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Empirical cumulant function based parameter estimation in stable laws

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Abstract. Stable distributions are a subclass of infinitely divisible distributions that form the only family of possible limiting distributions for sums of independent identically distributed random variables. A challenging problem is estimating their parameters because many have densities with no explicit form and infinite moments. To address this problem, a class of closed-form estimators, called cumulant estimators, has been introduced. Cumulant estimators are derived from the logarithm of empirical characteristic function at two arbitrary distinct positive real arguments. This paper extends cumulant estimators in two directions: (i) it is proved that they are asymptotically normal and (ii) a sample based rule for selecting the two arguments is proposed. Extensive simulations show that under the provided selection rule, the closed-form cumulant estimators generally outperform the well-known algorithmic methods.

1. Introduction

Stable laws are a subclass of infinitely divisible distributions that form the only family of possible limiting distributions for sums of independent identically distributed (i.i.d.) random variables. The theory of stable laws is amply described in [6], [37], [32], [34], [23], [26].

A general stable distribution on $\mathbb{R}$ is described by four parameters: a characteristic exponent (or index of stability or tail index) $\alpha \in (0, 2]$, a skewness parameter $\beta \in [-1, 1]$, and the scale and location parameters, denoted by $\gamma > 0$ and $\delta \in \mathbb{R}$, respectively.

The heavy-tailed flexible stable distributions can capture fuzzy dynamics and large fluctuations in data that result from stochastic processes occurring

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in diverse fields of business, science, and engineering (see, e.g., [34, Chapter II]). In practice, there are several alternatives to stable distributions. However, when solving applied problems involving limit distributions of sums with heavy tailed random variables, then general stable laws have the most appropriate structure. Parameter estimation of stable distributions is often complicated due to the lack of availability of an explicit form of the density function and that not all moments exist. For $\alpha \in (1, 2)$ the second and higher order moments are infinite while for $\alpha \in (0, 1]$ the first and higher order moments are infinite (see, e.g., [26]). Two types of methods have been proposed for estimating all parameters of a general (with no restrictions on the parameter space): algorithmic methods and closed-form estimators. The primary algorithmic methods used with general stable laws include: the quantile based look-up method by [22]; the empirical characteristic function based methods by [18, 19] and [16]; and the (numerical) maximum likelihood estimation methods (e.g., [24] and [25]). The primary closed-form estimators in general stable laws include empirical characteristic function based estimators by [29] and [20], and the logarithmic moments, fractional lower order moments, and extreme value theory based estimators by [21].

Of the aforementioned methods, the maximum likelihood method has high computational complexity; the quantile method has restrictions in tail index α and low accuracy; the logarithmic moments, fractional lower order moments, and extreme value theory based estimators do not provide estimators for the location parameter δ and their asymptotic properties are not provided.

In this paper we study empirical characteristic function based estimators of the parameters of stable laws in $\mathbb{R}$. The approach was introduced by [29], where he derived closed-form estimators based on the logarithm of empirical characteristic function $\varphi_n(u)$, given by (10), at four arbitrary different non-zero arguments $u_k, k = 1, \ldots, 4$ along the real line. Unfortunately, [29] provided no guidance on how to choose these four arguments. This problem has remained unresolved ever since thus making the method not very useful in practice. The papers [27], [1] point out that at various values of $u_k, k = 1, \ldots, 4$ the estimates by [29] notably vary and in may cases may yield estimates outside of the parameter space. To get around the difficulties of selecting $u_k, k = 1, \ldots, 4$, several algorithmic modifications along the real line have been proposed. For example, [27] provided a projection method that minimizes (by a gradient projection routine) the integrated squared error of $\varphi_n(u)$ for standardized data along the real line, where $\int |\varphi(u) - \varphi_n(u)|^2 \mathrm{d}t$ is approximated by a 20-point Hermitian quadrature at specified points u_k, $k = 1, \ldots, 20$. A similar approach is studied in [10]. A method due to [18] is based on regressing $\varphi_n(u)$ of standardized data onto $\varphi(u)$ at points $u_k = \pi k/25$ for $k = 1, 2, \ldots, K$ where K has values from $K = 10$ to $K = 134$, depending on sample size n and parameter α. A procedure due to [16] is based on ordinary least squares regression of $\varphi_n(u)$ of standardized data onto $\varphi(u)$ at points $u_k = k/10$ for $k = 1, 2, \dots, 10$ (see also [14]). In [16] it was empirically explored that using more than 10 points in $[0.1, 1]$ does not remarkably improve the estimates. Recently, [20] extended the [29] approach and provided closed-form estimators, called cumulant (for clarity, the term *cumulant estimators* refers to estimators obtained from the logarithm of the empirical characteristic function, called the cumulant function (e.g., [17]), estimators, that require only two distinct arguments, $u_k, k = 1, 2$, along the positive real line to determine the four parameters of a general stable law. In order to provide some guidance on how to choose these two arguments, [20] performed an empirical search over various pairs of positive real arguments, but no formal solution was provided.

This paper extends the [20] work by two contributions:

(i) it is proven that cumulant estimators are asymptotically normal, and
(ii) an Argument-Selection-Rule for selecting the arguments u_1, u_2 is proposed.

The advantage of cumulant estimators is their computational simplicity and, to the best of our knowledge, they are the only asymptotically normal closed-form estimators for four parameters of stable laws with no restrictions in the parameter space.

The paper is organized as follows: In Section 2 we briefly describe the cumulant function of stable laws. In Section 3 the empirical cumulant function based (ECuF) estimators for stable laws are presented. In Section 4 we establish the asymptotic normality of empirical cumulant function in general and for ECuF estimators. In Section 5 we introduce a sample based rule for selecting the two arguments needed for cumulant estimators. In Section 6 we report on comprehensive Monte–Carlo simulations. Concluding comments are given in Section 7.

2. Preliminaries: the cumulant function of stable laws

All stable laws are characterized by terms of their characteristic or cumulant functions (see, e.g., [23, Theorem 3.1.2]). A characteristic function of a random variable X on $\mathbb{R}$, denoted by $\varphi(u)$, is a complex-valued function, $\varphi(u) : \mathbb{R} \to \mathbb{C}$, defined as

$$\varphi(u) = \mathbb{E}\exp\{iuX\} = \mathbb{E}\cos(uX) + i\mathbb{E}\sin(uX),\ u \in \mathbb{R}. \tag{1}$$

A cumulant function of a random variable X on $\mathbb{R}$, denoted by $\psi(u)$, is a complex-valued function, $\psi(u) : \mathbb{R} \to \mathbb{C}$, defined as the principal value of the logarithm of the characteristic function,

$$\psi(u) \equiv \ln|\varphi(u)| + \operatorname{atan2}(\Im\varphi(u), \Re\varphi(u)), \tag{2}$$

where atan2 denotes the the arctangent function with two arguments, that is principal value[1] of a complex number (see, e.g., [12]).

There are several ways to determine the closed canonic form of the characteristic functions of stable laws (see, e.g., [6, p. 564–565], [33, p. 38–39], [28, p. 268]). The Lévy–Khintchine form [13], also known as form (A) by [37] has become a standard (see, e.g., [33, Theorem 14.15], [4, Theorem 2.2.3]). However, in that form the characteristic function of stable laws (as function of parameters) is not continuous in the whole parameter space. On real line an alternative represenation is advocated by [6, Theorem 1], [37], [28, Corollary 4.1, p. 269], and by [26, Definition 1.7]. In that representation stable laws on $\mathbb{R}$ are continuous in all parameters, and it is suggested to use when exploring the asymptotic convergence of the parameters. In this paper we follow [26, Definition 1.7], called a 0-parametrization. The cumulant function of a stable distribution in 0-parametrization [26], denoted by $S(\alpha, \beta, \gamma, \delta; 0)$, is given by

$$\psi(u) = \begin{cases} -\gamma^\alpha |u|^\alpha [1 + i\beta (\operatorname{sign} u) \tan \frac{\pi\alpha}{2} (|\gamma u|^{1-\alpha} - 1)] + i\delta u & \alpha \neq 1, \\ -\gamma |u| [1 + i\beta \frac{2}{\pi} (\operatorname{sign} u) \ln(\gamma |u|)] + i\delta u & \alpha = 1, \end{cases} \tag{3}$$

where $u \in \mathbb{R}$, $\alpha \in (0, 2]$ is the characteristic exponent or tail index, $\beta \in [-1, 1]$ is the skewness parameter, $\gamma > 0$ is scale parameter and $\delta \in \mathbb{R}$ is the location parameter. The corresponding characteristic function is $\varphi(u; \alpha, \beta, \gamma, \delta) = \varphi(u) = \exp\{\psi(u)\}$ with $\psi(u)$ given by (3).

The parameters of different representations of stable laws are uniquely related (see, e.g., [34, Section 3.6], [26, Equation (1.7)]). Moreover, the cumulant estimators given by [20, Definition 2] are easily adaptable in all representations of stable laws.

Tail index α and skewness parameter β can be introduced through the generalized central limit theorem on $\mathbb{R}$ (see, e.g., [34, p. 62–63]). For i.i.d. random variables $X_1, \ldots, X_n$ with the distribution function $F(x)$ satisfying the conditions $1 - F(x) \sim ax^{-k}$ as $x \to \infty$, and $F(x) \sim c|x|^{-k}$ as $x \to -\infty$ with $a, c \geq 0$, $a + c > 0$, and $k > 0$, there exist sequences $a_n \in \mathbb{R}$ and $b_n > 0$ such that the centred and normalized sum $(X_1 + \cdots + X_n - a_n)/b_n$ converges in distribution to a stable random variable with $\alpha = k$ for $k < 2$, $\alpha = 2$ for $k \geq 2$, and $\beta = (a - c)/(a + c)$ as $n \to \infty$. The tail index α characterises the rate of the decay of the tails with smaller values of α in the case of heavier tails and $\alpha = 2$ corresponding to the light-tailed normal distribution. The skewness parameter β illustrates the degree of asymmetry with $\beta = 0$ (i.e., $a = c$) denoting symmetric stable distributions and $\beta = \pm 1$ (i.e., $c = 0$ or $a = 0$) denoting maximally asymmetric distributions that are often called totally skewed stable laws.

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[1] In a variety of computer languages the principal value of complex number, the arctangent function with two arguments, is provided under the function name of atan2 (e.g., `R` [30], that is, $\operatorname{Arg} \varphi(u) = \operatorname{atan2}(\Im\varphi(u), \Re\varphi(u))$.

For simulations of stable laws (see, e.g., [3]) a R-package `stable` (part of the STABLE® software by [31]) is used.

3. Cumulant estimators for stable laws on $\mathbb{R}$

Let $\Re$ and $\Im$ denote the real and imaginary operators, respectively, i.e., given a complex number $z = x + iy$ then $\Re z = x$, $\Im z = y$, $\Re^2 z = x^2$, and $\Im^2 z = y^2$. From (3) the real part of the cumulant function of stable distribution is

$$\Re\psi(u) = -\gamma^\alpha |u|^\alpha, \tag{4}$$

and the imaginary part of the cumulant function is

$$\Im\psi(u) = \begin{cases} u[\beta\gamma \tan\frac{\pi\alpha}{2}(|\gamma u|^{\alpha-1} - 1) + \delta] & \alpha \neq 1 \\ u[-\beta\gamma\frac{2}{\pi}\ln(\gamma|u|) + \delta] & \alpha = 1. \end{cases}$$

We draw your attention to the fact that, unlike the case of the characteristic function, the parameters β and δ have no influence on the real part of the cumulant function of stable laws. This forms the basis of [20, Theorem 1] which states that for two fixed real values $u_1 > 0, u_2 > 0, u_1 \neq u_2$ the parameters α, γ, β, δ of stable laws (in [26] 1-parametrization (i.e., the Lévy–Khintchine form) can be expressed via real and imaginary part of the corresponding cumulant function. The method [20, Theorem 1] is easily adaptable for any representation of stable laws. We present a slightly modified version of [20, Theorem 1] for the stable laws in [26] 0-parametrization.

