x) > u, F_3(x) > u, F_4(x) < u)}{P(F_2(x) > u, F_3(x) > u, F_4(x) < u)}$$
$$+ \lim_{u \to 1} \frac{P(F_1(x) > u, F_2(x) > u, F_4(x) > u, F_3(x) < u)}{P(F_2(x) > u, F_4(x) > u, F_3(x) < u)} \qquad (3.10)$$
$$+ \lim_{u \to 1} \frac{P(F_1(x) > u, F_3(x) > u, F_4(x) > u, F_2(x) < u)}{P(F_3(x) > u, F_4(x) > u, F_2(x) < u)},$$

and

$$\lambda_U^T(X_1) = \lim_{u \to 1} \frac{P(F_1(x) > u, F_2(x) > u, F_3(x) > u, F_4(x) > u)}{P(F_2(x) > u, F_3(x) > u, F_4(x) > u)}. \qquad (3.11)$$

Finally, formula (3.5) of the multivariate upper tail dependence coefficient for variable X_1 is

$$\lambda_U(X_1) = \lambda_U^1(X_1) + \lambda_U^2(X_1) + \lambda_U^T(X_1). \qquad (3.12)$$

4. Simulation

4.1. Estimation of tail dependence coefficients for different marginals. We will investigate the behaviour of the defined multivariate tail dependence coefficients for t-distribution using t-copula. Upper and lower tail dependence coefficients in this case are equal. Therefore, we shall examine only upper tail dependence coefficient. The simulation experiment in the four-dimensional case has been carried out using R software and the following simulation algorithm:

(1) Take $i = 1$.
(2) Generate four random values from t-distribution with the number of degrees of freedom 2 and the correlation matrix:

$$\begin{pmatrix} 1 & 0.8 & 0.5 & 0.2 \\ 0.8 & 1 & 0.7 & 0.4 \\ 0.5 & 0.7 & 1 & 0.2 \\ 0.2 & 0.4 & 0.2 & 1 \end{pmatrix}$$

(3) Repeat step two 10^6 times.
(4) Take $i = 1$.
(5) Calculate tail dependence coefficients for X_i, using (3.9)–(3.11) and add them together.
(6) $i := i + 1$.
(7) If $i \le 4$, repeat steps 2–5.

Results of calculated conditional probabilities are presented in Table 1. Simulation results show that coefficients are stable and they are different for each marginal if correlation is not the same. The largest tail dependence coefficient is for the second marginal with correlations 0.8, 0.7 and 0.4. The smallest tail dependence coefficient has the fourth marginal with correlations 0.2, 0.4 and 0.2.

Coefficients for probabilities $P(U_1 > u | U_2 > u, U_3 < u, U_4 < u)$ and $P(U_2 > u | U_1 > u, U_3 < u, U_4 < u)$ are not equal (1st and 9th row). This is due to different correlations: conditional marginals in the first probability expression have correlations 0.7, 0.4 and 0.2, but in the second probability expression we have correlations 0.5, 0.2 and 0.2.

TABLE 1. Estimates of tail dependence coefficients.

Calculated probabilities	No. of rows	Percentiles			
		0.900	**0.950**	**0.990**	**0.995**
$P(U_1 > u \| U_2 > u, U_3 < u, U_4 < u)$	1	0.5944	0.5499	0.5022	0.4828
$P(U_1 > u \| U_3 > u, U_2 < u, U_4 < u)$	2	0.1219	0.1037	0.0853	0.0753
$P(U_1 > u \| U_4 > u, U_2 < u, U_3 < u)$	3	0.0614	0.0501	0.0369	0.0325
$P(U_1 > u \| U_2 > u, U_3 > u, U_4 < u)$	4	0.6285	0.5976	0.5575	0.5673
$P(U_1 > u \| U_2 > u, U_4 > u, U_3 < u)$	5	0.5410	0.5142	0.4947	0.4969
$P(U_1 > u \| U_3 > u, U_4 > u, U_2 < u)$	6	0.0924	0.0864	0.0833	0.0497
$P(U_1 > u \| U_2 > u, U_3 > u, U_4 > u)$	7	0.6318	0.6006	0.5788	0.6190
Sum	**8**	**2.6713**	**2.5025**	**2.3387**	**2.3235**
$P(U_2 > u \| U_1 > u, U_3 < u, U_4 < u)$	9	0.3942	0.3791	0.3623	0.3524
$P(U_2 > u \| U_3 > u, U_1 < u, U_4 < u)$	10	0.2575	0.2402	0.2229	0.2035
$P(U_2 > u \| U_4 > u, U_1 < u, U_3 < u)$	11	0.1095	0.1000	0.0828	0.0848
$P(U_2 > u \| U_1 > u, U_3 > u, U_4 < u)$	12	0.8088	0.8023	0.7950	0.8045
$P(U_2 > u \| U_1 > u, U_4 > u, U_3 < u)$	13	0.6888	0.6901	0.6978	0.7312
$P(U_2 > u \| U_3 > u, U_4 > u, U_1 < u)$	14	0.5251	0.5265	0.5391	0.5474
$P(U_2 > u \| U_1 > u, U_3 > u, U_4 < u)$	15	0.9491	0.9465	0.9465	0.9741
Sum	**16**	**3.7329**	**3.6847**	**3.6464**	**3.6978**
$P(U_3 > u \| U_2 > u, U_1 < u, U_4 < u)$	17	0.4886	0.4362	0.3828	0.3545
$P(U_3 > u \| U_1 > u, U_3 < u, U_4 < u)$	18	0.1451	0.1239	0.1019	0.0925
$P(U_3 > u \| U_4 > u, U_1 < u, U_2 < u)$	19	0.0915	0.0749	0.0555	0.0467
$P(U_3 > u \| U_2 > u, U_1 > u, U_4 < u)$	20	0.5245	0.4846	0.4365	0.4354
$P(U_3 > u \| U_2 > u, U_4 > u, U_1 < u)$	21	0.4753	0.4476	0.4323	0.3902
$P(U_3 > u \| U_1 > u, U_4 > u, U_3 < u)$	22	0.1354	0.1267	0.1225	0.0709
$P(U_3 > u \| U_2 > u, U_1 > u, U_4 > u)$	23	0.5687	0.5351	0.5166	0.5129
Sum	**24**	**2.4292**	**2.2291**	**2.0482**	**1.9032**
$P(U_4 > u \| U_2 > u, U_3 < u, U_1 < u)$	25	0.3449	0.2874	0.2183	0.2147
$P(U_4 > u \| U_3 > u, U_2 < u, U_1 < u)$	26	0.1354	0.1072	0.0776	0.0630
$P(U_4 > u \| U_1 > u, U_2 < u, U_3 < u)$	27	0.1107	0.0874	0.0625	0.0547
$P(U_4 > u \| U_2 > u, U_3 > u, U_1 < u)$	28	0.3329	0.2970	0.2554	0.2416
$P(U_4 > u \| U_2 > u, U_1 > u, U_3 < u)$	29	0.2974	0.2589	0.2132	0.2243
$P(U_4 > u \| U_3 > u, U_1 > u, U_2 < u)$	30	0.1030	0.0895	0.0758	0.0415
$P(U_4 > u \| U_2 > u, U_3 > u, U_1 > u)$	31	0.3360	0.2996	0.2722	0.2831
Sum	**32**	**1.6604**	**1.4270**	**1.1750**	**1.1228**

