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Pseudo-symmetric structures on almost Kenmotsu manifolds with nullity distributions

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Abstract. The object of the present paper is to characterize Ricci pseudosymmetric and Ricci semisymmetric almost Kenmotsu manifolds with (k,μ)-, $(k,\mu)'$-, and generalized (k,μ)-nullity distributions. We also characterize (k,μ)-almost Kenmotsu manifolds satisfying the condition $R \cdot S = L_S Q(g, S^2)$.

1. Introduction and preliminaries

A differentiable $(2n+1)$-dimensional manifold M is said to have a (ϕ, ξ, η)-structure, or an almost contact structure, if it admits a $(1,1)$-tensor field ϕ, a characteristic vector field ξ, and a 1-form η satisfying (see [1], [2])

$$\phi^2 = -I + \eta \otimes \xi, \ \eta(\xi) = 1, \tag{1.1}$$

where I denotes the identity endomorphism. Here $\phi\xi = 0$ and $\eta \circ \phi = 0$; both can be derived from (1.1) easily.

If a manifold M with a (ϕ, ξ, η)-structure admits a Riemannian metric g such that

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y)$$

for any vector fields X, Y in M, then M is said to be an almost contact metric manifold. The fundamental 2-form Φ on an almost contact metric manifold is defined by $\Phi(X, Y) = g(X, \phi Y)$ for any vector fields X, Y in M. The condition for an almost contact metric manifold of being normal is equivalent to the vanishing of the $(1,2)$-type torsion tensor

$$N_\phi = [\phi, \phi] + 2d\eta \otimes \xi,$$

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where $[\phi, \phi]$ is the Nijenhuis tensor of ϕ (see [1]). Recently (see, for example, [6], [7], [9]) almost contact metric manifolds such that η is closed and $d\Phi = 2\eta \wedge \Phi$ have been studied; they are called almost Kenmotsu manifolds. Obviously, a normal almost Kenmotsu manifold is a Kenmotsu manifold. It is well known (see [8]) that a $(2n+1)$-dimensional Kenmotsu manifold M^{2n+1} is locally a warped product $I \times_f N^{2n}$, where N^{2n} is a Kähler manifold, I is an open interval with coordinate t, and the warping function $f(t) = ce^t$ for some positive constant c.

In the present time, the study of nullity distributions is a very interesting topic on almost contact metric manifolds. Blair et al. [3] introduced the notion of a (k, μ)-nullity distribution on a contact metric manifold $(M^{2n+1}, \phi, \xi, \eta, g)$, which is defined for any $p \in M$ and $k, \mu \in \mathbb{R}$ by

$$\begin{aligned} N_p(k, \mu) = \{Z \in T_pM : R(X, Y)Z = k[g(Y, Z)X - g(X, Z)Y] \\ + \mu[g(Y, Z)hX - g(X, Z)hY]\}, \end{aligned} \tag{1.2}$$

where $h = \frac{1}{2}\pounds_\xi\phi$ and $\pounds$ denotes the Lie differentiation.

Dileo and Pastore [7] introduced the notion of a $(k, \mu)'$-nullity distribution on an almost Kenmotsu manifold $(M^{2n+1}, \phi, \xi, \eta, g)$, which is defined for any $p \in M^{2n+1}$ and $k, \mu \in \mathbb{R}$ as follows:

$$\begin{aligned} N_p(k, \mu)' = \{Z \in T_pM : R(X, Y)Z = k[g(Y, Z)X - g(X, Z)Y] \\ + \mu[g(Y, Z)h'X - g(X, Z)h'Y]\}, \end{aligned} \tag{1.3}$$

where $h' = h \circ \phi$.

Let M be a $(2n+1)$-dimensional Riemannian manifold with metric g, and let $T(M)$ be the Lie algebra of differentiable vector fields in M. The Ricci operator Q of type $(1, 1)$ and the $(0, 2)$-tensor S^2 are defined, respectively, by

$$g(QX, Y) = S(X, Y)$$

and

$$S^2(X, Y) = S(QX, Y), \tag{1.4}$$

where S denotes the Ricci tensor of type $(0, 2)$ on M and $X, Y \in T(M)$. We define an endomorphism $X \wedge_A Y$ of $T(M)$ by

