# Some Hermite–Hadamard and Ostrowski type inequalities for fractional integral operators with exponential kernel

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Abstract. We firstly establish Hermite–Hadamard type integral inequalities for fractional integral operators. Secondly, we give new generalizations of fractional Ostrowski type inequalities through convex functions via Hölder and power means inequalities. In accordance with this purpose, we use fractional integral operators with exponential kernel.

## 1. Introduction

The study of various types of integral inequalities has been the focus of great attention for well over a century by a number of scientists, interested both in pure and applied mathematics. On the other hand, the concept of fractional calculus got wide use in numerous physical and other applications, as viscoelastic materials, fluid flow, diffusive transport, electrical networks, electromagnetic theory, probability, and others. We will summarize these two concepts below.

**1.1. Hermite–Hadamard inequality.** The inequalities discovered by C. Hermite and J. Hadamard for convex functions are of considerable significance in the literature (see, e.g., [8, p. 137], [2]). These inequalities state that if $f\colon I \to \mathbb{R}$ is a convex function on an interval $I$ of real numbers and $a, b \in I$ with $a < b$, then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \qquad (1.1)$$

Both inequalities hold in the reverse direction if $f$ is concave. We note that Hadamard's inequality may be regarded as a refinement of the concept of convexity, and it follows easily from Jensen's inequality. Hadamard's inequality for convex functions has received renewed attention in recent years and a remarkable variety of refinements and generalizations have been found (see, for example, [3], [9], [10], and the references therein).

**1.2. Ostrowski inequality.** One of the many fundamental mathematical discoveries of A. M. Ostrowski [7] is the following classical integral inequality associated with differentiable mappings.

**Theorem 1.** *Let* $f \colon [a, b] \to \mathbb{R}$ *be a differentiable mapping on* $(a, b)$ *whose derivative* $f' \colon (a, b) \to \mathbb{R}$ *is bounded on* $(a, b)$, *i.e.,* $\|f'\|_\infty := \sup\limits_{t \in (a,b)} |f'(t)| < \infty$. *Then, we have the inequality*

$$\left| f(x) - \frac{1}{b-a} \int\limits_a^b f(t)dt \right| \leq \left[ \frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right] (b-a) \|f'\|_\infty$$

*for all* $x \in [a, b]$. *The constant* $\frac{1}{4}$ *is the best possible.*

Ostrowski's inequality has applications in quadrature, probability and optimization theory, stochastic, statistics, information, and integral operator theory. During the past few years, a number of scientists have focused on Ostrowski's type inequalities for functions of bounded variation (see, for example, [1], [4], [6], [11], and the references therein).

**1.3. Fractional integral operators with exponential kernel.** Recently, Kirane and Torebek [5] introduced a new class of fractional integrals as follows.

**Definition 1.** Let $f \in L_1(a, b)$. The fractional integrals $\mathcal{I}_{a+}^\alpha$ and $\mathcal{I}_{b-}^\alpha$ of order $\alpha \in (0, 1)$ are defined by

$$\mathcal{I}_{a+}^\alpha f(x) := \frac{1}{\alpha} \int\limits_a^x \exp\left\{ -\frac{1-\alpha}{\alpha}(x-t) \right\} f(t)dt, \quad x > a,$$

and

$$\mathcal{I}_{b-}^\alpha f(x) := \frac{1}{\alpha} \int\limits_x^b \exp\left\{ -\frac{1-\alpha}{\alpha}(t-x) \right\} f(t)dt, \quad x < b.$$

The remainder of this work is organized as follows. In Section 2, we will present a new Hermite–Hadamard type integral inequality via fractional calculus mentioned above. In Section 3, new Ostrowski type integral inequalities are proved via fractional integral operators with exponential kernel.

## 2. Hermite–Hadamard type inequalities via fractional integral operators with exponential kernel

In this section, we will present some significant results for Hermite–Hadamard type inequalities with fractional integral operators with exponential kernel. Throughout this section, we denote $A := \frac{1-\alpha}{\alpha}\left(\frac{b-a}{2}\right)$ for $\alpha \in (0,1)$.