Theorem 1. *Let $X \sim S(\alpha, \beta, \gamma, \delta; 0)$ with cumulant function be given by (3), and let, for every fixed real $u_1 > 0, u_2 > 0, u_1 \neq u_2$,*

$$\boldsymbol{b} = \boldsymbol{b}(u_1, u_2; X) = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix} = \begin{pmatrix} \exp\{\Re\psi(u_1)\}\cos\Im\psi(u_1) \\ \exp\{\Re\psi(u_2)\}\cos\Im\psi(u_2) \\ \exp\{\Re\psi(u_1)\}\sin\Im\psi(u_1) \\ \exp\{\Re\psi(u_2)\}\sin\Im\psi(u_2) \end{pmatrix} \tag{5}$$

be a 4-dimensional real valued vector. The parameters of X are expressed as $\alpha = g_1(\boldsymbol{b})$, $\gamma = \exp\{g_2(\boldsymbol{b})\}$, $\beta = g_3(\boldsymbol{b})$, $\delta = g_4(\boldsymbol{b})$, where

$$g_1(\boldsymbol{b}) = \frac{\ln\left(-\ln\sqrt{b_1^2 + b_3^2}\right) - \ln\left(-\ln\sqrt{b_2^2 + b_4^2}\right)}{\ln u_1 - \ln u_2}, \tag{6}$$

$$g_2(\boldsymbol{b}) = \frac{\ln u_1 \ln\left(-\ln\sqrt{b_2^2 + b_4^2}\right) - \ln u_2 \ln\left(-\ln\sqrt{b_1^2 + b_3^2}\right)}{\ln\left(-\ln\sqrt{b_1^2 + b_3^2}\right) - \ln\left(-\ln\sqrt{b_2^2 + b_4^2}\right)}, \tag{7}$$

if $\alpha \neq 1$, then

$$g_3(\boldsymbol{b}) = \frac{u_2 \operatorname{atan2}(b_3, b_1) - u_1 \operatorname{atan2}(b_4, b_2)}{\exp\{g_1(\boldsymbol{b})g_2(\boldsymbol{b})\}\left(u_2 u_1^{g_1(\boldsymbol{b})} - u_1 u_2^{g_1(\boldsymbol{b})}\right)\tan(\pi g_1(\boldsymbol{b})/2)} \tag{8a}$$

while if $\alpha = 1$, *then*

$$g_3(\boldsymbol{b}) = \frac{\pi}{2} \frac{u_2 \operatorname{atan2}(b_3, b_1) - u_1 \operatorname{atan2}(b_4, b_2)}{\exp\{g_2(\boldsymbol{b})\} u_1 u_2 (\ln u_2 - \ln u_1)}, \tag{8b}$$

if $\alpha \neq 1$, *then*

$$\begin{aligned} g_4(\boldsymbol{b}) = {} & \frac{u_2^{g_1(\boldsymbol{b})} \operatorname{atan2}(b_3, b_1) \left(\exp\{g_2(\boldsymbol{b})\} u_1\right)^{1-g_1(\boldsymbol{b})} - 1)}{u_2 u_1^{g_1(\boldsymbol{b})} - u_1 u_2^{g_1(\boldsymbol{b})}} \\ & - \frac{u_1^{g_1(\boldsymbol{b})} \operatorname{atan2}(b_4, b_2) \left(\exp\{g_2(\boldsymbol{b})\} u_2\right)^{1-g_1(\boldsymbol{b})} - 1)}{u_2 u_1^{g_1(\boldsymbol{b})} - u_1 u_2^{g_1(\boldsymbol{b})}} \end{aligned} \tag{9a}$$

while if $\alpha = 1$, *then*

$$g_4(\boldsymbol{b}) = \frac{u_2 \operatorname{atan2}(b_3, b_1)\, g_2(\boldsymbol{b}) \ln u_2 - u_1 \operatorname{atan2}(b_4, b_2)\, g_2(\boldsymbol{b}) \ln u_1}{u_1 u_2 (\ln u_2 - \ln u_1)}. \tag{9b}$$

Proof. From elementary complex analysis

$$\boldsymbol{b} = (\Re\varphi(u_1), \Re\varphi(u_2), \Im\varphi(u_1), \Im\varphi(u_2))',$$

where $\varphi(u) = \exp\{\psi(u)\}$. Then (6) to (9b) easily follow from in [20, Theorem 1] by replacing the cumulant function of 1-parametrization [26] with the cumulant function of 0-parametrization [26]. □

The key to cumulant estimators [20, Definition 2] is the substitution principle (see, e.g., [15]): the values of theoretical cumulant function, $\psi(u)$, are replaced by those of the empirical cumulant function. The empirical cumulant function, denoted by ψ_n, is defined as the principal value of the logarithm of the empiricial characteristic function. Empiricial characteristic function, denoted by $\varphi_n(u)$, is a complex valued function (e.g., [35]),

$$\begin{aligned} \varphi_n(u) = \varphi_n(u|Y_1, \ldots, Y_n) &= \frac{1}{n} \sum_{j=1}^{n} \exp\{iuY_j\} \\ &= \frac{1}{n} \sum_{j=1}^{n} \cos(uY_j) + i \frac{1}{n} \sum_{j=1}^{n} \sin(uY_j), \end{aligned} \tag{10}$$

where $Y_1, \ldots, Y_n$ form a real valued random sample of i.i.d. variables, $u \in \mathbb{R}$, $i^2 = -1$. Hereby, the empirical cumulant function is

$$\psi_n(u) = \ln \varphi_n(u), \tag{11}$$

where ln is the principal value of the logarithm.

Definition 1. Let $Y_1, \dots, Y_n$ form a real valued random sample of i.i.d. variables, and let, for every fixed real $u_1 > 0, u_2 > 0, u_1 \neq u_2$,

$$\boldsymbol{b}_n = \boldsymbol{b}_n(u_1, u_2; Y_1, \dots, Y_n) = \begin{pmatrix} \exp\{\Re\psi_n(u_1)\} \cos \Im\psi_n(u_1) \\ \exp\{\Re\psi_n(u_2)\} \cos \Im\psi_n(u_2) \\ \exp\{\Re\psi_n(u_1)\} \sin \Im\psi_n(u_1) \\ \exp\{\Re\psi_n(u_2)\} \sin \Im\psi_n(u_2) \end{pmatrix} \tag{12}$$

be a 4-dimensional real valued vector where $\psi_n(u)$ is the empirical cumulant function given by (11). Empirical cumulant function (ECuF) based estimators of the parameters of $S(\alpha, \beta, \gamma, \delta; 0)$, denoted by α_n, β_n, γ_n, δ_n, are given by

$$\begin{aligned} \alpha_n &= \alpha_n(u_1, u_2; Y_1, \dots, Y_n) = g_1(\boldsymbol{b}_n), \\ \gamma_n &= \gamma_n(u_1, u_2; Y_1, \dots, Y_n) = \exp\{g_2(\boldsymbol{b}_n)\}, \\ \beta_n &= \beta_n(u_1, u_2; Y_1, \dots, Y_n) = g_3(\boldsymbol{b}_n), \\ \delta_n &= \delta_n(u_1, u_2; Y_1, \dots, Y_n) = g_4(\boldsymbol{b}_n), \end{aligned}$$

where the g_j's are given in Theorem 1 by (6) through (9b).

In general, ECuF estimators may give non-admissible values, that is, one or more estimates may turn out of the parameter space: $\alpha \in (0, 2]$, $\beta \in [-1, 1]$, $\gamma > 0$, $\delta \in \mathbb{R}$. In simulations in Section (6) the values of estimators are truncated: replace α_n with $\min(\max(\alpha_n, 0.01), 2)$, β_n with $\beta_n = \min(\max(\beta_n, -1), 1)$ and γ with $\gamma_n = \max(0, \gamma_n)$. However, asymptotic normality is provided for the non-truncated estimators.

From elementary complex analysis

$$\boldsymbol{b}_n = (\Re\varphi_n(u_1), \Re\varphi_n(u_2), \Im\varphi_n(u_1), \Im\varphi_n(u_2))',$$

with $\varphi_n(u) = \exp\{\psi_n(u)\}$ given by (10). Then Definition 1 follows from Theorem 1 and [20, Definition 2] by replacing the cumulant function in [26] 1-parametrization with its form in the [26] 0-parametrization using g_j's given by (6) through (9b).

4. Asymptotic normality of cumulant estimators

Before deriving the asymptotic normality of ECuF estimators the asymptotic normality of the real and imaginary parts of the empirical cumulant function for an arbitrary distribution is established.

Theorem 2. *Let X be a real valued random variable (of some distribution) with the characteristic function $\varphi(u)$, given by* (1)*, and cumulant function $\psi = \psi(u)$, given by* (2)*. Let ψ_n be the empirical cumulant function given by* (11)*. For every fixed $u \in \mathbb{R}$,*

(i) $\Re\psi_n(u)$ is a strongly consistent estimator for $\Re\psi(u)$ and $\Im\psi_n(u)$ is a strongly consistent estimator for $\Im\psi(u)$;

(ii) $\Re\psi_n(u)$ *is asymptotically normal for* $\Re\psi(u)$ *and* $\Im\psi_n(u)$ *is asymptotically normal for* $\Re\psi(u)$,

$$\sqrt{n}(\Re\psi_n(u) - \Re\psi(u)) \xrightarrow{\mathcal{D}} N_1(0, \kappa_{\Re}(u)), \text{ as } n \to \infty,$$
$$\sqrt{n}(\Im\psi_n(u) - \Im\psi(u)) \xrightarrow{\mathcal{D}} N_1(0, \kappa_{\Im}(u)), \text{ as } n \to \infty,$$

with

$$\kappa_{\Re}(u) = \frac{1}{2\Re^2\psi(u)}\left(1 + \exp\{\Re\psi(2u)\} - 2\exp\{2\Re\psi(u)\}\right), \tag{13}$$
$$\kappa_{\Im}(u) = \frac{1}{2\Re^2\psi(u)}\left(1 - \exp\{\Re\psi(2u)\}\right). \tag{14}$$

Proof. For a real vector $\boldsymbol{x} = (x_1, x_2)' \neq \mathbf{0}$ define $h_1(\boldsymbol{x}) = \ln\sqrt{x_1^2 + x_2^2}$ and $h_2(\boldsymbol{x}) = \mathrm{atan2}(x_2, x_1)$. Let

$$\boldsymbol{a} = (\Re\varphi(u), \Im\varphi(u))'$$

with $\varphi(u)$ given by (1), and

$$\boldsymbol{a}_n = (\Re\varphi_n(u), \Im\varphi_n(u))'$$

with $\varphi_n(u)$ given by (10). The real and imaginary parts of cumulant function in (2) are the functions of the elements of $\boldsymbol{a}$,

$$h_1(\boldsymbol{a}) = \Re\psi(u),\ h_2(\boldsymbol{a}) = \Im\psi(u),$$

and the real and imaginary parts of the empirical cumulant function in (11) are the functions of the elements of $\boldsymbol{a}_n$,

$$h_1(\boldsymbol{a}_n) = \Re\psi_n(u),\ h_2(\boldsymbol{a}_n) = \Im\psi_n(u).$$

Clearly, h_1 is a continuous function. Function $h_2(\boldsymbol{a})$ has a discontinuity at $\Re\varphi(u) = 0$ and $\Im\varphi(u) = 0$, i.e., at $|\varphi(u)| = 0$. However, the term $|\varphi(u)|$ tends to 0 as $|u| \to \infty$ (see, e.g. [35]). However, in the theorem assumption it is said that $u \in \mathbb{R}$ is fixed, $|u| < \infty$, and $h_2(\boldsymbol{a})$ is continuous on any bounded interval.

(i) For every fixed $u \in \mathbb{R}$,

$$\boldsymbol{a}_n \xrightarrow{a.s.} \boldsymbol{a}, \text{ as } n \to \infty.$$

By the continuous mapping theorem (e.g., [36, Theorem 2.3]),

$$h_j(\boldsymbol{a}_n) \xrightarrow{a.s.} h_j(\boldsymbol{a}), \text{ as } n \to \infty,$$

and the estimators $h_j(\boldsymbol{a}_n)$ are consistent for $h_j(\boldsymbol{a})$, $j = 1, 2$.