$$(X \wedge_A Y)Z = A(Y, Z)X - A(X, Z)Y, \tag{1.5}$$

where A is a symmetric $(0, 2)$-tensor and $X, Y, Z \in T(M)$. The $(0, 4)$-tensor $\tilde{R}$ of M is defined by

$$\tilde{R}(X, Y, Z, W) = g(R(X, Y)Z, W),$$

where R is the Riemannian curvature tensor defined by

$$R(X, Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z.$$

For a $(0,k)$-tensor field T, $k \geq 1$ and a $(0,2)$-tensor field A on M, we define the tensors $R.T$ and $Q(A,T)$, respectively, by

$$\begin{aligned}(R \cdot T)(X_1, X_2, \ldots, X_k; X, Y) = &-T(R(X,Y)X_1, X_2, \ldots, X_k)\\ &- \cdots - T(X_1, X_2, \ldots, R(X,Y)X_k)\end{aligned}$$

and

$$\begin{aligned}Q(A,T)(X_1, X_2, \ldots, X_k; X, Y) = &-T((X \wedge_A Y)X_1, X_2, \ldots, X_k)\\ &- \cdots - T(X_1, X_2, \ldots, (X \wedge_A Y)X_k).\end{aligned} \tag{1.6}$$

A Riemannian manifold M is said to be Ricci pseudosymmetric (see [10]) if the tensor fields $R \cdot S$ and $Q(g,S)$ are linearly dependent, i.e., there exists a function $L_S \colon M \to \mathbb{R}$ such that $R \cdot S = L_S Q(g,S)$ holds on M. In particular, a Ricci pseudosymmetric manifold with $L_S = 0$ reduces to a Ricci semisymmetric manifold.

Let M^{2n+1} be a $(2n+1)$-dimensional almost Kenmotsu manifold. The tensor fields $l = R(\cdot, \xi)\xi$ and $h = \frac{1}{2}\pounds_\xi \phi$ are symmetric operators and they satisfy the relations (see [9])

$$h\xi = 0,\ l\xi = 0,\ tr(h) = 0,\ tr(h\phi) = 0,\ h\phi + \phi h = 0,$$

$$\nabla_X \xi = X - \eta(X)\xi - \phi h X (\Rightarrow \nabla_\xi \xi = 0),$$

$$l\phi - l = 2(h^2 - \phi^2),$$

$$R(X,Y)\xi = \eta(X)(Y - \phi h Y) - \eta(Y)(X - \phi h X) + (\nabla_Y \phi h)X - (\nabla_X \phi h)Y$$

for any vector fields X, Y. The $(1,1)$-type symmetric tensor field $h' = h \circ \phi$ is anticommuting with ϕ and $h'\xi = 0$. Also, it is clear that (see [7], [11])

$$h = 0 \Leftrightarrow h' = 0,\ \ h'^2 = (k+1)\phi^2\ (\Leftrightarrow h^2 = (k+1)\phi^2). \tag{1.7}$$

Almost Kenmotsu manifolds have been studied by several authors. Among them, Wang and Liu [11] study ξ-Riemannian semisymmetric almost Kenmotsu manifolds satisfying $(k,\mu)'$-nullity and (k,μ)-nullity distributions. Recently, Deshmukh et al. [5] studied Ricci semisymmetric almost Kenmotsu manifolds with nullity distributions. Pseudosymmetric almost Kenmotsu manifolds have been studied by Wang et al. [13]. In the present paper, we study some curvature conditions imposed on the Ricci curvature tensor of almost Kenmotsu manifolds with (k,μ)-, $(k,\mu)'$- and generalized (k,μ)-nullity distributions, by generalizing the results of Deshmukh et al. [5], and Wang et al. [13].

The paper is organized as follows. In Section 1, we introduce the notation and give a brief account on almost Kenmotsu manifolds with ξ belonging to the (k,μ)-nullity distribution and ξ belonging to the $(k,\mu)'$-nullity distribution. Section 2 deals with Ricci pseudosymmetric almost Kenmotsu manifolds and almost Kenmotsu manifolds satisfying the curvature condition $R \cdot S = L_S Q(g, S^2)$ with the characteristic vector field ξ belonging to the (k,μ)-nullity distribution. Section 3 is devoted to the study of Ricci pseudosymmetric almost Kenmotsu manifolds with the characteristic vector field ξ belonging to the $(k,\mu)'$-nullity distribution. Finally, in Section 4, we discuss Ricci semisymmetric almost Kenmotsu manifolds and Ricci pseudosymmetric almost Kenmotsu manifolds with generalized (k,μ)-nullity distributions.

