Some (p, q)-analogues of Apostol type numbers and polynomials

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Abstract. We consider a new class of generating functions of the generalizations of Bernoulli and Euler polynomials in terms of (p, q)-integers. By making use of these generating functions, we derive (p, q)-generalizations of several old and new identities concerning Apostol–Bernoulli and Apostol–Euler polynomials. Finally, we define the (p, q)-generalization of Stirling polynomials of the second kind of order v, and provide a link between the (p, q)-generalization of Bernoulli polynomials of order v and the (p, q)-generalization of Stirling polynomials of the second kind of order v.

1. Introduction

Let $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{R}$, and $\mathbb{C}$ be the sets of natural numbers, nonnegative integers, real numbers, and complex numbers, respectively.

The (p, q)-analog of a number n is known as (see [2–4, 13])

$$[n]_{p,q} := \frac{p^n - q^n}{p - q} \quad (p \neq q),$$

representing the relation $[n]_{p,q} = p^{n-1} [n]_{q/p}$, where $[n]_{q/p}$ is the q-number from q-calculus given by $[n]_{q/p} = ((q/p)^n - 1)/((q/p) - 1)$. Using this obvious relation between the q-notation and its (p, q)-variant, most (if not all) of the (p, q)-results can be derived from the corresponding known q-results. In the case when $p = 1$, (p, q)-numbers reduce to q-numbers (cf. [9]). In theory of operators, approximations, and other related fields, the (p, q)-variant has been investigated extensively by many mathematicians and also physicists (see [2–4, 13]).

A few (p, q)-notations are listed below which will be used in this paper. The (p, q)-derivative of a function f (with respect to x) is given by

$$D_{p,q}f(x) := \frac{f(px) - f(qx)}{(p - q)x} \quad (x \neq 0; p \neq q);\tag{1.1}$$

it satisfies the condition

$$\lim_{x \to 0} D_{p,q}f(x) := f'(0) \ where \ f'(0) = \frac{d}{dx}f(x)\mid_{x=0}.$$

The (p, q)-binomial formulae is

$$(x \oplus a)_{p,q}^n := \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\binom{k}{2}} q^{\binom{n-k}{2}} x^k a^{n-k}$$

with the (p, q)-binomial coefficients

$$\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} = \frac{[n]_{p,q}!}{[n - k]_{p,q}! [k]_{p,q}!} \quad (n \geq k)$$

and (p, q)-factorials

$$[n]_{p,q}! = [n]_{p,q} [n - 1]_{p,q} \cdots [2]_{p,q} [1]_{p,q} \quad (n \in \mathbb{N}).$$

The (p, q)-exponential functions are defined by

$$e_{p,q}(x) = \sum_{n=0}^{\infty} \frac{p^{\binom{n}{2}} x^n}{[n]_{p,q}!} \ \text{ and } \ E_{p,q}(x) = \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}} x^n}{[n]_{p,q}!}\tag{1.2}$$

under the condition

$$e_{p,q}(x)E_{p,q}(-x) = 1.\tag{1.3}$$

It follows from (1.1) and (1.2) that

$$D_{p,q}e_{p,q}(x) = e_{p,q}(px) \ \text{ and } \ D_{p,q}E_{p,q}(x) = E_{p,q}(qx).\tag{1.4}$$

The definite (p, q)-integral of a function f is determined by

$$\int_0^a f(x) \, d_{p,q}x = (p - q) a \sum_{k=0}^{\infty} \frac{p^k}{q^{k+1}} f\left(\frac{p^k}{q^{k+1}} a \right);$$

it satisfies the condition

$$\int_a^b f(x) \, d_{p,q}x = \int_0^b f(x) \, d_{p,q}x - \int_0^a f(x) \, d_{p,q}x.\tag{1.5}$$

One can find these notations (together with all the details) in the references [2], [3], [4], and [13].