(ii) The quantities $\Re\varphi_n(u_j)$, $\Im\varphi_n(u_j)$, $j = 1, 2$ are sample means of i.i.d. random variables with $\mathbf{E}\,\boldsymbol{a}_n = \boldsymbol{a}$ and with finite variance. Therefore (see, e.g., [8, Theorem 3.1]), for every fixed $u \in \mathbb{R}$,

$$\sqrt{n}(\boldsymbol{a}_n - \boldsymbol{a}) \xrightarrow{\mathcal{D}} N_2(\mathbf{0}, \boldsymbol{\Sigma}(u)), \text{ as } n \to \infty,$$

where $\mathbf{\Sigma}(u) \equiv \mathbf{\Sigma} = (\sigma_{kl})$ is a 2×2 covariance matrix with the structure following from (16),

$$\begin{aligned} 2\sigma_{11} &= 1 + \Re\varphi(2u) - 2\Re^2\varphi(u), \\ 2\sigma_{22} &= 1 - \Re\varphi(2u) - 2\Im^2\varphi(u), \\ 2\sigma_{12} = 2\sigma_{21} &= \Im\varphi(2u) - 2\Re\varphi(u)\Im\varphi(u), \end{aligned}$$

where $\varphi(u) = \exp\psi(u)$. From [17, Theorem 3.1.3] it immediately follows, that for every fixed $u \in \mathbb{R}$,

$$\begin{aligned} \sqrt{n}(h_1(\boldsymbol{a}_n) - h_1(\boldsymbol{a})) &\xrightarrow{\mathcal{D}} N_1(0, \boldsymbol{\nu}'\mathbf{\Sigma}\boldsymbol{\nu}), \text{ as } n \to \infty, \\ \sqrt{n}(h_2(\boldsymbol{a}_n) - h_2(\boldsymbol{a})) &\xrightarrow{\mathcal{D}} N_1(0, \boldsymbol{\eta}'\mathbf{\Sigma}\boldsymbol{\eta}), \text{ as } n \to \infty, \end{aligned}$$

with $\boldsymbol{\nu}$ and $\boldsymbol{\eta}$ as matrix derivatives ([17, Definition 1.4.1]),

$$\boldsymbol{\nu} = \frac{\mathrm{d}\, h_1(\boldsymbol{x})}{\mathrm{d}\boldsymbol{x}}\bigg|_{\boldsymbol{x}=\boldsymbol{a}} \text{ and } \boldsymbol{\eta} = \frac{\mathrm{d}\, h_2(\boldsymbol{x})}{\mathrm{d}\boldsymbol{x}}\bigg|_{\boldsymbol{x}=\boldsymbol{a}}.$$

It is easy to see that

$$\frac{\mathrm{d}\, h_1(\boldsymbol{x})}{\mathrm{d}\boldsymbol{x}} = \frac{\mathrm{d} \ln \sqrt{x_1^2 + x_2^2}}{\mathrm{d}\boldsymbol{x}} = \left(\frac{x_1}{x_1^2 + x_2^2}, \frac{x_2}{x_1^2 + x_2^2} \right)',$$

and

$$\frac{\mathrm{d}\, h_2(\boldsymbol{x})}{\mathrm{d}\boldsymbol{x}} = \frac{\mathrm{d}\, \mathrm{atan2}(x_2, x_1)}{\mathrm{d}\boldsymbol{x}} = \left(-\frac{x_2}{x_1^2 + x_2^2}, \frac{x_1}{x_1^2 + x_2^2} \right)'.$$

Hereby,

$$\boldsymbol{\nu} = \frac{\mathrm{d}\, h_1(\boldsymbol{x})}{\mathrm{d}\boldsymbol{x}}\bigg|_{\boldsymbol{x}=\boldsymbol{a}} = \left(\frac{\Re\varphi(u)}{|\varphi(u)|^2}, \frac{\Im\varphi(u)}{|\varphi(u)|^2} \right)' \neq \mathbf{0},$$

and

$$\boldsymbol{\eta} = \frac{\mathrm{d}\, h_2(\boldsymbol{x})}{\mathrm{d}\boldsymbol{x}}\bigg|_{\boldsymbol{x}=\boldsymbol{a}} = \left(-\frac{\Im\varphi(u)}{|\varphi(u)|^2}, \frac{\Re\varphi(u)}{|\varphi(u)|^2} \right)' \neq \mathbf{0}.$$

Obtaining $\boldsymbol{\nu}'\boldsymbol{\Sigma}\boldsymbol{\nu}$ is straightforward,

$$\begin{aligned}
\boldsymbol{\nu}'\boldsymbol{\Sigma}\boldsymbol{\nu} &= (\nu_1)^2\sigma_{11} + \nu_1\nu_2(\sigma_{12}+\sigma_{21}) + (\nu_2)^2\sigma_{22} \\
&= \frac{1}{2|\varphi(u)|^4}\Big[\Re^2\varphi(u)\big(1+\Re\varphi(2u)-2\Re^2\varphi(u)\big) \\
&\quad + \Im^2\varphi(u)\big(1-\Re\varphi(2u)-2\Im^2\varphi(u)\big) \\
&\quad + 2\Re\varphi(u)\Im\varphi(u)\big(\Im\varphi(2u)-2\Re\varphi(u)\Im\varphi(u)\big)\Big] \\
&= \frac{1}{2|\varphi(u)|^4}\Big[|\varphi(u)|^2 - 2|\varphi(u)|^4 \\
&\quad + \underbrace{\Re\varphi(2u)\big(\Re^2\varphi(u)-\Im^2\varphi(u)\big) + 2\Im\varphi(2u)\Re\varphi(u)\Im\varphi(u)}_{(*)}\Big],
\end{aligned}$$

with $\Re\varphi(u) + \Im\varphi(u) = |\varphi(u)|^2$ and

$$\begin{aligned}
(*) &= \Re\varphi(2u)2\big(\Re^2\varphi(u)-\Im^2\varphi(u)\big) + 2\Im\varphi(2u)\Re\varphi(u)\Im\varphi(u) \\
&= \exp\{\Re\psi(2u)\}\exp\{2\Re\psi(u)\}\big[\cos(2u)\big(\cos^2(u) \\
&\quad - \sin^2(u)\big) + 2\sin(u)\cos(u)\sin(2u)\big] \\
&= \exp\{\Re\psi(2u)\}\exp\{2\Re\psi(u)\}. \qquad (15)
\end{aligned}$$

Since $|\varphi(u)| = \exp\{\Re\psi(u)\}$, we have

$$\boldsymbol{\nu}'\boldsymbol{\Sigma}\boldsymbol{\nu} = \frac{1}{2\exp\{2\Re\psi(u)\}}\left(1-2\exp\{2\Re\psi(u)\}+\exp\{\Re\psi(2u)\}\right),$$

which is the same as (13) and $\kappa_\Re = \boldsymbol{\nu}'\boldsymbol{\Sigma}\boldsymbol{\nu}$. In a similar manner,

$$\begin{aligned}
\boldsymbol{\eta}'\boldsymbol{\Sigma}\boldsymbol{\eta} &= (\eta_1)^2\sigma_{11} + \eta_1\eta_2(\sigma_{12}+\sigma_{21}) + (\eta_2)^2\sigma_{22} \\
&= \frac{1}{2|\varphi(u)|^4}\Big[\Im^2\varphi(u)\big(1+\Re\varphi(2u)-2\Re^2\varphi(u)\big) \\
&\quad + \Re^2\varphi(u)\big(1-\Re\varphi(2u)-2\Im^2\varphi(u)\big) \\
&\quad - 2\Re\varphi(u)\Im\varphi(u)\big(\Im\varphi(2u)-2\Re\varphi(u)\Im\varphi(u)\big)\Big] \\
&= \frac{1}{2|\varphi(u)|^4}\Big[|\varphi(u)|^2 - (*)\Big],
\end{aligned}$$

where (*) is given by (15). Replacing $|\varphi(u)|$ by $\exp\{\Re\psi(u)\}$ yields

$$\boldsymbol{\nu}'\boldsymbol{\Sigma}\boldsymbol{\nu} = \frac{1}{2\exp\{2\Re\psi(u)\}}\left(1-\exp\{\Re\psi(2u)\}\right),$$

which is the same as (14) and $\kappa_\Im = \boldsymbol{\eta}'\boldsymbol{\Sigma}\boldsymbol{\eta}$.

□

As $\Re\psi(u) = \ln|\varphi(u)| \to 0$ for $|u| \to 0$ (see, e.g., [35]) then the asymptotic variances given by (13) and (14) tend to 0 as $|u| \to 0$. Hereby, the smaller the argument u, and the greater the sample size n, the better the real and imaginary parts of empirical cumulant function estimate those of the cumulant function of X. However, as $\psi_n(0) = \psi(0) = 0$ then at $u = 0$ cumulant function holds no info about the various parameters of the distribution. In the framework of the least squares estimation the asymptotic covariance of empirical cumulant function for a finite interval is discussed in [14]. From Theorem 2 the asymptotic normality of real and imaginary part of empirical cumulant function of stable distributions immediately follows.

Proposition 1. *Let $X \sim S(\alpha, \beta, \gamma, \delta; 0)$ be a stable random variable with cumulant function given by* (3) *and ψ_n be as given by* (11)*. Then, for every fixed $u \in \mathbb{R}$,*

$$\sqrt{n}(\Re\psi_n(u) - \Re\psi(u)) \xrightarrow{\mathcal{D}} N_1(0, \kappa_{\Re}(u)),\ \text{as } n \to \infty,$$
$$\sqrt{n}(\Im\psi_n(u) - \Im\psi(u)) \xrightarrow{\mathcal{D}} N_1(0, \kappa_{\Im}(u)),\ \text{as } n \to \infty,$$

where

$$\kappa_{\Re}(u) = \exp\{2(\gamma u)^\alpha\}\left(1 + \exp\{-(2\gamma u)^\alpha)\} - 2\exp\{-2(\gamma u)^\alpha\}\right)/2,$$

and

$$\kappa_{\Im}(u) = \exp\{2(\gamma u)^\alpha\}\left(1 - \exp\{-(2\gamma u)^\alpha)\}\right)/2.$$

The asymptotic normality of ECuF estimators, given by Definition 1, is established in a similar manner as Theorem 2. Note that the parameter space of stable laws has a boundary at $\alpha = 2$ and $\beta = \pm 1$ and the asymptotic normality does not make sense there. To obtain asymptotic distribution of estimators we assume that the true parameter is in the interior of the parameter space. The distribution on the boundary is not considered.

Theorem 3. *Let $\boldsymbol{b}$ be given by* (5)*, $\boldsymbol{b_n}$ by* (12)*, and functions $g_j(b)$, $j = 1, 2, 3, 4$ in Theorem 1. The ECuF estimators given by Definition 1 are consistent and $g_1(\boldsymbol{b_n})$ is asymptotically normal for $g_1(\boldsymbol{b}) \in (0, 2)$, $g_2(\boldsymbol{b_n})$ is asymptotically normal for $g_2(\boldsymbol{b})$, $g_3(\boldsymbol{b_n})$ is asymptotically normal for $g_3(\boldsymbol{b}) \in (-1, 1)$, and $g_4(\boldsymbol{b_n})$ is asymptotically normal for $g_4(\boldsymbol{b})$,*

$$\sqrt{n}\left(g_j(\boldsymbol{b_n}) - g_j(\boldsymbol{b})\right) \xrightarrow{d} N_1(0, \boldsymbol{\xi}_j' \boldsymbol{\Lambda} \boldsymbol{\xi}_j), \quad \text{as } n \to \infty,$$

where

$$\boldsymbol{\Lambda} = (\lambda_{ij}) \tag{16}$$

is the 4×4 *covariance matrix with components*

$$
\begin{aligned}
2\lambda_{11} &= 1 + \Re\varphi(2u_1) - 2\Re^2\varphi(u_1),\\
2\lambda_{22} &= 1 + \Re\varphi(2u_2) - 2\Re^2\varphi(u_2),\\
2\lambda_{33} &= 1 - \Re\varphi(2u_1) - 2\Im^2\varphi(u_1),\\
2\lambda_{44} &= 1 - \Re\varphi(2u_2) - 2\Im^2\varphi(u_2),\\
2\lambda_{12} &= 2\lambda_{21} = \Re\varphi(u_1 - u_2) + \Re\varphi(u_1 + u_2) - 2\Re\varphi(u_1)\Re\varphi(u_2),\\
2\lambda_{13} &= 2\lambda_{31} = \Im\varphi(2u_1) - 2\Re\varphi(u_1)\Im\varphi(u_1),\\
2\lambda_{14} &= 2\lambda_{41} = \Im\varphi(u_1 + u_2) - \Im\varphi(u_1 - u_2) - 2\Re\varphi(u_1)\Im\varphi(u_2),\\
2\lambda_{23} &= 2\lambda_{32} = \Im\varphi(u_1 + u_2) - \Im\varphi(u_2 - u_1) - 2\Re\varphi(u_2)\Im\varphi(u_1),\\
2\lambda_{24} &= 2\lambda_{42} = \Im\varphi(2u_2) - 2\Re\varphi(u_2)\Im\varphi(u_2),\\
2\lambda_{34} &= 2\lambda_{43} = \Re\varphi(u_1 - u_2) - \Re\varphi(u_1 + u_2) - 2\Im\varphi(u_1)\Im\varphi(u_2);
\end{aligned}
$$