2. Manifolds with (k,μ)-nullity distributions

In this section we study almost Kenmotsu manifolds with ξ belonging to the (k,μ)-nullity distribution, and satisfying the curvature condition

$$R\cdot S = L_S Q(g, S^2). \tag{2.1}$$

From (1.2) we obtain

$$R(X,Y)\xi = k[\eta(Y)X - \eta(X)Y] + \mu[\eta(Y)hX - \eta(X)hY], \tag{2.2}$$

where $k,\mu \in \mathbb{R}$. Before proving our main results in this section, we first state the following lemma.

Lemma 2.1 (see [7]). *Let M^{2n+1} be an almost Kenmotsu manifold of dimension $2n+1$. Suppose that the characterstic vector field ξ belongs to the (k,μ)-nullity distribution. Then $k=-1$, $h=0$, and M^{2n+1} is locally a warped product of an open interval and an almost Kähler manifold.*

In view of Lemma 2.1, from (2.2) it follows that

$$R(X,Y)\xi = \eta(X)Y - \eta(Y)X,$$

$$R(\xi,X)Y = -g(X,Y)\xi + \eta(Y)X, \tag{2.3}$$

$$S(X,\xi) = -2n\eta(X), \tag{2.4}$$

$$Q\xi = -2n\xi$$

for any vector fields X, Y on M^{2n+1}.

Theorem 2.1. *Let M^{2n+1} be an almost Kenmotsu manifold with the characeristic vector field ξ belonging to the (k,μ)-nullity distribution. Then the following conditions are equivalent, provided $L_S \neq -1$.*

(a) $\nabla S = 0$.
(b) $R\cdot S = 0$.
(c) *M^{2n+1} is an Einstein manifold.*
(d) $R\cdot S = L_S Q(g,S)$.

Proof. It is obvious (a) implies (b), and that (c) implies (a). It is proved by Deshmukh et al. [5] that (b) implies (c). To complete the proof it remains to prove that (c) implies (d), and that (d) implies (c).

First we prove that (d) implies (c). Using (2.3) and (2.4), we have

$$\begin{aligned}(R(\xi,X)\cdot S)(Y,Z) &= -S(R(\xi,X)Y,Z) - S(Y,R(\xi,X)Z)\\ &= -2ng(X,Y)\eta(Z) - \eta(Y)S(X,Z)\\ &\quad - \eta(Z)S(X,Y) - 2ng(X,Z)\eta(Y).\end{aligned} \tag{2.5}$$

Again, using (1.5), (1.6), and (2.4), we have

$$\begin{aligned}Q(g,S)(Y,Z;\xi,X) &= 2ng(X,Y)\eta(Z) + S(X,Z)\eta(Y)\\ &\quad + 2ng(X,Z)\eta(Y) + S(X,Y)\eta(Z).\end{aligned} \tag{2.6}$$

Since the condition $R\cdot S = L_S Q(g,S)$ is realized on M, we have

$$(R(\xi,X)\cdot S)(Y,Z) = L_S Q(g,S)(Y,Z;\xi,X). \tag{2.7}$$

So, substituting (2.5) and (2.6) into (2.7), and taking $Z=\xi$, we get

$$(L_S+1)S(X,Y) = -2n(L_S+1)g(X,Y),$$

which implies that $S(X,Y) = -2ng(X,Y)$, provided $L_S \neq -1$. This shows that M^{2n+1} is an Einstein manifold.

Conversely, if the manifold M^{2n+1} is an Einstein manifold, then $R\cdot S = 0$ and $Q(g,S)(Y,Z;U,X) = 0$, which implies that the relation $R\cdot S = L_S Q(g,S)$ holds. This completes the proof. □

Since $R\cdot R = 0$ implies $R\cdot S = 0$, the above theorem generalizes the result of Wang et al. [13].