Apostol [1] introduced a class of the classical Bernoulli polynomials and numbers (called Apostol–Bernoulli polynomials and numbers), when he studied the Lipschitz–Lerch zeta functions and investigated some elementary properties of these polynomials and numbers. From Apostol's time to the

present, Apostol type polynomials and several generalizations of them have been considered and discussed by many mathematicians, for example, by Luo [8], Srivastava [14], Tremblay et al. [15], Mahmudov et al. [10], and Duran et al. [2].

The Apostol–Bernoulli and Apostol–Euler polynomials of order $\alpha \in \mathbb{C}$ are defined by the generating functions (see [1, 6, 7, 8, 15])

$$\sum_{n=0}^{\infty} B_n^{(\alpha)}(x;\lambda) \frac{z^n}{n!} = \left(\frac{z}{\lambda e^z - 1} \right)^{\alpha} e^{xz} \quad (|z| < 2\pi \text{ if } \lambda = 1, \ |z| < |\log \lambda| \text{ if } \lambda \neq 1)$$

and

$$\sum_{n=0}^{\infty} E_n^{(\alpha)}(x;\lambda) \frac{z^n}{n!} = \left(\frac{2}{\lambda e^z + 1} \right)^{\alpha} e^{xz} \quad (|z| < \pi \text{ if } \lambda = 1, \ |z| < |\log(-\lambda)| \text{ if } \lambda \neq 1).$$

Upon setting $\lambda = 1$, the polynomials above reduce to the classical forms (cf. [11]).

For $m \in \mathbb{N}$ and $\alpha \in \mathbb{C}$, the generalized Apostol type Bernoulli polynomials $B_n^{[m-1,\alpha]}(x)$ of order α and the generalized Apostol type Euler polynomials $E_n^{[m-1,\alpha]}(x)$ of order α are defined, in a suitable neighborhood of $z = 0$, by means of the generating functions (see [3, 6, 12])

$$\sum_{n=0}^{\infty} B_n^{[m-1,\alpha]}(x;\lambda) \frac{z^n}{n!} = \left(\frac{z^m}{\lambda e^z - \sum_{h=0}^{m-1} \frac{z^h}{h!}} \right)^{\alpha} e^{xz} \qquad (1.6)$$

and

$$\sum_{n=0}^{\infty} E_n^{[m-1,\alpha]}(x;\lambda) \frac{z^n}{n!} = \left(\frac{2^m}{\lambda e^z + \sum_{h=0}^{m-1} \frac{z^h}{h!}} \right)^{\alpha} e^{xz}. \qquad (1.7)$$

In the next section, we give a new class of generating functions of the (p,q)-generalizations of Bernoulli and Euler polynomials. We derive (p,q)-generalizations of several known identities concerning Apostol–Bernoulli and Apostol–Euler polynomials. Finally, we consider the (p,q)-generalization of Stirling polynomials of the second kind of order v whose classical form can be found in [11], and then provide a link between the (p,q)-generalization of Bernoulli polynomials of order v and the (p,q)-generalization of Stirling polynomials of the second kind of order v.

2. On a (p,q)-analog of some polynomials

We begin with the following definition.

Definition 1. Let $p, q, \alpha \in \mathbb{C}$ with $0 < |q| < |p| \leq 1$, and let $m \in \mathbb{N}$. The generalized Apostol type (p,q)-Bernoulli polynomials $\mathcal{B}_n^{[m-1,\alpha]}(x, y : p, q)$ of order α and the generalized Apostol type (p,q)-Euler polynomials

$\mathcal{E}_n^{[m-1,\alpha]}(x, y : p, q)$ of order α are defined, in a suitable neighborhood of $z = 0$, by the generating functions

$$\sum_{n=0}^{\infty} \mathcal{B}_n^{[m-1,\alpha]}(x, y; \lambda : p, q) \frac{z^n}{[n]_{p,q}!} = \left(\frac{z^m}{\lambda e_{p,q}(z) - T_{m-1}^{p,q}(z)} \right)^{\alpha} e_{p,q}(xz) E_{p,q}(yz)$$

(2.1)

and

$$\sum_{n=0}^{\infty} \mathcal{E}_n^{[m-1,\alpha]}(x, y; \lambda : p, q) \frac{z^n}{[n]_{p,q}!} = \left(\frac{2^m}{\lambda e_{p,q}(z) + T_{m-1}^{p,q}(z)} \right)^{\alpha} e_{p,q}(xz) E_{p,q}(yz),$$

(2.2)

where $T_{m-1}^{p,q}(z) = \sum_{h=0}^{m-1} \frac{z^h}{[h]_{p,q}!}$.