**Theorem 2.** *Let $f\colon [a,b] \to \mathbb{R}$ be a function with $0 \leq a < b$ and $f \in L_1[a,b]$. If $f$ is a convex function on $[a,b]$, then we have the following inequalities for fractional integral operators with exponential kernel:*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1-\alpha}{2\left[1-\exp\{-A\}\right]}\left[\mathcal{I}^{\alpha}_{\left(\frac{a+b}{2}\right)+}f(b) + \mathcal{I}^{\alpha}_{\left(\frac{a+b}{2}\right)-}f(a)\right] \leq \frac{f(a)+f(b)}{2}. \tag{2.1}$$

*Proof.* Since $f$ is a convex function on $[a,b]$, we have

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(x)+f(y)}{2}, \quad x,y \in [a,b].$$

For $x = \frac{t}{2}a + \frac{2-t}{2}b$ and $y = \frac{2-t}{2}a + \frac{t}{2}b$, we obtain

$$2f\left(\frac{a+b}{2}\right) \leq f\left(\frac{t}{2}a + \frac{2-t}{2}b\right) + f\left(\frac{2-t}{2}a + \frac{t}{2}b\right). \tag{2.2}$$

Multiplying both sides of (2.2) by $\exp\{-At\}$ and integrating the resulting inequality with respect to $t$ over $[0,1]$, we get

$$2f\left(\frac{a+b}{2}\right)\int_0^1 \exp\{-At\}\,dt \leq \int_0^1 \exp\{-At\}\,f\left(\frac{t}{2}a + \frac{2-t}{2}b\right)dt$$

$$+ \int_0^1 \exp\{-At\}\,f\left(\frac{2-t}{2}a + \frac{t}{2}b\right)dt.$$

For $u = \frac{t}{2}a + \frac{2-t}{2}b$ and $v = \frac{2-t}{2}a + \frac{t}{2}b$, we obtain

$$\frac{2}{A}f\left(\frac{a+b}{2}\right)\left[1 - \exp\{-A\}\right] \leq \frac{2}{b-a}\int_{\frac{a+b}{2}}^{b} \exp\left\{-\frac{1-\alpha}{\alpha}(b-u)\right\}f(u)\,du$$

$$+ \frac{2}{b-a}\int_{a}^{\frac{a+b}{2}} \exp\left\{-\frac{1-\alpha}{\alpha}(v-a)\right\}f(v)\,dv$$

$$= \frac{2\alpha}{b-a}\left[\mathcal{I}^{\alpha}_{\left(\frac{a+b}{2}\right)+}f(b) + \mathcal{I}^{\alpha}_{\left(\frac{a+b}{2}\right)-}f(a)\right],$$

and the first inequality is proved.

For the proof of the second inequality in (2.1), we first note that if $f$ is a convex function, then

$$f\left(\frac{t}{2}a + \frac{2-t}{2}b\right) \leq \frac{t}{2}f(a) + \frac{2-t}{2}f(b)$$

and

$$f\left(\frac{2-t}{2}a + \frac{t}{2}b\right) \leq \frac{2-t}{2}f(a) + \frac{t}{2}f(b).$$

By adding these inequalities, we have

$$f\left(\frac{t}{2}a + \frac{2-t}{2}b\right) + f\left(\frac{2-t}{2}a + \frac{t}{2}b\right) \leq f(a) + f(b). \qquad (2.3)$$

Then multiplying both sides of (2.3) by $\exp\{-At\}$ and integrating the resulting inequality with respect to $t$ over $[0,1]$, we obtain

$$\int_0^1 \exp\{-At\} f\left(\frac{t}{2}a + \frac{2-t}{2}b\right) dt + \int_0^1 \exp\{-At\} f\left(\frac{2-t}{2}a + \frac{t}{2}b\right) dt$$

$$\leq [f(a) + f(b)] \int_0^1 \exp\{-At\} dt = \frac{1}{A}[1 - \exp\{-A\}][f(a) + f(b)].$$

That is,

$$\frac{2\alpha}{b-a}\left[\mathcal{I}^\alpha_{\left(\frac{a+b}{2}\right)+}f(b) + \mathcal{I}^\alpha_{\left(\frac{a+b}{2}\right)-}f(a)\right] \leq \frac{1}{A}[1 - \exp\{-A\}][f(a) + f(b)].$$

The proof is complete. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

**Remark 1.** Since $\lim\limits_{\alpha\to 1}\frac{1-\alpha}{2[1-\exp\{-A\}]} = \frac{1}{b-a}$, the inequality (2.1) reduces to the classical Hermite–Hadamard inequality (1.1).