$\boldsymbol{\xi}_1$ *has components*

$$
\begin{aligned}
\xi_{1i} &= \frac{b_i}{|\varphi(u_1)|^2\Re\phi(u_1)\ln(u_1/u_2)} && \text{for } i = 1,3,\\
\xi_{1i} &= \frac{-b_i}{|\varphi(u_2)|^2\Re\phi(u_2)\ln(u_1/u_2)} && \text{for } i = 2,4,
\end{aligned}
$$

$\boldsymbol{\xi}_2$ *has components*

$$
\begin{aligned}
\xi_{2i} &= \frac{1}{|\varphi(u_1)|^2\Re\phi(u_1)}\frac{-b_i(\ln u_2 + g_2(\boldsymbol{b}))}{\ln(\Re\psi(u_1)/\Re\psi(u_2))} && \text{for } i = 1,3,\\
\xi_{2i} &= \frac{1}{|\varphi(u_2)|^2\Re\phi(u_2)}\frac{b_i(\ln u_1 + g_2(\boldsymbol{b}))}{\ln(\Re\psi(u_1)/\Re\psi(u_2))} && \text{for } i = 2,4;
\end{aligned}
$$

if $\alpha \neq 1$, *then* $\boldsymbol{\xi}_3$ *has components*

$$
\begin{aligned}
\xi_{31} &= \frac{g_3(\boldsymbol{b})}{|\varphi(u_1)|^2\Re\phi(u_1)}\left(\frac{-b_3u_2\ln|\varphi(u_1)|}{C_1} - \frac{b_1C_2}{\ln(u_1/u_2)} - \frac{b_1u_2u_1^{g_1(\boldsymbol{b})}}{C_3}\right),\\
\xi_{32} &= \frac{g_3(\boldsymbol{b})}{|\varphi(u_2)|^2\Re\phi(u_2)}\left(\frac{-b_4u_1\ln|\varphi(u_2)|}{C_1} + \frac{b_2C_2}{\ln(u_1/u_2)} + \frac{b_2u_1u_2^{g_1(\boldsymbol{b})}}{C_3}\right),\\
\xi_{33} &= \frac{g_3(\boldsymbol{b})}{|\varphi(u_1)|^2\Re\phi(u_1)}\left(\frac{b_1u_2\ln|\varphi(u_1)|}{C_1} - \frac{b_3C_2}{\ln(u_1/u_2)} - \frac{b_3u_2u_1^{g_1(\boldsymbol{b})}}{C_3}\right),\\
\xi_{34} &= \frac{g_3(\boldsymbol{b})}{|\varphi(u_2)|^2\Re\phi(u_2)}\left(\frac{b_2u_1\ln|\varphi(u_2)|}{C_1} + \frac{b_4C_2}{\ln(u_1/u_2)} + \frac{b_4u_1u_2^{g_1(\boldsymbol{b})}}{C_3}\right),
\end{aligned}
$$

and $\boldsymbol{\xi}_4$ *has components*

$$\xi_{41} = \frac{u_2^{g_1(\boldsymbol{b})}}{C_3^2|\varphi(u_1)|^2}\Big(\ \ b_1 u_1^{g_1(\boldsymbol{b})}C_1 - b_3\Big) + \frac{C_1(\xi_{31}/g_3(\boldsymbol{b}) + \xi_{21} + \xi_{11}C_2)}{C_3\exp\{g_2(\boldsymbol{b})(g_1(\boldsymbol{b})-1)\}},$$
$$\xi_{42} = \frac{u_1^{g_1(\boldsymbol{b})}}{C_3^2|\varphi(u_2)|^2}\Big(-b_2 u_2^{g_1(\boldsymbol{b})}C_1 + b_4\Big) + \frac{C_1(\xi_{32}/g_3(\boldsymbol{b}) + \xi_{22} + \xi_{12}C_2)}{C_3\exp\{g_2(\boldsymbol{b})(g_1(\boldsymbol{b})-1)\}},$$
$$\xi_{43} = \frac{u_2^{g_1(\boldsymbol{b})}}{C_3^2|\varphi(u_1)|^2}\Big(\ \ b_3 u_1^{g_1(\boldsymbol{b})}C_1 + b_1\Big) + \frac{C_1(\xi_{33}/g_3(\boldsymbol{b}) + \xi_{23} + \xi_{13}C_2)}{C_3\exp\{g_2(\boldsymbol{b})(g_1(\boldsymbol{b})-1)\}},$$
$$\xi_{44} = \frac{u_1^{g_1(\boldsymbol{b})}}{C_3^2|\varphi(u_2)|^2}\Big(-b_4 u_2^{g_1(\boldsymbol{b})}C_1 - b_2\Big) + \frac{C_1(\xi_{34}/g_3(\boldsymbol{b}) + \xi_{24} + \xi_{14}C_2)}{C_3\exp\{g_2(\boldsymbol{b})(g_1(\boldsymbol{b})-1)\}},$$

where $C_1 = u_2\Im\psi(u_1) - u_1\Im\psi(u_2)$, $C_2 = \pi/2[\cot(\pi g_1(\boldsymbol{b})/2) + \tan(\pi g_1(\boldsymbol{b})/2)]$, $C_3 = u_2\exp\{g_1(\boldsymbol{b})\ln u_1\} - u_1\exp\{g_1(\boldsymbol{b})\ln u_2\}$. *If* $\alpha = 1$, *then* $\boldsymbol{\xi}_3$ *has components*

$$\xi_{31} = \frac{\pi}{2\exp\{g_2(\boldsymbol{b})\}}\left(\xi_{21} - \frac{b_3}{u_1|\varphi(u_1)|^2\ln(u_2/u_1)}\right),$$
$$\xi_{32} = \frac{\pi}{2\exp\{g_2(\boldsymbol{b})\}}\left(\xi_{22} + \frac{b_4}{u_2|\varphi(u_2)|^2\ln(u_2/u_1)}\right),$$
$$\xi_{33} = \frac{\pi}{2\exp\{g_2(\boldsymbol{b})\}}\left(\xi_{23} + \frac{b_1}{u_1|\varphi(u_1)|^2\ln(u_2/u_1)}\right),$$
$$\xi_{34} = \frac{\pi}{2\exp\{g_2(\boldsymbol{b})\}}\left(\xi_{24} - \frac{b_2}{u_2|\varphi(u_2)|^2\ln(u_2/u_1)}\right),$$

and $\boldsymbol{\xi}_4$ *has components*

$$\xi_{41} = \frac{-b_3\ln u_2}{u_1\ln(u_2/u_1)} + \frac{2}{\pi}(\xi_{31}g_2(\boldsymbol{b})\exp\{g_2(\boldsymbol{b})\} + \xi_{31}g_4(\boldsymbol{b})(g_2(\boldsymbol{b})+1)),$$
$$\xi_{42} = \frac{b_4\ln u_1}{u_2\ln(u_2/u_1)} + \frac{2}{\pi}(\xi_{32}g_2(\boldsymbol{b})\exp\{g_2(\boldsymbol{b})\} + \xi_{32}g_4(\boldsymbol{b})(g_2(\boldsymbol{b})+1)),$$
$$\xi_{43} = \frac{b_1\ln u_2}{u_1\ln(u_2/u_1)} + \frac{2}{\pi}(\xi_{33}g_2(\boldsymbol{b})\exp\{g_2(\boldsymbol{b})\} + \xi_{33}g_4(\boldsymbol{b})(g_2(\boldsymbol{b})+1)),$$
$$\xi_{44} = \frac{-b_2\ln u_1}{u_2\ln(u_2/u_1)} + \frac{2}{\pi}(\xi_{34}g_2(\boldsymbol{b})\exp\{g_2(\boldsymbol{b})\} + \xi_{34}g_4(\boldsymbol{b})(g_2(\boldsymbol{b})+1)).$$

Proof. From elementary complex analysis,

$$\boldsymbol{b} = (\Re\varphi(u_1), \Re\varphi(u_2), \Im\varphi(u_1), \Im\varphi(u_2))',$$

$\varphi(u) = \exp\{\psi(u)\}$, and

$$\boldsymbol{b}_n = (\Re\varphi_n(u_1), \Re\varphi_n(u_2), \Im\varphi_n(u_1), \Im\varphi_n(u_2))'$$

with $\varphi_n(u) = \exp\{\psi_n(u)\}$ given by (10). It is easy to see, that for every $u \in \mathbb{R}$ the empirical characteristic function is an unbiased estimator of the corresponding characteristic function

$$\mathbb{E}\varphi_n(u) = \frac{1}{n}\sum_{j=1}^{n}\mathbb{E}[e^{iuY_j}] = \varphi_{Y_1}(u) = \varphi(u),$$

where $Y_1, \ldots, Y_n$ form a real valued random sample of i.i.d. variables with $\varphi_{Y_1}(u) = \varphi(u)$. Thereby, $\mathbb{E}\boldsymbol{b}_n = \boldsymbol{b}$. At every fixed point $u \in \mathbb{R}$, $\varphi_n(u) \to \varphi(u)$ both almost surely and in mean square as $n \to \infty$ (e.g., [35]). Then, for every fixed pair of positive real arguments (u_1, u_2) with $u_1 \neq u_2$,

$$\boldsymbol{b}_n \overset{a.s.}{\to} \boldsymbol{b}, \text{ as } n \to \infty.$$

By continuous mapping theorem (e.g., [36, Theorem 2.3]),

$$g_j(\boldsymbol{b}_n) \overset{a.s.}{\to} g_j(\boldsymbol{b}), \text{ as } n \to \infty,$$

and the estimators $g_j(\boldsymbol{b}_n)$ are consistent for $g_j(\boldsymbol{b})$, $j = 1, \ldots, 4$. The quantities $\Re\varphi_n(u_1)$, $\Re\varphi_n(u_2)$, $\Im\varphi_n(u_1)$, $\Im\varphi_n(u_2)$ in $\boldsymbol{b}_n$ are sample means of i.i.d. random variables with $\mathbb{E}\boldsymbol{b}_n = \boldsymbol{b}$ and finite variance. Applying the multivariate central limit theorem to $\boldsymbol{b}_n$ for fixed u_1, u_2 it yields

$$\sqrt{n}(\boldsymbol{b}_n - \boldsymbol{b}) \overset{d}{\to} N_4(\boldsymbol{0}, \boldsymbol{\Lambda}), \text{ as } n \to \infty,$$

where $\boldsymbol{\Lambda}$ is the 4×4 asymptotic covariance matrix with components as given by (16) (that easily follow from the product-to-sum identities of cosine and sine functions, see, also, [35, p. 162]). For more on the convergence we refer, e.g., to [8, Theorem 3.1] and [7, p. 22]. Applying the Theorem 3.1.3 in [17] we have for $j = 1, 2, 3, 4$

$$\sqrt{n}\,(g_j(\boldsymbol{b_n}) - g_j(\boldsymbol{b})) \overset{d}{\to} N_1(0, \boldsymbol{\xi}_j'\boldsymbol{\Lambda}\boldsymbol{\xi}_j), \quad \text{as } n \to \infty$$

where

$$\boldsymbol{\xi}_j = \frac{\mathrm{d}}{\mathrm{d}\,\boldsymbol{x}} g_j(\boldsymbol{x})\bigg|_{\boldsymbol{x}=\boldsymbol{b}} \neq 0$$

are the matrix derivatives [17, Definition 1.4.1]. The components of ξ_j, $j = 1, 2, 3, 4$ easily follow from the matrix derivatives of $g_j(\boldsymbol{b})$, $j = 1, 2, 3, 4$, given by (6)–(9b), and elementary complex analysis. □