Theorem 2.2. *Let M^{2n+1} be a $(2n+1)$-dimensional almost Kenmotsu manifold with ξ belonging to the (k,μ)-nullity distribution. If the curvature condition* (2.1) *holds on M^{2n+1}, then either M^{2n+1} is an Einstein manifold or S^2 satisfies the condition*

$$S^2(X,Y) = \frac{2n(2nL_S-1)}{L_S}g(X,Y) - \frac{1}{L_S}S(X,Y). \tag{2.8}$$

Proof. Using (2.3) and (2.4), we have

$$\begin{aligned}(R(\xi,X)\cdot S)(Y,Z) &= -S(R(\xi,X)Y,Z) - S(Y,R(\xi,X)Z)\\ &= -2ng(X,Y)\eta(Z) - \eta(Y)S(X,Z)\\ &\quad - \eta(Z)S(X,Y) - 2ng(X,Z)\eta(Y).\end{aligned} \tag{2.9}$$

On the other hand, by (1.6) we get

$$\begin{aligned}Q(g,S^2)(Y,Z;\xi,X) &= -S^2(g(X,Y)\xi - g(\xi,Y)X,Z)\\ &\quad -S^2(Y,g(X,Z)\xi - g(\xi,Z)X).\end{aligned}$$

Again, using (1.4), from the above equation we get

$$\begin{aligned}Q(g,S^2)(Y,Z;\xi,X) &= -4n^2g(X,Y)\eta(Z) + \eta(Y)S^2(X,Z)\\ &\quad - 4n^2g(X,Z)\eta(Y) + \eta(Z)S^2(X,Y).\end{aligned} \tag{2.10}$$

Since the condition (2.1) is realized on M^{2n+1}, we have

$$(R(\xi, X) \cdot S)(Y, Z) = L_S Q(g, S^2)(Y, Z; \xi, X).$$

From (2.9) and (2.10) we have

$$\begin{aligned}
-2ng(X,Y)\eta(Z) - \eta(Y)S(X,Z) - \eta(Z)S(X,Y) - 2ng(X,Z)\eta(Y) = \\
-4L_S n^2 g(X,Y)\eta(Z) + L_S \eta(Y) S^2(X,Z) \\
-4L_S n^2 g(X,Z)\eta(Y) + L_S \eta(Z) S^2(X,Y).
\end{aligned}$$

Putting $Z = \xi$ in the above equation, we obtain

$$L_S S^2(X,Y) = 2n(2nL_S - 1)g(X,Y) - S(X,Y). \tag{2.11}$$

If $L_S = 0$, then from the above equation we have

$$S(X,Y) = -2ng(X,Y), \tag{2.12}$$

that is, M is an Einstein manifold.

If $L_S \neq 0$, then from (2.11) we get (2.8). This completes the proof. □

3. Manifolds with $(k, \mu)'$-nullity distributions

In this section we study almost Kenmotsu manifolds with ξ belonging to the $(k, \mu)'$-nullity distribution, which are also Ricci pseudosymmetric and satisfy the curvature conditions (2.1). Let $X \in \mathcal{D}$ be the eigenvector of h' corresponding to the eigenvalue λ. Then from (1.7) it is clear that $\lambda^2 = -(k+1)$. Therefore, $k \leq -1$ and $\lambda = \pm\sqrt{-k-1}$. We denote by $[\lambda]'$ and $[-\lambda]'$ the corresponding eigenspaces related to the non-zero eigenvalues λ and $-\lambda$ of h', respectively. Before presenting our main theorem we recall some known results.

Lemma 3.1 (see [7], Propositions 4.1 and 4.3)**.** *Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be an almost Kenmotsu manifold such that ξ belongs to the $(k, \mu)'$-nullity distribution and $h' \neq 0$. Then $k < -1$, $\mu = -2$, and $Spec(h') = \{0, \lambda, -\lambda\}$ with 0 as simple eigenvalue and $\lambda = \sqrt{-k-1}$. The distributions $[\xi] \oplus [\lambda]'$ and $[\xi] \oplus [-\lambda]'$ are integrable with totally geodesic leaves. The distributions $[\lambda]'$ and $[-\lambda]'$ are integrable with totally umbilical leaves. Furthermore, the sectional curvatures are given by the following:*

(a) $K(X, \xi) = \begin{cases} k - 2\lambda & \textit{if } X \in [\lambda]', \\ k + 2\lambda & \textit{if } X \in [-\lambda]', \end{cases}$

(b) $K(X, Y) = \begin{cases} k - 2\lambda & \textit{if } X, Y \in [\lambda]', \\ k + 2\lambda & \textit{if } = X, Y \in [-\lambda]', \\ -(k+2) & \textit{if } X \in [\lambda]', Y \in [-\lambda]', \end{cases}$

(c) M^{2n+1} *has constant negative scalar curvature* $r = 2n(k - 2n)$.