Remark 1. The order α of the Apostol type (p, q)-polynomials in Definition 1 (and also in all analogous situations occuring elsewhere in this paper) is tacitly assumed to be a nonnegative integer except possibly in those cases in which the right-hand side of the generating functions (2.1) and (2.2) turns out to be a power series in z. Only in these latter cases, we can safely assume that $\alpha \in \mathbb{C}$.

In the case $x = 0$ and $y = 0$ in Definition 1, we get

$$\mathcal{B}_n^{[m-1,\alpha]}(\lambda : p, q) := \mathcal{B}_n^{[m-1,\alpha]}(0, 0; \lambda : p, q)$$

and

$$\mathcal{E}_n^{(\alpha)}(\lambda : p, q) := \mathcal{E}_n^{[m-1,\alpha]}(0, 0; \lambda : p, q),$$

which are termed, respectively, the n-th generalized Apostol type (p, q)-Bernoulli numbers of order α and the n-th generalized Apostol type (p, q)-Euler numbers of order α.

For $\alpha = 1$ in Definition 1, we have

$$\mathcal{B}_n^{[m-1]}(x, y; \lambda : p, q) := \mathcal{B}_n^{[m-1,1]}(x, y; \lambda : p, q)$$

and

$$\mathcal{E}_n^{[m-1]}(x, y; \lambda : p, q) := \mathcal{E}_n^{[m-1,1]}(x, y; \lambda : p, q),$$

which are called, respectively, the n-th generalized Apostol type (p, q)-Bernoulli polynomial and the n-th generalized Apostol type (p, q)-Euler polynomial.

Remark 2. Upon setting $\lambda = 1$ in Definition 1, we obtain the generalized (p, q)-Bernoulli and Euler polynomials of order α defined in [4].

Remark 3. If we put $m = \lambda = 1$, then the polynomials in Definition 1 reduce to the known (p, q)-polynomials given in [3].

In the following corollaries, we discuss some particular situations of Definition 1.

Corollary 1 (see [10]). *If we take $p = 1$ in Definition 1, then we get*

$$\sum_{n=0}^{\infty} \mathfrak{B}_{n,q}^{[m-1,\alpha]}(x, y; \lambda) \frac{z^n}{[n]_q!} = \left(\frac{z^m}{\lambda e_q(z) - \sum_{k=0}^{m-1} \frac{z^k}{[k]_q!}} \right)^{\alpha} e_q(xz) E_q(yz),$$

$$\sum_{n=0}^{\infty} \mathfrak{E}_{n,q}^{[m-1,\alpha]}(x, y; \lambda) \frac{z^n}{[n]_q!} = \left(\frac{2^m}{\lambda e_q(z) + \sum_{k=0}^{m-1} \frac{z^k}{[k]_q!}} \right)^{\alpha} e_q(xz) E_q(yz),$$

where $\mathfrak{B}_{n,q}^{[m-1,\alpha]}(x, y; \lambda)$ and $\mathfrak{E}_{n,q}^{[m-1,\alpha]}(x, y; \lambda)$ are called the n-th generalized q-Apostol–Bernoulli polynomial of order α and the n-th generalized q-Apostol–Euler polynomial of order α, respectively.

Corollary 2 (see [6, 8, 12]). *When $q = 1$ and $y = 0$, the polynomials in Corollary 1 reduce to the polynomials in (1.6) and (1.7).*

The following proposition follows from Definition 1.