5. Selection of the arguments u_1 and u_2

First we illustrate the ECuF estimates for the tail index α at various choice of $(u_1, u_2) \in (0, 1] \times (0, 1]$. A single replicate from $S(\alpha = 1.5, \beta = 0.5; 0)$ and a single replicate from $S(\alpha = 0.5, \beta = 0.5; 0)$ are simulated with the sample size $n = 10000$. The ECuF estimator $\alpha_n(u_1, u_2) = g_1(b_n)$ is given by Definition 1. The ECuF estimates $\hat{\alpha}_n(u_1, u_2) = g_1(\hat{b}_n)$ are obtained by `R` [30] at $(u_1, u_2) \in (0, 1] \times (0, 1]$ with step size 0.01 (i.e., at 100×100 pairs of arguments). Results are presented in Figure 1. Note that ECuF estimators are not defined for $u_1 = u_2$ and in all illustrative figures those pairs (i.e., the diagonal) have no values. Due to the definition of cumulant function of stable laws ECuF estimates behave symmetrically with respect to the diagonal of $(0,1]\times(0,1]$. In Figure 1 smaller values relate to underestimating and bigger

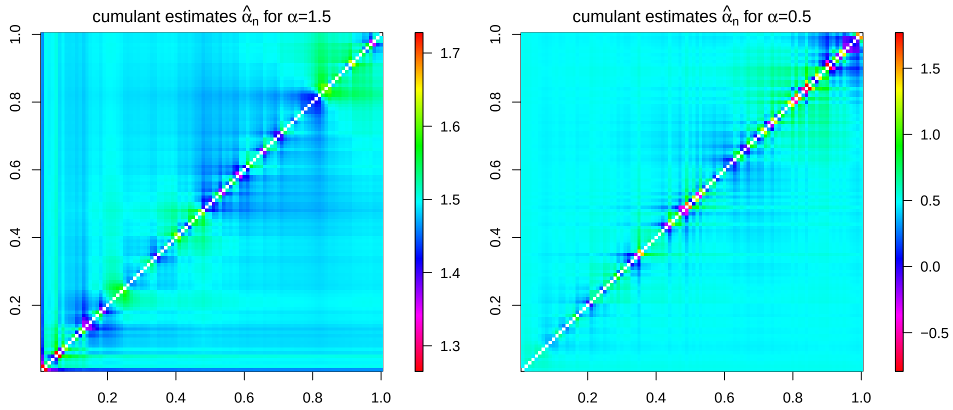

FIGURE 1. ECuF estimates at $(u_1, u_2) \in (0,1] \times (0,1]$ for $\alpha = 1.5$ (on left) and $\alpha = 0.5$ (on right).

values to overestimating, comparing to the actual values of $\alpha = 1.5$ (on right) and $\alpha = 0.5$ (on left). For $\alpha = 0.5$ the ECuF estimates turn out negative, that is, outside of the parameter space of $\alpha \in (0,2]$. Based on Figures 1 the

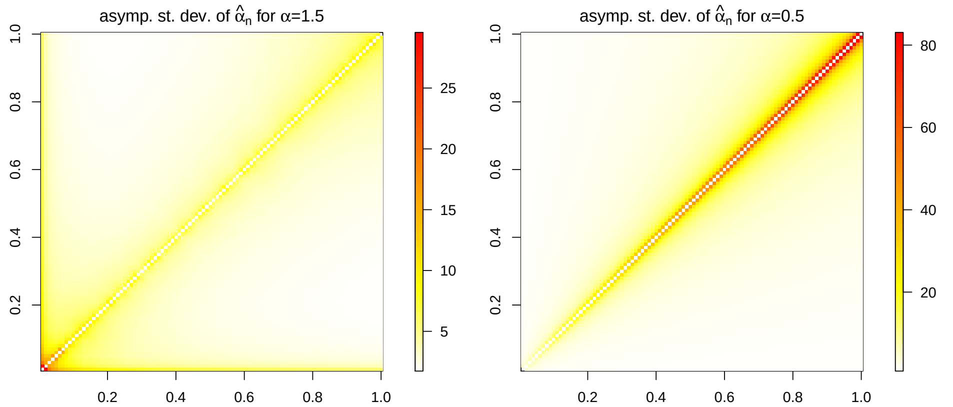

FIGURE 2. The asymptotic standard deviation of the ECuF estimates at $(u_1, u_2) \in (0,1] \times (0,1]$ for $\alpha = 1.5$ (on left) and $\alpha = 0.5$ (on right).

ECuF estimates at $(u_1, u_2) \in (0,1] \times (0,1]$ vary a lot form the actual values $\alpha = 1.5$ and $\alpha = 0.5$ even for a quite large sample such as $n = 10000$. ECuF estimates for $\alpha = 1.5$ give relatively smaller range of values comparing to the ECuF estimates for $\alpha = 0.5$.

In addition we obtain for each $\hat{\alpha}_n = \hat{\alpha}_n(u_1, u_2)$ the corresponding asymptotic standard deviation. Note that the asymptotic variance of α_n is given by Theorem 3, $v_1(u_1, u_2) = \boldsymbol{\xi}_1' \boldsymbol{\Lambda} \boldsymbol{\xi}_1$, where the matrix calculations are made by `R` [30]. The results are presented in Figure 2. The asymptotic standard deviations of ECuF estimates for $\alpha = 1.5$ give remarkably smaller values comparing to the asymptotic standard deviation of ECuF estimates for $\alpha = 0.5$. For both $\alpha = 1.5$ and $\alpha = 0.5$ bigger values are around the diagonal of $(0, 1] \times (0, 1]$ while for $\alpha = 1.5$ the biggest values occur when at least one of the arguments is close to 0, while for $\alpha = 0.5$ the highest values occur when both arguments are close to 1.

From Figures 1 and 2 it is not clear whether the pairs of $(u_1, u_2) \in (0, 1] \times (0, 1]$ yielding small asymptotic variance also yield better ECuF estimates (comparing to other pairs). For a better comparison the absolute errors of $\hat{\alpha}_n$ are obtained and a more robust overview is provided by grouping the values of absolute errors of $\hat{\alpha}_n$ and asymptotic standard deviation of $\hat{\alpha}_n$ within their quartiles. That is, within the quartiles of the 100×100 results obtained from the single replicate with $n = 10000$. Figure 3 corresponds to $\alpha = 1.5$ and Figure 4 to $\alpha = 0.5$ with absolute errors of the ECuF estimates on left and standard deviations of the ECuF estimates on right. It is expected that

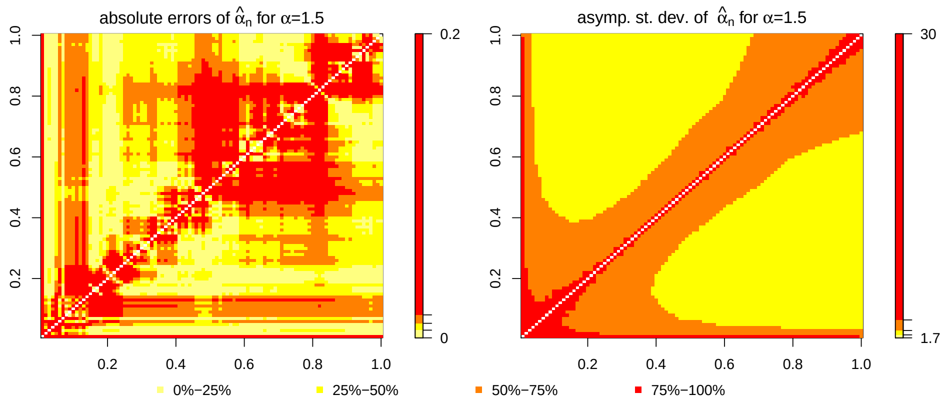

FIGURE 3. Absolute errors and asymptotic standard deviation (grouped between their quartiles) of ECuF estimates at $(u_1, u_2) \in (0, 1] \times (0, 1]$ for $\alpha = 1.5$.

if the smaller asymptotic variance would indicate the more accurate ECuF estimates then the patterns of colors on the left and right half of Figures 3 and 4 would look similar. However, it can be noticed, that the pairs of arguments u_1 and u_2 at which the asymptotic standard deviation of ECuF estimates has smallest values (lightest yellow) do not guarantee the smallest absolute errors (lightest yellow) of the estimates for α. In fact, many of the pairs of (u_1, u_2) that minimize the asymptotic standard deviation lead to quite poor estimates of α while other such pairs lead to rather good estimates of α. That leads to conclusion, that selecting u_1, u_2 for ECuF estimates based on minimizing the asymptotic variance of α_n may not be a useful approach. Note that in simulations of samples with smaller size that disparateness is even more noticeable. In conclusion, based on Figures 3 and 4, and more

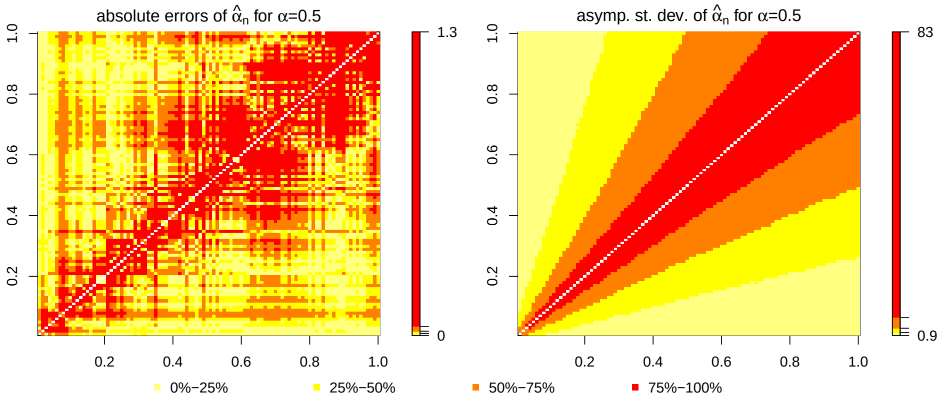

FIGURE 4. Absolute errors and asymptotic standard deviation (grouped between their quartiles) of ECuF estimates at $(u_1, u_2) \in (0, 1] \times (0, 1]$ for $\alpha = 0.5$

simulations not presented here, there is no evidence of existing a consistently good area for selecting the arguments (u_1, u_2) for ECuF estimators and the selection of $u_1 > 0$, $u_2 > 0$, $u_1 \neq u_2$ should not be based on minimizing the asymptotic variance of one or more of $g_1(\boldsymbol{b}_n)$, $g_2(\boldsymbol{b}_n)$, $g_3(\boldsymbol{b}_n)$, $g_4(\boldsymbol{b}_n)$.

For applying [29] estimators in symmetric stable laws centered around zero with $\gamma = 1$ [5] used values $u_1 = 0.5$ and $u_2 = 1.5$ while [11, p. 157] suggested selecting u_1 from the 0.3-quantile and u_2 from the 0.7-quantile of the values of the empirical characteristic function $\varphi_n(u)$.

Perhaps the most important aim of our paper is to propose suggestions on the selection of the of arguments $u_1 > 0$, $u_2 > 0$, $u_1 \neq u_2$ for estimating the parameters of general (including symmetric as well as skewed) stable laws, that is, at which the cumulant estimators $g_1(\boldsymbol{b}_n)$, $g_2(\boldsymbol{b}_n)$, $g_3(\boldsymbol{b}_n)$, and $g_4(\boldsymbol{b}_n)$ would accurately estimate $\alpha = g_1(\boldsymbol{b})$, $\ln\gamma = g_2(\boldsymbol{b})$, $\beta = g_3(\boldsymbol{b})$, and $\delta = g_4(\boldsymbol{b})$. We propose the selection of $u_1 > 0$, $u_2 > 0$, $u_1 \neq u_2$ based on the real part of empirical cumulant function, $\Re\psi_n(u)$. The idea follows from the fact, that ECuF estimators are found step-by-step: the estimators for α and γ are evaluated through $\Re\psi_n(u)$ at $u_1 > 0$, $u_2 > 0$, $u_1 \neq u_2$ and then the estimators for β and δ are evaluated through $\Re\psi_n(u)$ and $\Im\psi_n(u)$ at the same values u_1, u_2. It is easy to see that for $u \geq 0$ the absolute estimation error of the real part of $\psi_n(u)$ is

$$|\Re\psi_n(u) - \Re\psi(u)| = |1/2\ln(\Im^2\varphi_n(u) + \Re^2\varphi_n(u)) + \gamma^\alpha u^\alpha|, \qquad (17)$$

where $\Re\psi(u)$ is given by (4). The estimation error in (17) decreases when $u \to 0$ (note that, the point $u = 0$ is uninformative because $\psi_n(0) = \psi(0) = 0$ and the cumulant function holds no information about its parameters). However, as it is discussed in [27], the scale parameter $\gamma >> 1$ may dominate the estimation error in (17).