**Lemma 1.** Let $f\colon [a,b] \to \mathbb{R}$ be a differentiable function on $(a,b)$ with $a < b$. If $f' \in L[a,b]$, then we have the following identity for generalized fractional integral operators with exponential kernel:

$$\frac{1-\alpha}{2[1-\exp\{-A\}]}\left[\mathcal{I}^\alpha_{\left(\frac{a+b}{2}\right)+}f(b) + \mathcal{I}^\alpha_{\left(\frac{a+b}{2}\right)-}f(a)\right] - f\left(\frac{a+b}{2}\right)$$

$$= \frac{1}{2[1-\exp\{-A\}]}\left[\int_{\frac{a+b}{2}}^b \left[1 - \exp\left\{-\frac{1-\alpha}{\alpha}(b-u)\right\}\right] f'(u)\, du \right.$$

$$\left. - \int_a^{\frac{a+b}{2}} \left[1 - \exp\left\{-\frac{1-\alpha}{\alpha}(u-a)\right\}\right] f'(u)\, du\right]. \qquad (2.4)$$

*Proof.* Integrating by parts gives

$$
\begin{aligned}
J_1 := & \int_{\frac{a+b}{2}}^{b} \left[ 1 - \exp\left\{ -\frac{1-\alpha}{\alpha} (b-u) \right\} \right] f'(u)\, du \\
= & \, f(u) \left[ 1 - \exp\left\{ -\frac{1-\alpha}{\alpha} (b-u) \right\} \right] \Big|_{\frac{a+b}{2}}^{b} \\
& + \frac{1-\alpha}{\alpha} \int_{\frac{a+b}{2}}^{b} \exp\left\{ -\frac{1-\alpha}{\alpha} (b-u) \right\} f(u)\, du \\
= & -[1 - \exp\{-A\}] f\left( \frac{a+b}{2} \right) + (1-\alpha)\, \mathcal{I}_{\left(\frac{a+b}{2}\right)+}^{\alpha} f(b).
\end{aligned}
\tag{2.5}
$$

Similarly, we get

$$
\begin{aligned}
J_2 := & \int_{a}^{\frac{a+b}{2}} \left[ 1 - \exp\left\{ -\frac{1-\alpha}{\alpha} (u-a) \right\} \right] f'(u)\, du \\
= & \, [1 - \exp\{-A\}] f\left( \frac{a+b}{2} \right) - (1-\alpha)\, \mathcal{I}_{\left(\frac{a+b}{2}\right)-}^{\alpha} f(a).
\end{aligned}
\tag{2.6}
$$

By subtracting (2.6) from (2.5), we have

$$
\begin{aligned}
J_1 - J_2 = & -2\,[1 - \exp\{-A\}] f\left( \frac{a+b}{2} \right) \\
& + (1-\alpha) \left[ \mathcal{I}_{\left(\frac{a+b}{2}\right)+}^{\alpha} f(b) + \mathcal{I}_{\left(\frac{a+b}{2}\right)-}^{\alpha} f(a) \right].
\end{aligned}
$$

By re-arranging the last equality above, we get the desired result.     □

**Theorem 3.** *Let* $f \colon [a,b] \to \mathbb{R}$ *be a function with* $0 \le a < b$ *and* $f \in L_1[a,b]$. *If* $f'$ *is bounded on* $[a,b]$, *then we have the inequality*

$$
\begin{aligned}
& \left| \frac{1-\alpha}{2\,[1 - \exp\{-A\}]} \left[ \mathcal{I}_{\left(\frac{a+b}{2}\right)+}^{\alpha} f(b) + \mathcal{I}_{\left(\frac{a+b}{2}\right)-}^{\alpha} f(a) \right] - f\left( \frac{a+b}{2} \right) \right| \\
& \le \frac{(1-\alpha)(b-a) - 2\alpha\,[1 - \exp\{-A\}]}{2(1-\alpha)\,[1 - \exp\{-A\}]} \, \|f'\|_\infty .
\end{aligned}
$$