Lemma 3.2 (see [12], Lemma 3). *Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be an almost Kenmotsu manifold with ξ belonging to the $(k, \mu)'$-nullity distribution. If $h' \neq 0$, then the Ricci operator Q of M^{2n+1} is given by*

$$Q = -2nid + 2n(k+1)\eta \otimes \xi - 2nh'.$$

Lemma 3.3 (see [7], Proposition 4.2). *Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be an almost Kenmotsu manifold such that ξ belongs to the $(k, -2)'$-nullity distribution and $h' \neq 0$. Then, for any $X_\lambda, Y_\lambda, Z_\lambda \in [\lambda]'$ and $X_{-\lambda}, Y_{-\lambda}, Z_{-\lambda} \in [-\lambda]'$, the Riemann curvature tensor satisfies the conditions*

$$R(X_\lambda, Y_\lambda)Z_{-\lambda} = 0, \quad R(X_{-\lambda}, Y_{-\lambda})Z_\lambda = 0,$$

$$R(X_\lambda, Y_{-\lambda})Z_\lambda = (k+2)g(X_\lambda, Z_\lambda)Y_{-\lambda},$$

$$R(X_\lambda, Y_{-\lambda})Z_{-\lambda} = -(k+2)g(Y_{-\lambda}, Z_{-\lambda})X_\lambda,$$

$$R(X_\lambda, Y_\lambda)Z_\lambda = (k-2\lambda)[g(Y_\lambda, Z_\lambda)X_\lambda - g(X_\lambda, Z_\lambda)Y_\lambda],$$

$$R(X_{-\lambda}, Y_{-\lambda})Z_{-\lambda} = (k+2\lambda)[g(Y_{-\lambda}, Z_{-\lambda})X_{-\lambda} - g(X_{-\lambda}, Z_{-\lambda})Y_{-\lambda}].$$

From (1.3) we have,

$$R(X, Y)\xi = k[\eta(Y)X - \eta(X)Y] + \mu[\eta(Y)h'X - \eta(X)h'Y], \tag{3.1}$$

where $k, \mu \in \mathbb{R}$. Also, from (3.1) we get

$$R(\xi, X)Y = k[g(X, Y)\xi - \eta(Y)X] + \mu[g(h'X, Y)\xi - \eta(Y)h'X]. \tag{3.2}$$

Contracting X in (3.1), we have

$$S(Y, \xi) = 2nk\eta(Y). \tag{3.3}$$

Moreover, in an almost Kenmotsu manifold with $(k, \mu)'$-nullity distribution, one has

$$\nabla_X \xi = X - \eta(X)\xi + h'X$$

and

$$(\nabla_X \eta)Y = g(X, Y) - \eta(X)\eta(Y) + g(h'X, Y).$$

Theorem 3.1. *Let M^{2n+1} be a $(2n+1)$-dimensional almost Kenmotsu manifold with ξ belonging to the $(k, \mu)'$-nullity distribution. If M^{2n+1} is Ricci pseudosymmetric, then it is an η-Einstein manifold, provided $L_S \neq -(k+2)$.*

Proof. Using (3.2) and (3.3), we have

$$\begin{aligned}(R(\xi, X) \cdot S)(Y, Z) = &-2nk^2 g(X, Y)\eta(Z) + k\eta(Y)S(X, Z) \\ &+ k\eta(Z)S(X, Y) - 2nk^2 g(X, Z)\eta(Y) \\ &+ 4nkg(h'X, Y)\eta(Z) - 2\eta(Y)S(h'X, Z) \\ &+ 4nkg(h'X, Z)\eta(Y) - 2\eta(Z)S(h'X, Y).\end{aligned} \tag{3.4}$$

Moreover,

$$Q(g,S)(Y,Z;\xi,X) = -2nkg(X,Y)\eta(Z) + S(X,Z)\eta(Y) \\ - 2nkg(X,Z)\eta(Y) + S(X,Y)\eta(Z). \tag{3.5}$$