Proposition 1. *The following relations hold true:*

$$\mathcal{B}_n^{[m-1,\alpha]}(x, y; \lambda : p, q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} q^{(n-k)(n-k-1)/2} \mathcal{B}_k^{[m-1,\alpha]}(x, 0; \lambda : p, q) y^{n-k},$$

$$= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{(n-k)(n-k-1)/2} \mathcal{B}_k^{[m-1,\alpha]}(0, y; \lambda : p, q) x^{n-k},$$

and

$$\mathcal{E}_n^{[m-1,\alpha]}(x, y; \lambda : p, q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} q^{(n-k)(n-k-1)/2} \mathcal{E}_k^{[m-1,\alpha]}(x, 0; \lambda : p, q) y^{n-k},$$

$$= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{(n-k)(n-k-1)/2} \mathcal{E}_k^{[m-1,\alpha]}(0, y : p, q) x^{n-k},$$

Corollary 3. *Setting $y = 1$ (or $x = 1$) in Proposition 1 yields*

$$\mathcal{B}_n^{[m-1,\alpha]}(x, 1; \lambda : p, q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} q^{(n-k)(n-k-1)/2} \mathcal{B}_k^{[m-1,\alpha]}(x, 0; \lambda : p, q),$$

$$(2.3)$$

$$\mathcal{B}_n^{[m-1,\alpha]}(1, y; \lambda : p, q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{(n-k)(n-k-1)/2} \mathcal{B}_k^{[m-1,\alpha]}(0, y; \lambda : p, q),$$

$$(2.4)$$

and

$$\mathcal{E}_n^{[m-1,\alpha]}(x, 1; \lambda : p, q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} q^{(n-k)(n-k-1)/2} \mathcal{E}_k^{[m-1,\alpha]}(x, 0; \lambda : p, q),$$

$$(2.5)$$

$$\mathcal{E}_n^{[m-1,\alpha]}(1, y; \lambda : p, q) = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{(n-k)(n-k-1)/2} \mathcal{E}_k^{[m-1,\alpha]}(0, y; \lambda : p, q),$$

$$(2.6)$$

Notice that formulae (2.3)–(2.6) are (p, q)-analogues of the following formulae in [6, 8, 12]:

$$B_n^{[m-1,\alpha]}(x+1; \lambda) = \sum_{k=0}^n \binom{n}{k} B_k^{[m-1,\alpha]}(x; \lambda)$$

and

$$E_n^{[m-1,\alpha]}(x+1; \lambda) = \sum_{k=0}^n \binom{n}{k} E_k^{[m-1,\alpha]}(x; \lambda).$$

Now we present the addition properties of the generalized Apostol type (p, q)-Bernoulli and (p, q)-Euler polynomials of order α.

Proposition 2. *Let* $n \in \mathbb{N}$. *Then*

$$\mathcal{B}_n^{[m-1,\alpha+\beta]}(x, y; \lambda : p, q) = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} \mathcal{B}_{n-k}^{[m-1,\alpha]}(x, 0; \lambda : p, q) \mathcal{B}_k^{[m-1,\beta]}(0, y; \lambda : p, q)$$

and

$$\mathcal{E}_n^{[m-1,\alpha+\beta]}(x, y; \lambda : p, q) = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} \mathcal{E}_{n-k}^{[m-1,\alpha]}(x, 0; \lambda : p, q) \mathcal{E}_k^{[m-1,\beta]}(0, y; \lambda : p, q).$$

The (p, q)-derivatives of $\mathcal{B}_n^{[m-1,\alpha]}(x, y; \lambda : p, q)$ and $\mathcal{E}_n^{[m-1,\alpha]}(x, y; \lambda : p, q)$, with respect to x and y, are given as follows.

Proposition 3. *We have*

$$D_{p,q;x} \mathcal{B}_n^{[m-1,\alpha]}(x, y; \lambda : p, q) = [n]_{p,q} \mathcal{B}_{n-1}^{[m-1,\alpha]}(px, y; \lambda : p, q),$$

$$D_{p,q;y} \mathcal{B}_n^{[m-1,\alpha]}(x, y; \lambda : p, q) = [n]_{p,q} \mathcal{B}_{n-1}^{[m-1,\alpha]}(x, qy; \lambda : p, q),$$

$$D_{p,q;x} \mathcal{E}_n^{[m-1,\alpha]}(x, y; \lambda : p, q) = [n]_{p,q} \mathcal{E}_{n-1}^{[m-1,\alpha]}(px, y; \lambda : p, q),$$

$$D_{p,q;y} \mathcal{E}_n^{[m-1,\alpha]}(x, y; \lambda : p, q) = [n]_{p,q} \mathcal{E}_{n-1}^{[m-1,\alpha]}(x, qy; \lambda : p, q).$$