4.2. Comparison of bivariate tail dependence coefficients with multivariate tail dependence coefficients.

Secondly, we have calculated, using formula (2.1), matrix of bivariate tail dependence coefficients:

$$\begin{pmatrix} 1 & 0.6042 & 0.3910 & 0.2522 \\ 0.6042 & 1 & 0.5195 & 0.3393 \\ 0.3910 & 0.5195 & 1 & 0.2522 \\ 0.2522 & 0.3393 & 0.2522 & 1 \end{pmatrix}.$$

If we compare bivariate tail dependence coefficient between the first and the second marginal 0.6042 with coefficients in Table 1 on lines from 1 and 7, then it is possible to see that it is larger than the first one and smaller than the last one. Similar situation occurs when we compare coefficient 0.3910 between the first and the third variable with coefficients on lines 9 and 15. More detailed study shows that all other correlation coefficients are playing essential role. For example, if we compare probabilities $P(U_1 > u | U_2 > u, U_3 > u, U_4 < u)$ and $P(U_1 > u | U_3 > u, U_4 > u, U_2 < u)$ (rows 4 and 6 in Table 1), then we can see that correlations between marginals 1 and 2 and 1 and 3 in row 4 are 0.8 and 0.5, respectively, while between marginals 1 and 3 and 1 and 4 in row 6 correlations are 0.5 and 0.2, respectively.

4.3. Tail dependence in the case of equal correlations.

We have investigated how large is total tail dependence coefficient in different situations. Finally, we have simulated four-dimensional t-copula with equal correlations and one 5-dimensional copula. Results are presented in Table 2.

TABLE 2. Comparison of tail dependence coefficients for copulas with different equal correlations and different number of coordinates.

	100000	50000	10000	5000	1000	100
Percentile	0.9	0.95	0.99	0.995	0.999	0.9999
$n = 4, r = 0.85, \nu = 2$	3.4359	3.3837	3.3337	3.2852	3.2972	3.2235
$n = 4, r = 0.50, \nu = 2$	2.7866	2.7083	2.6340	2.6177	2.6225	2.6655
$n = 4, r = 0.20, \nu = 2$	2.1795	2.0848	1.9819	1.9826	1.9340	2.5082
$n = 5, r = 0.85, \nu = 2$	5.3998	5.3998	4.6665	4.5755	5.6500	3.5000

We can see from Table 2 that total upper tail dependence coefficient becomes smaller if correlation coefficient decreases. We can also see that with the same correlation and degrees of freedom tail dependence coefficient is larger for five dimensional copula compared with the four dimensional copula.

5. Summary

From simulation study we have got some information about behaviour of the defined tail dependence coefficient. It is very important to know in practice how each marginal in multivariate distribution depends not only on all other marginals but how it depends on different combinations of other marginals. It is interesting to know, are simulated estimates of tail dependence coefficients larger or smaller than calculated tail dependence coefficient for two-dimensional copula evaluated by formula in Demarta and McNeil [7]. It depends on all other correlation coefficients. Tail dependence coefficients for different-dimensional copulas are difficult to compare because number of terms in final expression is growing fast. If terms are obtained in explicit form like in Table 1 then each coefficient gives us information about the tail behaviour of a given marginal with the other marginals. Dependence between different insurance lines of business is mostly described by a multivariate distribution. Therefore, we are planning to apply t-copula as an alternative method for risk assessment under Solvency II framework for insurance internal models in our future research.

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