Since M is Ricci pseudosymmetric, that is, $R \cdot S = L_S Q(g,S)$, we have

$$(R(\xi,X)\cdot S)(Y,Z) = L_S Q(g,S)(Y,Z;\xi,X). \tag{3.6}$$

By substituting (3.4) and (3.5) into (3.6) and taking $Z = \xi$, we get

$$-2nk^2 g(X,Y) + kS(X,Y) + 4nkg(h'X,Y) - 2S(h'X,Y) \\ = -2nkL_S g(X,Y) + L_S S(X,Y). \tag{3.7}$$

Now, from (3.1) we have

$$S(X,Y) = -2ng(X,Y) + 2n(k+1)\eta(X)\eta(Y) - 2ng(h'X,Y), \tag{3.8}$$

which gives

$$2ng(h'X,Y) = -2ng(X,Y) + 2n(k+1)\eta(X)\eta(Y) - S(X,Y). \tag{3.9}$$

Thus, using (1.7) and (3.9) in (3.8), we can write

$$S(h'X,Y) = 2n(k+2)g(X,Y) + S(X,Y) - 4n(k+1)\eta(X)\eta(Y). \tag{3.10}$$

Using (3.9) and (3.10) in (3.7), we get

$$S(X,Y) = Ag(X,Y) + B\eta(X)\eta(Y), \tag{3.11}$$

where

$$A = \frac{2nkL_S - 2nk^2 - 4nk - 4n(k+2)}{L_S + k + 2} \text{ and } B = \frac{4n(k+1)(k+2)}{L_S + k + 2}.$$

This shows that the manifold M^{2n+1} is η-Einstein. This completes the proof. □

If $k = -2$, then $L_S \neq 0$, so from the equation (3.11) we see that

$$S(X,Y) = -4ng(X,Y),$$

that is, the manifold M^{2n+1} is an Einstein manifold.

Also, $k = -2$ implies $\lambda = -1$. Then from Lemma 3.3 we have

$$R(X_\lambda, Y_\lambda)Z_\lambda = 0$$

and

$$R(X_{-\lambda}, Y_{-\lambda})Z_{-\lambda} = -4[g(Y_{-\lambda}, Z_{-\lambda})X_{-\lambda} - g(X_{-\lambda}, Z_{-\lambda})Y_{-\lambda}]$$

for any $X_\lambda, Y_\lambda, Z_\lambda \in [\lambda]'$ and $X_{-\lambda}, Y_{-\lambda}, Z_{-\lambda} \in [-\lambda]'$. Noticing $\mu = -2$, it follows from Lemma 3.1 that $K(X,\xi) = -4$ for any $X \in [-\lambda]'$, and $K(X,\xi) = 0$ for any $X \in [\lambda]'$. Again from Lemma 3.1, we see that $K(X,Y) = -4$ for any $X, Y \in [-\lambda]'$, and $K(X,Y) = 0$ for any $X, Y \in [\lambda]'$. As it is shown in [7], the distribution $[\xi] \oplus [\lambda]'$ is integrable with totally geodesic leaves, and the distribution $[-\lambda]'$ is integrable with totally umbilical leaves by $H = -(1-\lambda)\xi$, where H is the mean curvature tensor field for the leaves of $[-\lambda]'$ immersed in M^{2n+1}. Here $\lambda = -1$, then the two orthogonal distributions $[\xi] \oplus [\lambda]'$ and $[-\lambda]'$ are both integrable with totally geodesic leaves immersed in M^{2n+1}. Then we can say that M^{2n+1} is locally isometric to $\mathbb{H}^{n+1}(-4) \times \mathbb{R}^n$. Thus we arrive to the following corollary.

Corollary 3.1. *Let M^{2n+1} be a $(2n+1)$-dimensional almost Kenmotsu manifold with ξ belonging to the $(-2,-2)'$-nullity distribution. If M is Ricci pseudosymmetric, then M^{2n+1} is either an Einstein manifold or locally isometric to $\mathbb{H}^{n+1}(-4) \times \mathbb{R}^n$.*

This generalizes the result of Deshmukh et al. [5]. The above corollary can be verified by the example given in [4].