We nextly give the (p, q)-integral representations of $\mathcal{B}_n^{[m-1,\alpha]}(x, y; \lambda : p, q)$ and $\mathcal{E}_n^{[m-1,\alpha]}(x, y; \lambda : p, q)$.

Proposition 4. *We have the following integral representations:*

$$\int_a^b \mathcal{B}_n^{[m-1,\alpha]}(x, y; \lambda : p, q) \, d_{p,q}x$$

$$= p \, \frac{\mathcal{B}_{n+1}^{[m-1,\alpha]}\left(\frac{b}{p}, y; \lambda : p, q\right) - \mathcal{B}_{n+1}^{[m-1,\alpha]}\left(\frac{a}{p}, y; \lambda : p, q\right)}{[n+1]_{p,q}},$$

$$\int_a^b \mathcal{B}_n^{[m-1,\alpha]}(x,y;\lambda:p,q)\,d_{p,q}y$$

$$= p\,\frac{\mathcal{B}_{n+1}^{[m-1,\alpha]}\left(x,\frac{b}{q};\lambda:p,q\right) - \mathcal{B}_{n+1}^{[m-1,\alpha]}\left(x,\frac{a}{q};\lambda:p,q\right)}{[n+1]_{p,q}}$$

and

$$\int_a^b \mathcal{E}_n^{[m-1,\alpha]}(x,y;\lambda:p,q)\,d_{p,q}x$$

$$= p\,\frac{\mathcal{E}_{n+1}^{[m-1,\alpha]}\left(\frac{b}{p},y;\lambda:p,q\right) - \mathcal{E}_{n+1}^{[m-1,\alpha]}\left(\frac{a}{p},y;\lambda:p,q\right)}{[n+1]_{p,q}},$$

$$\int_a^b \mathcal{E}_n^{[m-1,\alpha]}(x,y;\lambda:p,q)\,d_{p,q}y$$

$$= p\,\frac{\mathcal{E}_{n+1}^{[m-1,\alpha]}\left(x,\frac{b}{q};\lambda:p,q\right) - \mathcal{E}_{n+1}^{[m-1,\alpha]}\left(x,\frac{a}{q};\lambda:p,q\right)}{[n+1]_{p,q}}.$$

Proof. The Claim follows from the property $\int_a^b D_{p,q}f(x)\,d_{p,q}x = f(b) - f(a)$ given in (1.5). $\qquad\square$

We next provide the following relations.

Proposition 5. *The following identities hold true:*

$$\lambda\mathcal{B}_n^{[m-1,\alpha]}(1,y;\lambda:p,q) - \sum_{k=0}^{\min(n,m-1)} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} \mathcal{B}_{n-k}^{[m-1,\alpha]}(0,y;\lambda:p,q)$$

$$= [n]_{p,q} \sum_{k=0}^{n-1} \begin{bmatrix} n-1 \\ k \end{bmatrix}_{p,q} \mathcal{B}_k^{[m-1,\alpha]}(0,y;\lambda:p,q)\,\mathcal{B}_{n-1-k}^{[0,-1]}(\lambda:p,q)$$

and

$$\lambda\mathcal{E}_n^{[m-1,\alpha]}(1,y;\lambda:p,q) + \sum_{k=0}^{\min(n,m-1)} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} \mathcal{E}_{n-k}^{[m-1,\alpha]}(0,y;\lambda:p,q)$$

$$= 2\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \mathcal{E}_k^{[m-1,\alpha-1]}(0,y;\lambda:p,q)\,\mathcal{E}_{n-k}^{[0,-1]}(\lambda:p,q).$$

Now we establish the following recurrence relationships.