To reduce the influence of γ in (17) it is suggested to pre-standardize the data (e.g., [27], [18, 19], and [16]) or reduce the scale by dividing the data by its median [20]. In this paper we follow a different approach. From (4) it follows that $\Re\psi(1/\gamma) = -1$. By the substitution principle (e.g., [15]), it is reasonable to believe that for large sample a similar empirical relation holds for empirical characteristic function (even when γ is unknown),

$$\Re\psi_n(1/\gamma) \approx -1. \qquad (18)$$

The empirical cumulant function can oscillate and there may be multiple solutions for 18. This empirical relationship forms the basis of our selection of u_1, u_2 for the [20] cumulant estimators. More exactly, we assume that the selection of u_1 and u_2 such that $\Re\psi_n(u_k) \in (0, -1], k = 1, 2$ would concur the domination of the scale parameter γ in (17). Let $p_1 \neq p_2 \in (0, -1]$. Suggestions for the values of p_1 and p_2 are as follows: by the definition of α_n in Definition 1, and simulation results in Figure 1, it is suggested that u_1, u_2 are not too close, that is, p_1 and p_2 should not be too close. Based on (17), and Figure 5, it is suggested that $u_1 \to, u_2 \to 0$, or equivalently $p_1 \to 0, p_2 \to 0$. However, based on Figures 3 and 4, and on the definition of α_n, it is suggested that u_1, u_2 are not too close to zero. As a conclusion, a following Argument-Selection-Rule is proposed.

Argument-Selection-Rule 1. For $S(\alpha, \beta, \gamma, \delta; 0)$ the cumulant estimators, given by Definition 1, are obtained at $u_1 > 0$ and $u_2 > 0$ numerically (approximately) satisfying

$$\Re\hat{\psi}_n(u_1) = -0.1, \qquad (19)$$

$$\Re\hat{\psi}_n(u_2) = -0.5, \qquad (20)$$

where $\Re\hat{\psi}_n(u)$ is the realization of $\Re\psi_n(u)$, given by (11).

In other words, Argument-Selection-Rule 1 suggests to select u_1 as the approximate numerical solutions of (19) with respect to u_1, and u_1 as the approximate numerical solutions of (20) with respect to u_2. The empirical cumulant function can oscillate and there may be multiple solutions for (19) and (20). The selection of $u_1 > 0$ and $u_2 > 0$ can be solved by a look-up procedure with the mid-range rule, or some simple one-dimensional search function (see, e.g., [2])). One-dimensional search function are available in a variety of computer programming languages, e.g., functions `uniroot` and `optimize` in `R`-package `stats4` [30]. We illustrate the choice of $u_1 > 0$ and $u_2 > 0$ for fixed $\gamma = 1$ and $\alpha = 0.2, 1.0, 1.8$. Single replicates from $S(\alpha, \beta = 0.5; 0)$, where $\alpha = 0.2, 1.0, 1.8$, with $n = 1000$ are simulated. In Figure 5 the graphs of corresponding $\Re\psi(u)$ (dashed blue lines) and graphs of $\Re\hat{\psi}_n(u)$ of the simulated samples (solid black lines) at $u > 0$ are presented. In Figure 5, the solid red lines show the levels where $\Re\psi(u) = -0.1$ and

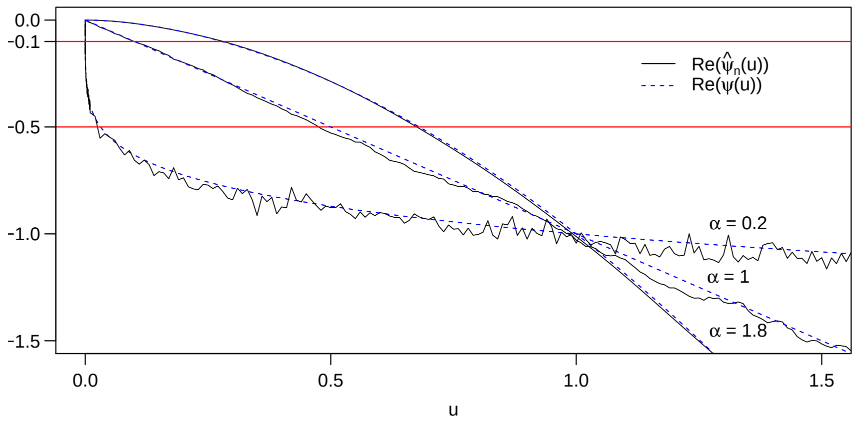

FIGURE 5. Argument-Selection-Rule 1 applied to a sample from $S(\alpha, \beta = 0.5; \cdot)$. For each sample, the values of u_1 are selected from $\Re\hat{\psi}_n(u_1) = -0.1$ and u_2 from $\Re\hat{\psi}_n(u_2) = -0.5$.

$\Re\psi(u) = -0.5$. The corresponding u_1 and u_2 where obtained through the look-up procedure by `R` [30] functions `which.min` and `which.max`) and the mid-range rule. Based on Argument-Selection-Rule 1, the ECuF estimators for the parameters of $X \sim S(\alpha, \beta = 0.5; \cdot)$ for $\alpha = 0.2$ are suggested to evaluate at $u_1 = 0.6 \times 10^{-5}$ and $u_2 = 0.03$; for $\alpha = 1$ at $u_1 = 0.11$ and $u_2 = 0.48$; and for $\alpha = 1.8$ at $u_1 = 0.29$ and $u_2 = 0.68$; and they all indeed are less than $1/\gamma$. Note that in addition to considering the unknown γ the selection by Argument-Selection-Rule 1 considers the unknown value of α. Indeed, assuming that data arises from some unknown stable distribution with $\Re\psi(u) = -\gamma^\alpha |u|^\alpha$ then for every fixed γ the solutions of (19) and (20) depend on the (unknown) tail index α.

6. Estimating $S(\alpha, \beta; 0)$ via ECuF and other estimators

In this section, Monte–Carlo simulations for assessing the quality of the empirical cumulant function (ECuF) based estimators, given by Definition 1, are carried out at the arguments u_1, u_2 selected by Argument-Selection-Rule 1. Without loss of generality, standard stable laws are studied, $\delta = 0$ and $\gamma = 1$, and by reflection property (e.g., [34, Property (2), p. 99], [26, Proposition 1.11, p. 12]) only non-negative values of β are used, $\beta \in [0, 1]$. In simulation study $K = 100$ replicates $S(\alpha, \beta; 0)$ are generated. For a simple comparison between various methods the number of replicates of $K = 100$ is considered sufficient. The ECuF estimates $\hat{\alpha}_n = g_1(\hat{\boldsymbol{b}}_n)$, $\ln \hat{\gamma}_n = g_2(\hat{\boldsymbol{b}}_n)$, $\hat{\beta}_n = g_3(\hat{\boldsymbol{b}}_n)$, and $\hat{\delta}_n = g_4(\hat{\boldsymbol{b}}_n)$ are obtained from (6)–(9b) at $\hat{\boldsymbol{b}}_n$. When $|\hat{\alpha}_n - 1| < 0.01$, then it is set $\hat{\alpha}_n = 1$. For $\hat{\alpha}_n = 1$ the estimates of β and δ are calculated by (8b) and (9b), respectively, and for $\hat{\alpha}_n \neq 1$ by (8a) and (9a), respectively. The arguments u_1 and u_2 are obtained by solving (19) and (20) by a search function combining the so-called gold section procedure with the parabolic interpolation [9], available by function `uniroot` in `R`-package `stats4` [30]. Note that in ECuF estimation procedure the values $\Re\psi_n(u_1)$ and $\Re\psi_n(u_2)$ are used, i.e., these are not replaced by the values of -0.1 and -0.5. Values of the possible parameter values are used: the values of $\hat{\beta}_n$ are taken using the equality $\hat{\beta}_n = \min(\max(\hat{\beta}_n, 0), 1)$ and the calculated values of $\hat{\alpha}_n$ are replaced with $\hat{\alpha}_n = \min(\max(\hat{\alpha}_n, 0.001), 2)$.

For comparison purposes, we calculated also estimates results for the following algorithmic methods: the maximum likelihood (ML) based [25], the empirical characteristic function (EChF) based [16], and the quantile based (QB) estimators [22]. We did not include the closed-form logarithmic moments (Log), fractional lower order moments (FLOM), nor extreme value theory (EVT) based methods by [21] for the following reasons: EVT, FLOM, Log methods do not provide estimators for the location parameter δ; as mentioned in [21], EVT methods, although very fast, have not shown as high a performance (in the sense of the estimation error) as FLOM and Log methods; in [21] the FLOM and Log methods, though well-performing in general, did not outperform the EChF methods in estimating parameters α and γ.

All statistical computing and graphics are done by open-source free software `R` [30] while simulations and the estimates of ML, EChF and QB methods are found by its package `STABLE`® [31].

All results are reported (following [16], [21]) in terms of the root mean-square error (RMSE) of the parameter estimates, which is given by

$$\mathrm{RMSE}(\hat{\theta}_n) = \sqrt{\tfrac{1}{K} \textstyle\sum_{k=1}^{K} \left(\theta - \hat{\theta}_n(k)\right)^2},$$

where $K = 100$ is the number of replications, θ is the true parameter value and $\hat{\theta}_n(k)$ is the estimate of the parameter from the k^{th} sample, $k = 1, 2, \ldots, K$.

6.1. Estimating $S(\alpha, \beta; 0)$ from samples with $n = 5000$. From $S(\alpha, \beta = 0.5; 0)$ with $\alpha = 0.2, 0.3, 0.5, 0.8, 1, 1.2, 1.5, 1.8, 2$ the $K = 100$ replicates with size $n = 5000$ are simulated. The RMSE of $\hat{\alpha}_n, \hat{\beta}_n, \hat{\gamma}_n, \hat{\delta}_n$ versus the values of tail index α for the ECuF, ML, EChF, QB (solid red line, dashed, dotted and dash-dot lines, respectively) estimates are shown in Figure 6.

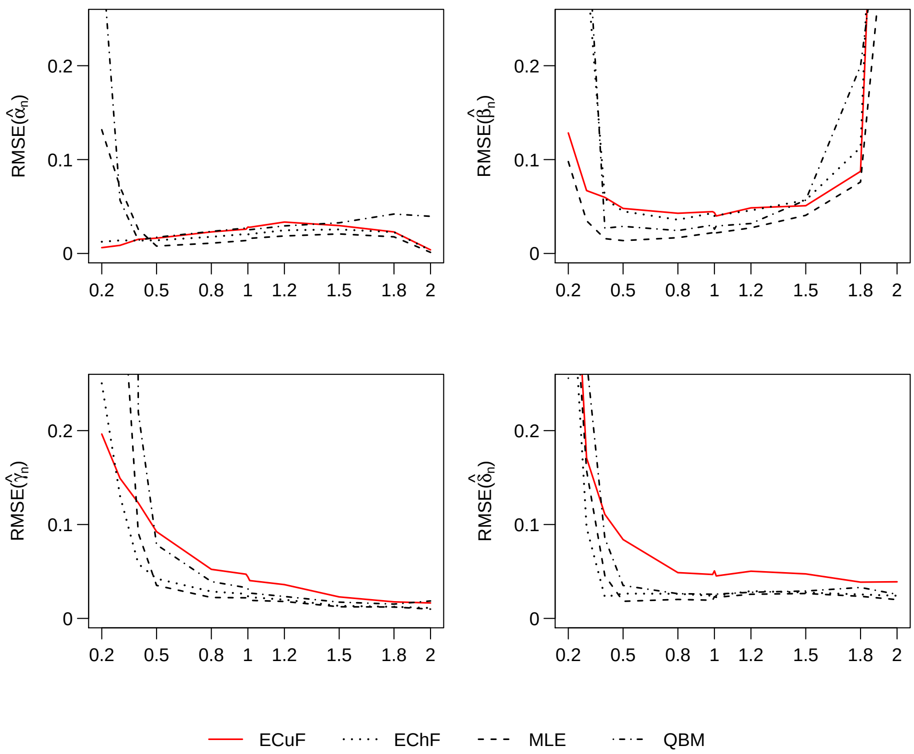

FIGURE 6. The performance of empirical cumulant function (ECuF), empirical characteristic function (EChF), maximum likelihood (ML) and quantile based (QB) estimators for α (upper left), β (upper right), γ (lower left) and δ (lower right) plotted as RMSE vs. tail index α ($n = 5000$).