4. Manifolds with generalized (k,μ)-nullity distributions

Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be an almost Kenmotsu manifold. If the characteristic vector field ξ satisfies the generalized (k,μ)-nullity condition

$$R(X,Y)\xi = k(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)hY) \tag{4.1}$$

for any vector fields X, Y and some smooth functions k and μ on M^{2n+1}, then we say that M^{2n+1} is a generalized (k,μ)-almost Kenmotsu manifold (see [9]). From (4.1) we get

$$R(\xi,X)Y = k[g(X,Y)\xi - \eta(Y)X] + \mu[g(hX,Y)\xi - \eta(Y)hX]. \tag{4.2}$$

Contracting X in (4.1), we have

$$S(Y,\xi) = 2nk\eta(X). \tag{4.3}$$

Proposition 4.1. *A generalized (k,μ)-almost Kenmotsu manifold M^{2n+1} with k, μ non-zero functions is Ricci semisymmetric if and only if it is an Einstein manifold.*

Proof. Let us first assume that the manifold M^{2n+1} is Ricci semisymmetric, that is, $R \cdot S = 0$. Then we have

$$(R(X,Y) \cdot S)(U,V) = 0,$$

which implies

$$S(R(X,Y)U,V) + S(U,R(X,Y)V) = 0.$$

Replacing X by ξ in the foregoing equation and using (4.2) and (4.3), we get

$$\begin{aligned} k[&2nkg(Y,U)\eta(V) - S(Y,V)\eta(U) + 2nkg(Y,V)\eta(U) \\ &- S(Y,U)\eta(V)] + \mu[2nkg(hY,U)\eta(V) - S(hY,V)\eta(U) \\ &+ 2nkg(hY,V)\eta(U) - S(hY,U)\eta(V)] = 0. \end{aligned}$$

Putting $U = \xi$ in the above equation and using (4.3) yields

$$k[2nkg(Y,V) - S(Y,V)] + \mu[2nkg(hY,V) - S(hY,V)] = 0. \tag{4.4}$$

Replacing Y by hY in the foregoing equation and using (1.7), we get

$$k[2nkg(hY,V) - S(hY,V)] + \mu(k+1)[-2nkg(Y,V) + S(Y,V)] = 0. \quad (4.5)$$

Multiplying the equation (4.4) by k and the equation (4.5) by μ, and then subtracting the resulting equations, we have

$$[\mu^2(k+1) - k^2][S(Y,V) - 2nkg(Y,V)] = 0.$$

Since $\lambda^2 = -(k+1)$ (see Proposition 3.1 of [9]), where λ is an non-zero eigenvalue of h, the above equation yields

$$(\lambda^2\mu^2 + k^2)(S(Y,V) - 2nkg(Y,V)) = 0.$$

Since k, μ are non-zero functions and λ is a non-zero eigenvalue of h, we have $\lambda^2\mu^2 + k^2 \neq 0$. Thus we get

$$S(Y,V) = 2nkg(Y,V), \quad (4.6)$$

which shows that the manifold is Einstein.

The converse part is obvious. □

Theorem 4.1. *Let M^{2n+1} be a generalized (k,μ)-almost Kenmotsu manifold with k, μ non-zero functions. Then the following conditions are equivalent, provided $L_S \neq k$.*

(a) $\nabla S = 0$.
(b) $R \cdot S = 0$.
(c) M^{2n+1} *is an Einstein manifold.*
(d) $R \cdot S = L_S Q(g,S)$.

Proof. Obviously, (a) implies (b) and (c) implies (a). Proposition 4.1 shows that (b) implies (c). To complete the proof it remains to prove that (c) implies (d), and that (d) implies (c).

At first we prove that (d) implies (c). Using (4.2) and (4.3), we have

$$\begin{aligned}(R(\xi,X)\cdot S)(Y,Z) = &-2nk^2g(X,Y)\eta(Z) + k\eta(Y)S(X,Z)\\ &- 2nk\mu g(hX,Y)\eta(Z) + \mu\eta(Y)S(hX,Z)\\ &- 2nk^2g(X,Z)\eta(Y) + k\eta(Z)S(X,Y)\\ &- 2nk\mu g(hX,Z)\eta(Y) + \mu\eta(Z)S(hX,Y).\end{aligned} \quad (4.7)$$

Again, making use of (1.5), (1.6), and (4.3), we get

$$\begin{aligned}Q(g,S)(Y,Z;\xi,X) = &-2nkg(X,Y)\eta(Z) + S(X,Z)\eta(Y)\\ &- 2nkg(X,Z)\eta(Y) + S(X,Y)\eta(Z).\end{aligned} \quad (4.8)$$