*Proof.* Using Lemma 1, we have

$$
M := \left| \frac{1-\alpha}{2\,[1 - \exp\{-A\}]} \left[ \mathcal{I}_{\left(\frac{a+b}{2}\right)+}^{\alpha} f(b) + \mathcal{I}_{\left(\frac{a+b}{2}\right)-}^{\alpha} f(a) \right] - f\left( \frac{a+b}{2} \right) \right|
$$

$$\leq \frac{1}{2\left[1 - \exp\{-A\}\right]} \left[ \int_{\frac{a+b}{2}}^{b} \left[1 - \exp\left\{-\frac{1-\alpha}{\alpha}(b-u)\right\}\right] |f'(u)| \, du \right.$$

$$\left. + \int_{a}^{\frac{a+b}{2}} \left[1 - \exp\left\{-\frac{1-\alpha}{\alpha}(u-a)\right\}\right] |f'(u)| \, du \right].$$

Since $f'$ is bounded on $[a, b]$, we deduce that

$$M \leq \frac{\|f'\|_{\infty}}{2\left[1 - \exp\{-A\}\right]} \left[ \int_{\frac{a+b}{2}}^{b} \left[1 - \exp\left\{-\frac{1-\alpha}{\alpha}(b-u)\right\}\right] du \right.$$

$$\left. + \int_{a}^{\frac{a+b}{2}} \left[1 - \exp\left\{-\frac{1-\alpha}{\alpha}(u-a)\right\}\right] du \right]$$

$$= \frac{(1-\alpha)(b-a) - 2\alpha\left[1 - \exp\{-A\}\right]}{2(1-\alpha)\left[1 - \exp\{-A\}\right]} \|f'\|_{\infty},$$

which completes the proof. $\qquad\square$

**Remark 2.** Since

$$\lim_{\alpha \to 1} \frac{1-\alpha}{2\left[1 - \exp\{-A\}\right]} = \frac{1}{b-a}$$

and

$$\lim_{\alpha \to 1} \frac{(1-\alpha)(b-a) - 2\alpha\left[1 - \exp\{-A\}\right]}{2(1-\alpha)\left[1 - \exp\{-A\}\right]} = \frac{b-a}{4},$$

we have the midpoint inequality

$$\left| \frac{1}{b-a} \int_{a}^{b} f(x)dx - f\left(\frac{a+b}{2}\right) \right| \leq \frac{b-a}{4} \|f'\|_{\infty}.$$

## 3. Ostrowski type inequalities via fractional integral operators with exponential kernel

Throughout this section, we denote $\theta_a := \frac{1-\alpha}{\alpha}(x-a)$ and $\theta_b := \frac{1-\alpha}{\alpha}(b-x)$ for $\alpha \in (0, 1)$.

**Lemma 2.** *Let $f: [a, b] \to \mathbb{R}$ be a differentiable function on $(a, b)$ with $a < b$. If $f' \in L[a, b]$, then we have the following identity for generalized*

*fractional integral operators with exponential kernel:*

$$f(x) - \frac{1-\alpha}{2 - \exp\{-\theta_a\} - \exp\{-\theta_b\}} \left[ \mathcal{I}_{x-}^{\alpha} f(a) + \mathcal{I}_{x+}^{\alpha} f(b) \right]$$

$$= \frac{x-a}{2 - \exp\{-\theta_a\} - \exp\{-\theta_b\}} \int_0^1 \left[ 1 - \exp\{-\theta_a t\} \right] f'(tx + (1-t)a) dt$$

$$- \frac{b-x}{2 - \exp\{-\theta_a\} - \exp\{-\theta_b\}} \int_0^1 \left[ 1 - \exp\{-\theta_b t\} \right] f'(tx + (1-t)b) dt.$$