Proposition 6. *We have*

$$
\lambda \mathcal{B}_n^{[m-1,\alpha]}(1,y;\lambda:p,q) - \sum_{k=0}^{\min(n,m-1)} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} \mathcal{B}_{n-k}^{[m-1,\alpha]}(0,y;\lambda:p,q)
$$

$$
= \frac{[n]_{p,q}!}{[n-m]_{p,q}!} \mathcal{B}_{n-m}^{[m-1,\alpha-1]}(0,y;\lambda:p,q) \quad (n \geq m),
$$

(2.7)

$$
\lambda \mathcal{B}_n^{[m-1,\alpha]}(x,0;\lambda:p,q) - \sum_{k=0}^{\min(n,m-1)} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} \mathcal{B}_{n-k}^{[m-1,\alpha]}(x,-1;\lambda:p,q)
$$

$$
= \frac{[n]_{p,q}!}{[n-m]_{p,q}!} \mathcal{B}_{n-m}^{[m-1,\alpha-1]}(x,-1;\lambda:p,q) \quad (n \geq m)
$$

and

$$
\lambda \mathcal{E}_n^{[m-1,\alpha]}(1,y;\lambda:p,q) + \sum_{k=0}^{\min(n,m-1)} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} \mathcal{E}_{n-k}^{[m-1,\alpha]}(0,y;\lambda:p,q)
$$

$$
= 2^m \mathcal{E}_n^{[m-1,\alpha-1]}(0,y;\lambda:p,q),
$$

(2.8)

$$
\lambda \mathcal{E}_n^{[m-1,\alpha]}(x,0;\lambda:p,q) + \sum_{k=0}^{\min(n,m-1)} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} \mathcal{E}_{n-k}^{[m-1,\alpha]}(x,-1;\lambda:p,q)
$$

$$
= 2^m \mathcal{E}_n^{[m-1,\alpha-1]}(x,-1;\lambda:p,q).
$$

Proof. By utilizing the idea of the proof in [9] and the formula

$$
T_{m-1}^{p,q}(z) \left(\frac{z^m}{\lambda e_{p,q}(z) - T_{m-1}^{p,q}(z)} \right)^\alpha E_{p,q}(yz)
$$

$$
= \sum_{n=0}^{m-1} \frac{z^n}{[n]_{p,q}!} \sum_{n=0}^\infty \mathcal{B}_n^{[m-1,\alpha]}(0,y;\lambda:p,q) \frac{z^n}{[n]_{p,q}!}
$$

$$
= \sum_{n=0}^\infty \mathcal{B}_n^{[m-1,\alpha]}(0,y;\lambda:p,q) \left(\frac{z^n}{[n]_{p,q}!} + \frac{z^{n+1}}{[n]_{p,q}!} + \frac{z^{n+2}}{[n]_{p,q}![2]_{p,q}!} \right.
$$

$$
\left. + \cdots + \frac{z^{n+m-1}}{[n]_{p,q}![m-1]_{p,q}!} \right)
$$

$$
= \sum_{n=0}^\infty \mathcal{B}_n^{[m-1,\alpha]}(0,y;\lambda:p,q) \frac{z^n}{[n]_{p,q}!} + \sum_{n=0}^\infty [n]_{p,q} \mathcal{B}_n^{[m-1,\alpha]}(0,y:p,q) \frac{z^n}{[n]_{p,q}!}
$$

$$
+ \cdots + \sum_{n=0}^\infty \frac{[n]_{p,q} \cdots [n-m+2]_{p,q}}{[m-1]_{p,q}!} \mathcal{B}_{n-m+1}^{[m-1,\alpha]}(0,y;\lambda:p,q) \frac{z^n}{[n]_{p,q}!}
$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{\min(n,m-1)} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} \mathcal{B}_{n-k}^{[m-1,\alpha]}(0,y;\lambda:p,q) \frac{z^n}{[n]_{p,q}!},$$

we arrive at the following:

$$\sum_{n=0}^{\infty} \left(\lambda \mathcal{B}_n^{[m-1,\alpha]}(1,y;\lambda:p,q) - \sum_{k=0}^{\min(n,m-1)} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} \mathcal{B}_{n-k}^{[m-1,\alpha]}(0,y;\lambda:p,q) \right) \frac{z^n}{[n]_{p,q}!}$$