The results for the tail index α: According to RMSEs of α, as shown in Figure 6 (upper left), the ECuF and EChF methods remarkably outperform other estimators at the lower values of α while the ECuF method performs slightly better than the EChF method. For $0.5 \leq \alpha \leq 1.8$, the methods give similar results while the ML method performs better. For $\alpha > 1.8$, the ECuF, EChF, and ML methods perform similarly while they all outperform the QB method.

The results for the asymmetry index β: According to RMSEs of β, as shown in Figure 6 (upper right), the ECuF and ML methods notably outperform the other estimates at the lower values of α while for $0.5 \leq \alpha \leq 1.5$

the ML and QB methods perform better than ECuF and EChF methods. For $1.5 \le \alpha \le 2$ the ECuF, EChF and ML methods outperform the QB method (the fact that RMSE $(\hat{\beta}_n) \to \infty$ as $\alpha \to 2$ is not relevant in practice as β means little when $\alpha \to 2$). In estimating the tail index α and asymmetry index β our ECuF method performs most steadily over the whole space of the values of parameter α.

The results for the scale parameter γ and the shift parameter δ: According to RMSEs of $\hat{\gamma}_n$, as shown in Figure 6 (lower left), the ECuF method outperforms other methods at the lower values of α. For $0.5 \le \alpha \le 1.2$, all methods perform similarly. Based on RMSEs of the shift parameter δ, as shown in in Figure 6 (lower right), the proposed ECuF method does not outperform others.

6.2. Estimating $S(\alpha, \beta; 0)$ from samples with various sizes. From stable laws $S(\alpha, \beta; 0)$ with $\alpha = 0.3, 1, 1.8$ and $\beta = 0, 1$ $K = 100$ replicates with sample sizes n = 200, 500, 1000, 3000, 5000 are generated. The RMSEs of $\hat{\alpha}_n$, $\hat{\beta}_n$, $\hat{\gamma}_n$, $\hat{\delta}_n$ of the ECuF, ML, EChF, QB (solid red line, dashed, dotted and dash-dot lines, respectively) estimates versus the sample size are presented in Appendix 7, Figures A1–A4 with Figure A1 showing results for α, Figure A2 presenting results for β, Figure A3 showing results for γ, and Figure A4 showing results for δ. Not surprisingly, the effectiveness of all methods under discussion depends on the sample size (the greater the sample size the better the performance). However, the results in Appendix 7 show that the ECuF method performs best at the lower values of α and in comparison to other methods, it is more robust to changes in values of β.

7. Conclusions

We provide new insights to the empirical cumulant function (ECuF) based estimation by [20]. Under proposed Argument-Selection-Rule 1 the simulation results show that ECuF method compares favourably with the quantile (QB), empirical characteristic function (EChF), and maximum likelihood (ML) based estimation methods. In estimating the tail index α, the closed-form ECuF estimators outperform other methods in the case when $\alpha < 0.5$, while at the higher values of α the the closed-form ECuF estimators perform similarly to the algorithmic methods. In estimating the parameters β, γ, δ the ECuF method outperforms algorithmic methods in some cases but not always.

The main argument in favour of the closed-form ECuF estimators is their computational simplicity: there is no need for data standardization, no restrictions have to be put to the parameter space, and they perform steadily across the values of the tail index α and skewness index β.

Appendix: Figures

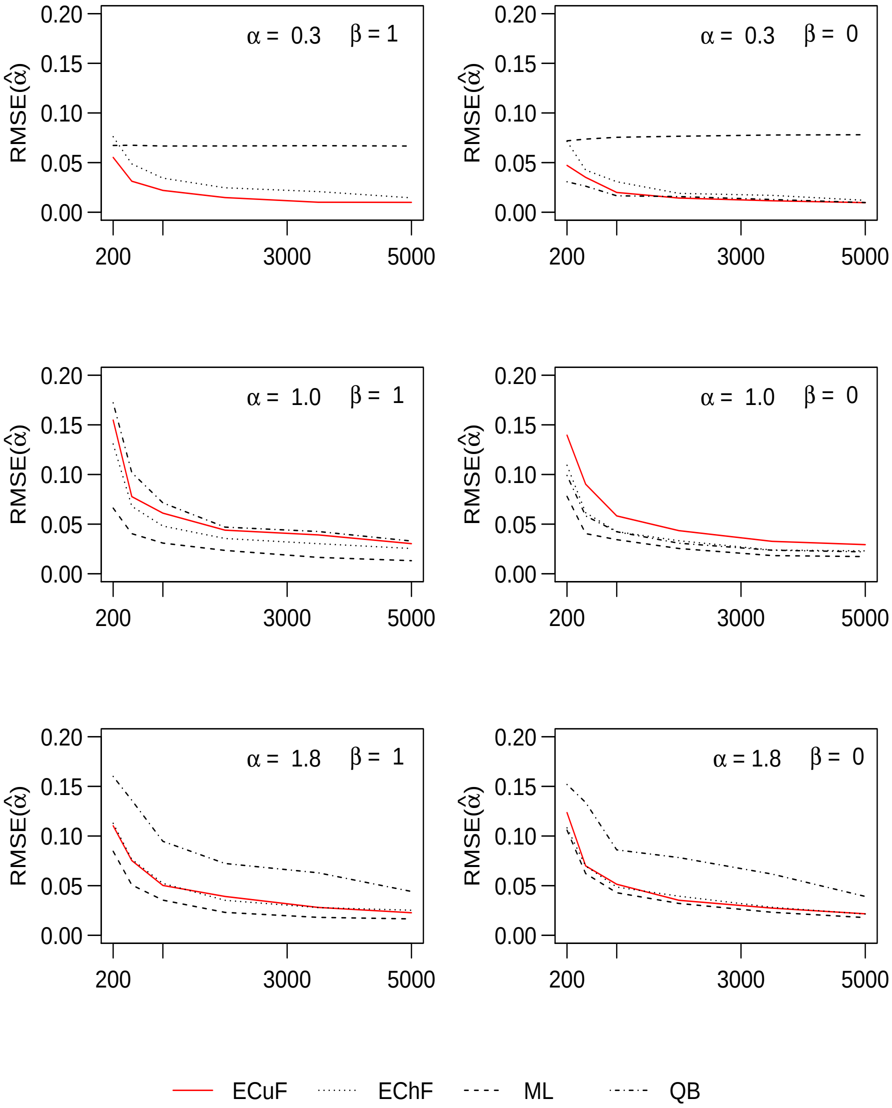

FIGURE A1. The performance of empirical cumulant function (ECuF), empirical characteristic function (EChF), maximum likelihood (ML) and quantile based (QB) estimators for parameter α of $K = 100$ replicates from $S(\alpha, \beta; 0)$ plotted as RMSE vs. sample size n.

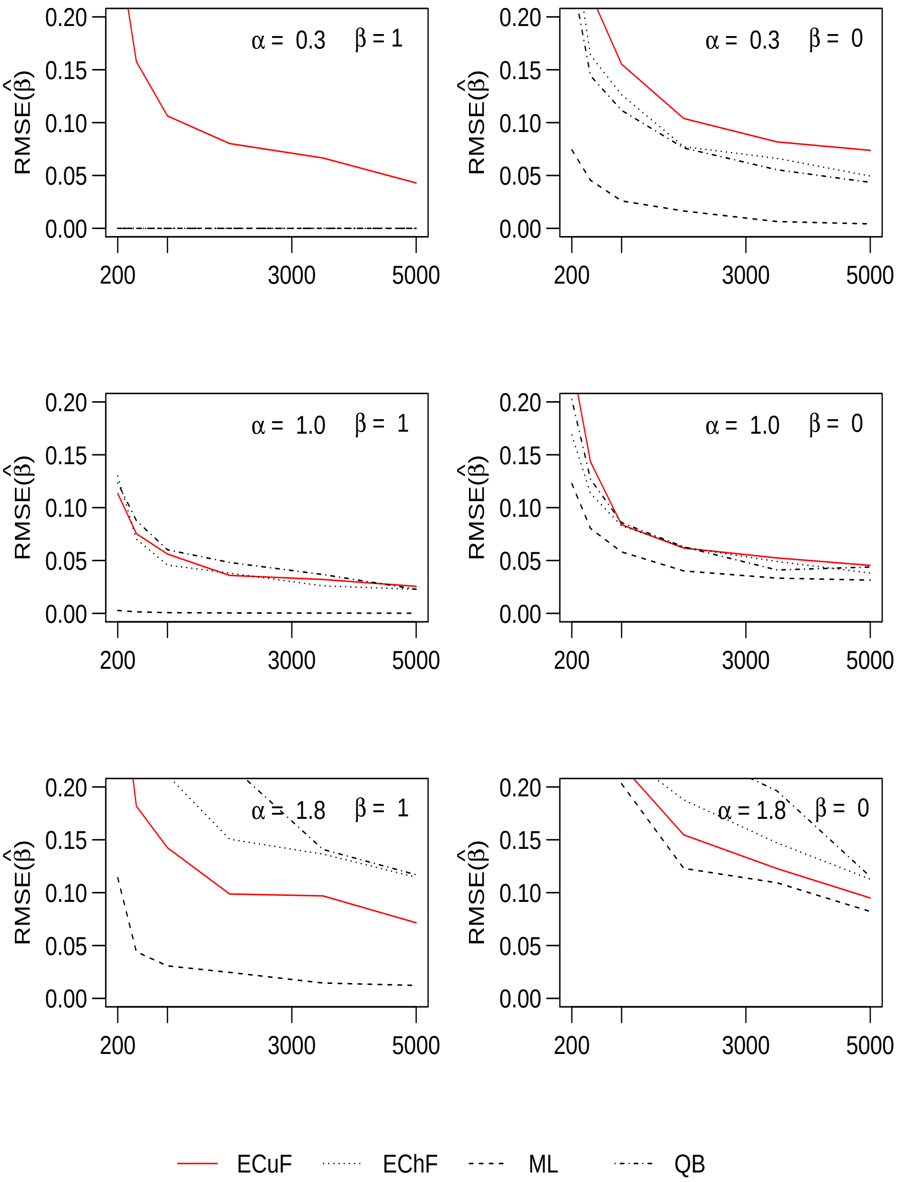

Figure A2. The performance of empirical cumulant function (ECuF), empirical characteristic function (EChF), maximum likelihood (ML) and quantile based (QB) estimators for parameter β of $K = 100$ replicates from $S(\alpha, \beta; 0)$ plotted as RMSE vs. sample size n.