*Proof.* Integrating by parts, we have

$$J_3 := \int_0^1 \left[ 1 - \exp\{-\theta_a t\} \right] f'(tx + (1-t)a) dt$$

$$= \frac{1}{x-a} \left[ 1 - \exp\{-\theta_a t\} \right] f(tx + (1-t)a) \Big|_0^1$$

$$- \frac{\theta_a}{x-a} \int_0^1 \exp\{-\theta_a t\} f(tx + (1-t)a) dt$$

$$= \frac{1 - \exp\{-\theta_a\}}{x-a} f(x) - \frac{1-\alpha}{x-a} \mathcal{I}_{x-}^{\alpha} f(a).$$

Similarly,

$$J_4 := \int_0^1 \left[ 1 - \exp\{-\theta_b t\} \right] f'(tx + (1-t)b) dt$$

$$= -\frac{1 - \exp\{-\theta_b\}}{b-x} f(x) + \frac{1-\alpha}{b-x} \mathcal{I}_{x+}^{\alpha} f(b).$$

It follows that

$$\frac{(x-a)J_3 - (b-x)J_4}{2 - \exp\{-\theta_a\} - \exp\{-\theta_b\}}$$

$$= f(x) - \frac{1-\alpha}{2 - \exp\{-\theta_a\} - \exp\{-\theta_b\}} \left[ \mathcal{I}_{x-}^{\alpha} f(a) + \mathcal{I}_{x+}^{\alpha} f(b) \right],$$

which completes the proof. □

**Theorem 4.** *Let $f \colon [a,b] \to \mathbb{R}$ be a function with $0 \leq a < b$ and $f \in L_1[a,b]$. If $|f'|^q$, $q > 1$, is a convex function on $[a,b]$, then we have the following inequality for fractional integral operators with exponential kernel:*

$$\left| f(x) - \frac{1-\alpha}{2 - \exp\{-\theta_a\} - \exp\{-\theta_b\}} \left[ \mathcal{I}_{x-}^{\alpha} f(a) + \mathcal{I}_{x+}^{\alpha} f(b) \right] \right|$$

$$\leq \frac{1}{2 - \exp\{-\theta_a\} - \exp\{-\theta_b\}} \times \left[ (x - a)A_1(\alpha, p) \left( \frac{|f'(x)|^q + |f'(a)|^q}{2} \right)^{\frac{1}{q}} \right.$$

$$\left. + (b - x)A_2(\alpha, p) \left( \frac{|f'(x)|^q + |f'(b)|^q}{2} \right)^{\frac{1}{q}} \right],$$

where $\frac{1}{p} + \frac{1}{q} = 1$, and

$$A_1(\alpha, p) := \left( \int_0^1 [1 - \exp\{-\theta_a t\}]^p \, dt \right)^{\frac{1}{p}}, \quad A_2(\alpha, p) := \left( \int_0^1 [1 - \exp\{-\theta_b t\}]^p \, dt \right)^{\frac{1}{p}}.$$

*Proof.* Taking the modulus in Lemma 2 and using the fact that $e^x > 1$, $x > 0$, we obtain

$$\left| f(x) - \frac{1 - \alpha}{2 - \exp\{-\theta_a\} - \exp\{-\theta_b\}} \left[ \mathcal{I}_{x-}^\alpha f(a) + \mathcal{I}_{x+}^\alpha f(b) \right] \right|$$

$$\leq \frac{x - a}{2 - \exp\{-\theta_a\} - \exp\{-\theta_b\}} J_3 + \frac{b - x}{2 - \exp\{-\theta_a\} - \exp\{-\theta_b\}} J_4. \tag{3.1}$$

Using Hölder's inequality and convexity of $|f'|^q$, we find that

$$\int_0^1 [1 - \exp\{-\theta_a t\}] \left| f'(tx + (1 - t)a) \right| dt$$

$$\leq \left( \int_0^1 [1 - \exp\{-\theta_a t\}]^p \, dt \right)^{\frac{1}{p}} \left( \int_0^1 \left| f'(tx + (1 - t)a) \right|^q dt \right)^{\frac{1}{q}} \tag{3.2}$$

$$\leq A_1(\alpha, p) \left( \frac{|f'(x)|^q + |f'(a)|^q}{2} \right)^{\frac{1}{q}}$$

and, similarly,

$$\int_0^1 [1 - \exp\{-\theta_b t\}] \left| f'(tx + (1 - t)b) \right| dt$$

$$\leq A_2(\alpha, p) \left( \frac{|f'(x)|^q + |f'(b)|^q}{2} \right)^{\frac{1}{q}}. \tag{3.3}$$