Since the condition $R \cdot S = L_S Q(g,S)$ is realized on M^{2n+1}, we have

$$(R(\xi,X)\cdot S)(Y,Z) = L_S Q(g,S)(Y,Z;\xi,X). \quad (4.9)$$

So, substituting (4.7) and (4.8) into (4.9) and taking $Z = \xi$, we get

$$\begin{aligned} -2nk^2 g(X,Y) - 2nk\mu g(hX,Y) + kS(X,Y) + \mu S(hX,Y) \\ = -2nkL_S g(X,Y) + L_S S(X,Y). \end{aligned} \tag{4.10}$$

Let $X, Y \in [\lambda]'$, then from (4.10) we have

$$(k + \lambda\mu - L_S)(S(X,Y) - 2nkg(X,Y)) = 0,$$

which implies that $L_S = k + \lambda\mu$ or $S(X,Y) = 2nkg(X,Y)$.
On the other hand, if $X, Y \in [-\lambda]'$, then from (4.10) we have

$$(k - \lambda\mu - L_S)(S(X,Y) - 2nkg(X,Y)) = 0, \tag{4.11}$$

which implies

$$L_S = k - \lambda\mu \text{ or } S(X,Y) = 2nkg(X,Y).$$

Equating these two values of L_S, we get $\mu = 0$ as $\lambda \neq 0$. Therefore, we have $L_S = k$, a contradiction to our hypothesis. Thus M is an Einstein manifold.

Conversely, if the manifold M^{2n+1} is Einstein, then $R \cdot S = 0$ and

$$Q(g,S)(Y,Z;U,X) = 0,$$

which implies that the relation $R \cdot S = L_S Q(g,S)$ holds. This completes the proof. □

Since $R \cdot R = 0$ implies $R \cdot S = 0$, the above theorem generalizes the result of Wang et al. [13].

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References

[1] D. E. Blair, *Contact Manifolds in Riemannian Geometry*, Lecture Notes in Mathematics **509**, Springer, Berlin, 1976.

[2] D. E. Blair, *Riemannian Geometry of Contact and Symplectic Manifolds*, Progress in Mathematics **203**, Birkhäuser Boston, Inc., Boston, 2010.

[3] D. E. Blair, T. Koufogiorgos, and B. J. Papantoniou, *Contact metric manifolds satisfying a nullity condition*, Israel J. Math. **91** (1995), 189–214.

[4] U. C. De and K. Mandal, *On a type of almost Kenmotsu manifolds with nullity distributions*, Arab J. Math. Sci. **23** (2017), 109–123.

[5] S. Deshmukh, U. C. De, and P. Zhao, *Ricci semisymmetric almost Kenmotsu manifolds with nullity distributions*, Filomat **32** (2018), 179–186.

[6] G. Dileo and A. M. Pastore, *Almost Kenmotsu manifolds and local symmetry*, Bull. Belg. Math. Soc. Simon Stevin **14** (2007), 343–354.

[7] G. Dileo and A. M. Pastore, *Almost Kenmotsu manifolds and nullity distributions*, J. Geom. **93** (2009), 46–61.

[8] K. Kenmotsu, *A class of almost contact Riemannian manifolds*, Tohoku Math. J. **24** (1972), 93–103.
[9] A. M. Pastore and V. Saltarelli, *Generalized nullity distributions on almost Kenmotsu manifolds*, Int. Electron. J. Geom. **4** (2011), 168–183.
[10] L. Verstraelen, *Comments on pseudosymmetry in the sense of Ryszard Deszcz*, in: Geometry and Topology of Submanifolds, VI (Leuven, 1993/Brussels, 1993), World Sci. Publ., River Edge, NJ, 1994, pp. 141–151.
[11] Y. Wang and X. Liu, *Riemannian semisymmetric almost Kenmotsu manifolds and nullity distributions*, Ann. Polon. Math. **112** (2014), 37–46.
[12] Y. Wang and X. Liu, *On ϕ-recurrent almost Kenmotsu manifolds*, Kuwait J. Sci. **42** (2015), 65–77.
[13] Y. Wang and W. Wang, *Curvature properties of almost Kenmotsu manifolds with generalized nullity conditions*, Filomat **30** (2016), 3807–3816.

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