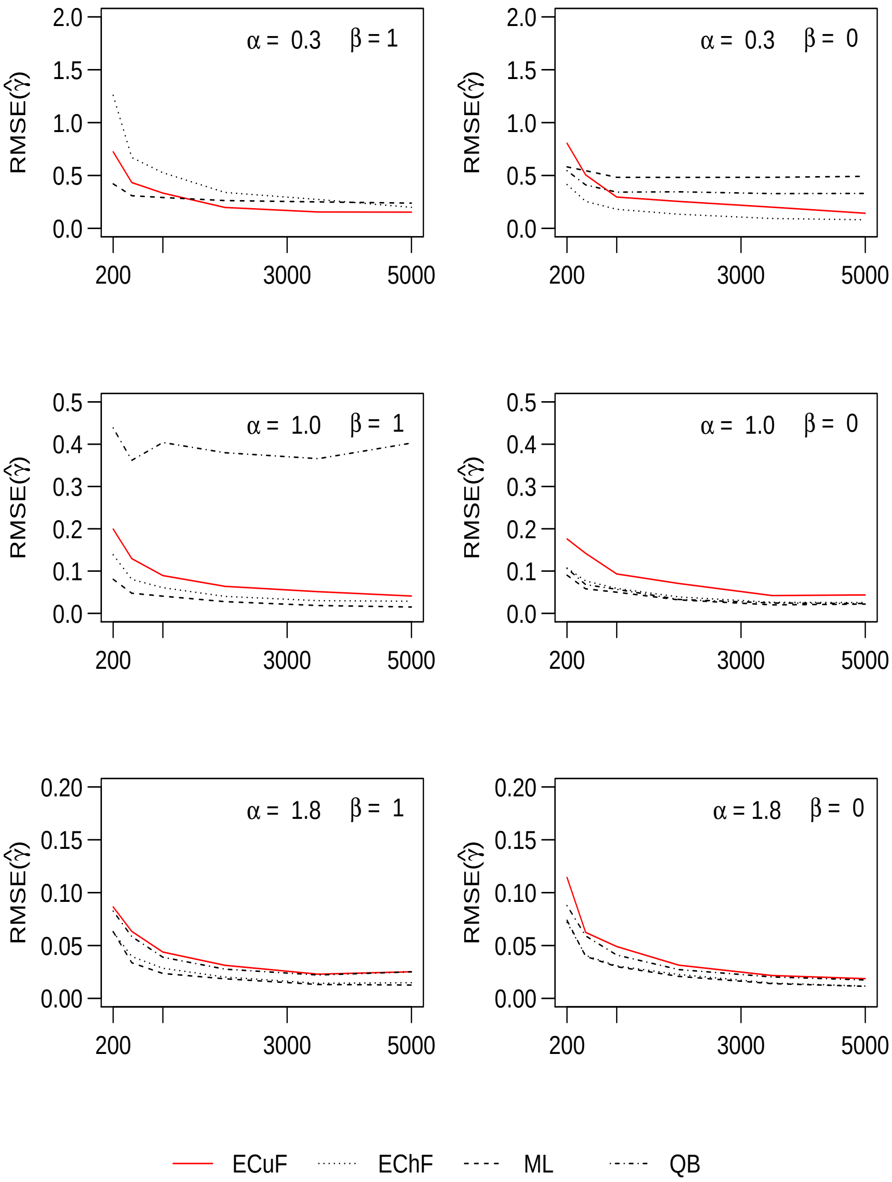

FIGURE A3. The performance of empirical cumulant function (ECuF), empirical characteristic function (EChF), maximum likelihood (ML) and quantile based (QB) estimators for parameter γ of $K = 100$ replicates from $S(\alpha, \beta; 0)$ plotted as RMSE vs. sample size n.

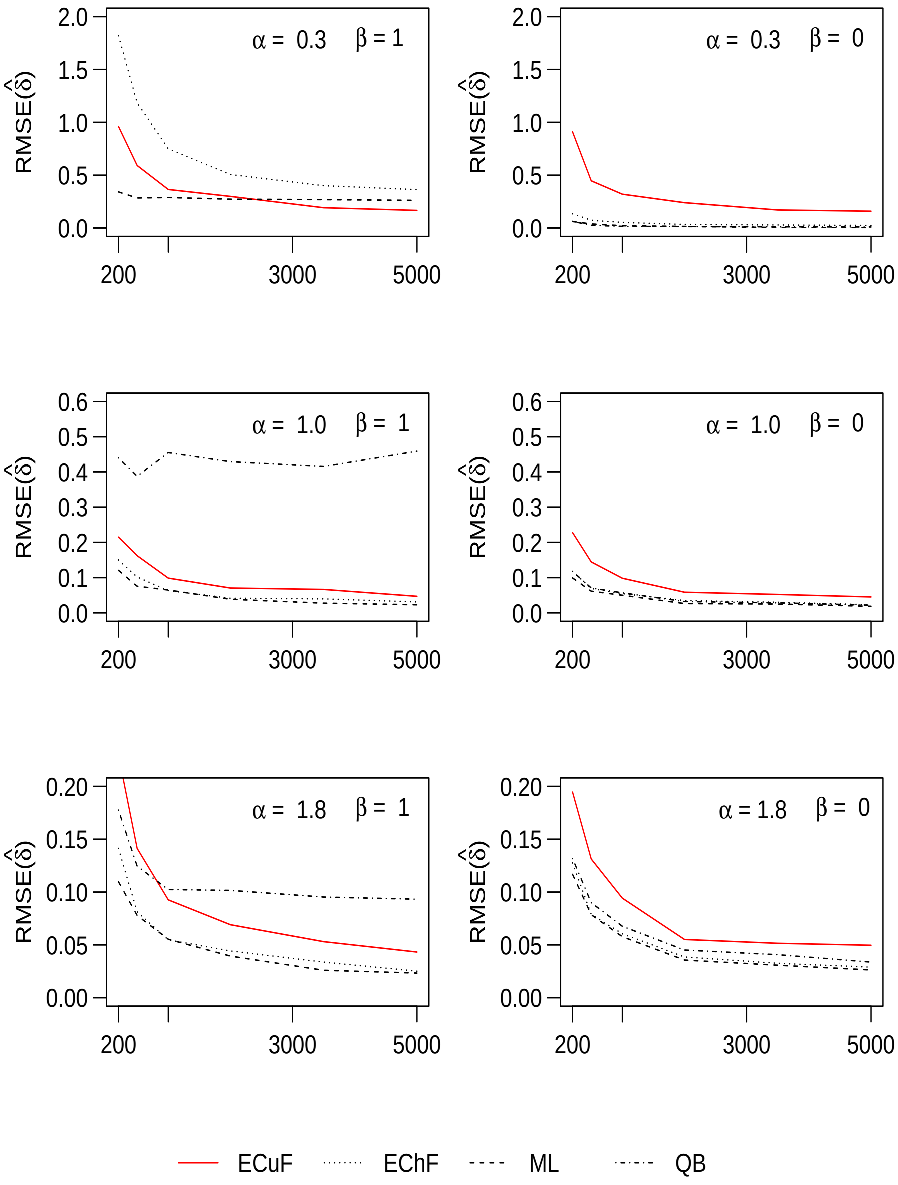

FIGURE A4. The performance of empirical cumulant function (ECuF), empirical characteristic function (EChF), maximum likelihood (ML) and quantile based (QB) estimators for parameter δ of $K = 100$ replicates from $S(\alpha, \beta; 0)$ plotted as RMSE vs. sample size n.

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References

[1] S. Borak, W. K. Härdle, and R. Weron, *Stable distributions*, in: P. Cizek, W. K. Härdle, and R. Weron (eds.) Statistical Tools for Finance and Insurance, Springer, Berlin Heidelberg, 2005.

[2] R. P. Brent, *Algorithms for Minimization Without Derivatives*, Dover Publications, 2013.

[3] J. M. Chambers, C. L. Mallows, and B. W. Stuck, *A method for simulating stable random variables*, J. Amer. Statist. Assoc. **71**(354) (1976), 340–344.

[4] P. Embrechts, C. Klüppelberg, and T. Mikosch, *Modelling Extremal Events: for Insurance and Finance*, Springer, Berlin Heidelberg, 2013.

[5] Z. Fan, *Parameter estimation of stable distributions*, Comm. Statist. Theory Methods **35**(2) (2006), 245–255.

[6] W. Feller, *An Introduction to Probability Theory and its Applications. Vol. II*, 2nd ed., 2nd ed., John Wiley & Sons, New York, 1971.

[7] A. Feuerverger and P. McDunnough, *On the efficiency of empirical characteristic function procedures*, J. R. Stat. Soc. Ser. B. Stat. Methodol. **43**(1) (1981), 20–27.

[8] A. Feuerverger and R. A. Mureika, *The empirical characteristic function and its applications*, Ann. Statist. **5**(1) (1977), 88–97.

[9] G. E. Forsythe, M. A. Malcolm, and C. B. Moler, *Computer Methods for Mathematical Computations*, Prentice Hall Professional Technical Reference, 1977.

[10] C. R. Heathcote, *The integrated squared error estimation of parameters*, Biometrika **64**(2) (1977), 255–264.

[11] R. Höpfner and L. Rüschendorf, *Comparison of estimators in stable models*, Math. Comput. Model. **29**(10) (1999), 145–160.

[12] H. S. Kasana, *Complex Variables: Theory and Applications*, Prentice-Hall of India Private Limited, New Delhi, 2005.

[13] A. Ya. Khintchine, A new derivation of a formula by P. Lèvy, Bull. Mosc. State Univ. 1(1) (1937), 1–5. (Russian).

[14] J. L. Knight and S. E. Satchell, *The cumulant generating function estimation method: Implementation and asymptotic efficiency*, Econometric Theory **13**(2) (1997), 170–184.

[15] K. Knight, *Mathematical Statistics*, CRC Press, 1999.

[16] S. M. Kogon and D. B. Williams, *Characteristic function based estimation of stable distribution parameters*, in: R. J. Adler, R. E. Feldman, and M. S. Taqqu (eds.) A Practical Guide to Heavy Tails, Birkhäuser Basel, Boston, 1998.

[17] T. Kollo and D. von Rosen, *Advanced Multivariate Statistics with Matrices*, Springer, Dordrecht, 2005.

[18] I. A. Koutrouvelis, *Regression-type estimation of the parameters of stable laws*, J. Amer. Statist. Assoc. **75**(372) (1980), 918–928.

[19] I. A. Koutrouvelis, *An iterative procedure for the estimation of the parameters of stable laws*, Comm. Statist. Simulation Comput. **10**(1) (1981), 17–28.

[20] A. Krutto, *Parameter estimation in stable law*, Risks **4**(4) (2016), 43.

[21] E. E. Kuruoglu, *Density parameter estimation of skewed alpha-stable distributions*, IEEE Trans. Signal Process. **49**(10) (2001), 2192–2201.

[22] J. H. McCulloch, *Simple consistent estimators of stable distribution parameters*, Comm. Statist. Simulation Comput. **15**(4) (1986), 1109–1136.

[23] M. M. Meerschaert and H.-P. Scheffler, *Limit Distributions for Sums of Independent Random Vectors: Heavy Tails in Theory and Practice* **321**, John Wiley & Sons, New York, 2001.

[24] S. Mittnik, S. T. Rachev, T. Doganoglu, and D. Chenyao, *Maximum likelihood estimation of stable Paretian models*, Mathematical and Computer Modelling **29**(10) (1999), 275 –293.

[25] J. P. Nolan, *Maximum likelihood estimation and diagnostics for stable distributions*, in: O. E. Barndorff-Nielsen, S. I. Resnick, and T. Mikosch (eds.) Lévy Processes, Birkhäuser, Boston, 2001.

[26] J. P. Nolan, *Stable Distributions: Models for Heavy Tailed Data*, Birkhäuser, Boston, 2018. In progress, Chapter 1 online at `http://fs2.american.edu/jpnolan/www/stable/stable.html`.

[27] A. S. Paulson, E. W. Holcomb, and R. A. Leitch, *The estimation of the parameters of the stable laws*, Biometrika **62**(1) (1975), 163–170.

[28] E. J. G. Pitman and J. Pitman, *A direct approach to the stable distributions*, Advances in Applied Probability **48**(A) (2016), 261–282.

[29] J. S. Press, *Estimation in univariate and multivariate stable distributions*, J. Amer. Statist. Assoc. **67**(340) (1972), 842–846.

[30] R Core Team, *R: A language and environment for statistical computing*, R Foundation for Statistical Computing, Vienna, Austria, 2016. `https://www.R-project.org/`.

[31] Robust Analysis Inc., *Stable 5.3 r version for windows*, Robust Analysis Inc., Washington, DC, 2017. `http://www.robustanalysis.com/`.

[32] G. Samorodnitsky and M. S. Taqqu, *Stable Non-Gaussian Random Processes: Stochastic Models with Infinite Variance*, Chapman & Hall, New York, 1994.

[33] K. Sato, *Lévy Processes and Infinitely Divisible Distributions*, Cambridge University Press, 1999.

[34] V. V. Uchaikin and V. M. Zolotarev, *Chance and Stability: Stable Distributions and their Applications*, Walter de Gruyter, Utrecht, 1999.

[35] N. G. Ushakov, *Selected Topics in Characteristic Functions*, Walter de Gruyter, Utrecht, 1999.

[36] A. W. van der Vaart, *Asymptotic Statistics*, Cambridge University Press, 1998.

[37] V. M. Zolotarev, *One-Dimensional Stable Distributions*, Translations of Mathematical Monographs **65**, American Mathemathical Society, Providence, Rhode Island, 1986.

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