By substituting the inequalities (3.2) and (3.3) into (3.1), we have

$$\left| f(x) - \frac{1 - \alpha}{2 - \exp\{-\theta_a\} - \exp\{-\theta_b\}} \left[ \mathcal{I}_{x-}^\alpha f(a) + \mathcal{I}_{x+}^\alpha f(b) \right] \right|$$

$$\leq \frac{1}{2 - \exp\left\{-\theta_a\right\} - \exp\left\{-\theta_b\right\}} \left[(x-a)A_1(\alpha,p)\left(\frac{|f'(x)|^q + |f'(a)|^q}{2}\right)^{\frac{1}{q}}\right.$$

$$\left. + (b-x)A_2(\alpha,p)\left(\frac{|f'(x)|^q + |f'(b)|^q}{2}\right)^{\frac{1}{q}}\right].$$

This completes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

**Theorem 5.** *Let* $f\colon [a,b] \to \mathbb{R}$ *be a function with* $0 \leq a < b$ *and* $f \in L_1[a,b]$. *If* $|f'|^q$, $q \geq 1$, *is a convex function on* $[a,b]$, *then we have the inequality*

$$\left| f(x) - \frac{1-\alpha}{2 - \exp\left\{-\theta_a\right\} - \exp\left\{-\theta_b\right\}}\left[\mathcal{I}_{x-}^\alpha f(a) + \mathcal{I}_{x+}^\alpha f(b)\right]\right|$$

$$\leq \frac{1}{2 - \exp\left\{-\theta_a\right\} - \exp\left\{-\theta_b\right\}}$$

$$\times \left\{ \frac{x-a}{\theta_a^{1-\frac{1}{q}}}\left[1 + \theta_a - \exp\left\{-\theta_a\right\}\right]^{1-\frac{1}{q}} \right.$$

$$\times \left(|f'(x)|^q\left[\frac{1}{2} - \frac{1}{\theta_a^2}\left[1 - \exp\left\{-\theta_a\right\}(\theta_a+1)\right]\right]\right.$$

$$\left. + |f'(a)|^q\left[\frac{1}{2} - \frac{1}{\theta_a^2}\left[\theta_a + \exp\left\{-\theta_a\right\} - 1\right]\right]\right)^{\frac{1}{q}}$$

$$+ \frac{b-x}{\theta_b^{1-\frac{1}{q}}}\left(1 + \theta_b - \exp\left\{-\theta_b\right\}\right)^{1-\frac{1}{q}}\left(|f'(x)|^q\left[\frac{1}{2} - \frac{1}{\theta_b^2}\left[1 - \exp\left\{-\theta_b\right\}(\theta_b+1)\right]\right]\right.$$

$$\left.\left. + |f'(b)|^q\left[\frac{1}{2} - \frac{1}{\theta_b^2}\left[\theta_b + \exp\left\{-\theta_b\right\} - 1\right]\right]\right)^{\frac{1}{q}}\right\}.$$

*Proof.* Using power means inequality and convexity of $|f'|^q$ in (3.1), we find that

$$\left| f(x) - \frac{1-\alpha}{2 - \exp\left\{-\theta_a\right\} - \exp\left\{-\theta_b\right\}}\left[\mathcal{I}_{x-}^\alpha f(a) + \mathcal{I}_{x+}^\alpha f(b)\right]\right|$$

$$\leq \frac{x-a}{2 - \exp\left\{-\theta_a\right\} - \exp\left\{-\theta_b\right\}}\left(\int_0^1 \left[1 - \exp\left\{-\theta_a t\right\}\right]dt\right)^{1-\frac{1}{q}}$$

$$\times \left(\int_0^1 \left[1 - \exp\left\{-\theta_a t\right\}\right]|f'(tx + (1-t)a)|^q\, dt\right)^{\frac{1}{q}} \qquad (3.4)$$

$$+ \frac{b - x}{2 - \exp\{-\theta_a\} - \exp\{-\theta_b\}} \left( \int_0^1 [1 - \exp\{-\theta_b t\}]\, dt \right)^{1 - \frac{1}{q}}$$

$$\times \left( \int_0^1 [1 - \exp\{-\theta_b t\}] \left| f'(tx + (1-t)b) \right|^q dt \right)^{\frac{1}{q}}.$$

Since $|f'|^q$ is a convex function, we have

$$\int_0^1 [1 - \exp\{-\theta_a t\}] \left| f'(tx + (1-t)a) \right|^q dt$$

$$\leq \int_0^1 [1 - \exp\{-\theta_a t\}] \left[ t \left| f'(x) \right|^q + (1-t) \left| f'(a) \right|^q \right] dt \tag{3.5}$$

$$= \left| f'(x) \right|^q \left[ \frac{1}{2} - \frac{1}{\theta_a^2} \left[ 1 - \exp\{-\theta_a\}(\theta_a + 1) \right] \right]$$

$$+ \left| f'(a) \right|^q \left[ \frac{1}{2} - \frac{1}{\theta_a^2} \left[ \theta_a + \exp\{-\theta_a\} - 1 \right] \right]$$

and, similarly,

$$\left| \int_0^1 [1 - \exp\{-\theta_b t\}] \left| f'(tx + (1-t)b) \right|^q dt \right|$$

$$\leq \left| f'(x) \right|^q \left[ \frac{1}{2} - \frac{1}{\theta_b^2} \left[ 1 - \exp\{-\theta_b\}(\theta_b + 1) \right] \right] \tag{3.6}$$

$$+ \left| f'(b) \right|^q \left[ \frac{1}{2} - \frac{1}{\theta_b^2} \left[ \theta_b + \exp\{-\theta_b\} - 1 \right] \right].$$

On the other hand, we have

$$\int_0^1 [1 - \exp\{-\theta_a t\}]\, dt = \frac{1}{\theta_a} \left( 1 + \theta_a - \exp\{-\theta_a\} \right) \tag{3.7}$$

and, similarly,

$$\int_0^1 [1 - \exp\{-\theta_b t\}]\, dt = \frac{1}{\theta_b} \left( 1 + \theta_b - \exp\{-\theta_b\} \right). \tag{3.8}$$

Then, substituting the inequalities (3.5)–(3.8) into (3.4), we obtain the desired result. $\qquad \square$

**Corollary 1.** *Suppose that the assumptions of Theorem 5 hold for $q = 1$. Then we have the following inequality for fractional integral operators with exponential kernel:*

$$\left| f(x) - \frac{1 - \alpha}{2 - \exp\{-\theta_a\} - \exp\{-\theta_b\}} \left[ \mathcal{I}_{x-}^{\alpha} f(a) + \mathcal{I}_{x+}^{\alpha} f(b) \right] \right|$$

$$\leq \frac{x - a}{2 - \exp\{-\theta_a\} - \exp\{-\theta_b\}}$$

$$\times \left\{ \left[ \frac{1}{2} - \frac{1}{\theta_a^2} \left[ 1 - \exp\{-\theta_a\} (\theta_a + 1) \right] \right] \left| f'(x) \right| \right.$$

$$+ \left. \left[ \frac{1}{2} - \frac{1}{\theta_a^2} \left[ \theta_a + \exp\{-\theta_a\} - 1 \right] \right] \left| f'(a) \right| \right\}$$

$$+ \frac{b - x}{2 - \exp\{-\theta_a\} - \exp\{-\theta_b\}}$$

$$\times \left\{ \left[ \frac{1}{2} - \frac{1}{\theta_b^2} \left[ 1 - \exp\{-\theta_b\} (\theta_b + 1) \right] \right] \left| f'(x) \right| \right.$$

$$+ \left. \left[ \frac{1}{2} - \frac{1}{\theta_b^2} \left[ \theta_b + \exp\{-\theta_b\} - 1 \right] \right] \left| f'(b) \right| \right\}.$$

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