$$= \lambda \left(\frac{z^m}{\lambda e_{p,q}(z) - T_{m-1}^{p,q}(z)} \right)^{\alpha} e_{p,q}(z) E_{p,q}(yz)$$

$$- T_{m-1}^{p,q}(z) \left(\frac{z^m}{\lambda e_{p,q}(z) - T_{m-1}^{p,q}(z)} \right)^{\alpha} E_{p,q}(yz)$$

$$= \left(\frac{z^m}{\lambda e_{p,q}(z) - T_{m-1}^{p,q}(z)} \right)^{\alpha} E_{p,q}(yz) \left(\lambda e_{p,q}(z) - T_{m-1}^{p,q}(z) \right)$$

$$= z^m \left(\frac{z^m}{\lambda e_{p,q}(z) - T_{m-1}^{p,q}(z)} \right)^{\alpha-1} E_{p,q}(yz) = \sum_{n=0}^{\infty} \mathcal{B}_n^{[m-1,\alpha-1]}(0,y;\lambda:p,q) \frac{z^{n+m}}{[n]_{p,q}!}.$$

Comparing the coefficients of z^n on both sides, we obtain (2.7). The other equalities in this theorem can be proved similarly. $\qquad\square$

It follows from Definition 1 in the case $\alpha = 0$ that

$$\mathcal{B}_n^{[m-1,0]}(x,y;\lambda:p,q) = \mathcal{E}_n^{[m-1,0]}(x,y;\lambda:p,q) = (x \oplus y)_{p,q}^n. \qquad (2.9)$$

By combining Proposition 6 and Corollary 3 with (2.9) in the case $\alpha = 1$, we acquire the following formulae for $n \geq m$:

$$y^{n-m} = \frac{[n-m]_{p,q}!}{q^{\binom{n-m}{2}} [n]_{p,q}!} \left(\lambda \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\binom{n-k}{2}} \mathcal{B}_k^{[m-1]}(0,y;\lambda:p,q) \right.$$

$$\left. - \sum_{k=0}^{\min(n,m-1)} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} \mathcal{B}_{n-k}^{[m-1]}(0,y;\lambda:p,q) \right)$$

and

$$y^n = \frac{2^{-m}}{q^{\binom{n}{2}}} \left(\lambda \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\binom{n-k}{2}} \mathcal{E}_k^{[m-1]}(0,y:p,q) \right.$$

$$\left. + \sum_{k=0}^{\min(n,m-1)} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} \mathcal{E}_{n-k}^{[m-1]}(0,y:p,q) \right).$$

In the following theorem, we give some relations between the old and new (p,q)-polynomials given in Definition 1.

Theorem 1. *For $n \in \mathbb{N}_0$ and $x, y \in \mathbb{C}$, the following relations hold true:*

$$\mathcal{B}_n^{[m-1,\alpha]}(x, y; \lambda : p, q) = \frac{1}{[n+1]_{p,q}} \sum_{u=0}^{n+1} \begin{bmatrix} n+1 \\ u \end{bmatrix}_{p,q} \mathcal{B}_{n-u+1}(0, ly; \lambda : p, q) l^{u-n}$$

$$\times \left(\lambda \sum_{s=0}^{u} \begin{bmatrix} u \\ s \end{bmatrix}_{p,q} \mathcal{B}_s^{[m-1,\alpha]}(x, 0; \lambda : p, q) l^{s-u} p^{\binom{u-s}{2}} - \mathcal{B}_u^{[m-1,\alpha]}(x, 0; \lambda : p, q) \right)$$

$$= \frac{1}{[n+1]_{p,q}} \sum_{u=0}^{n+1} \begin{bmatrix} n+1 \\ u \end{bmatrix}_{p,q} \mathcal{B}_{n-u+1}(lx, 0; \lambda : p, q) l^{u-n}$$

$$\times \left(\lambda \sum_{s=0}^{u} \begin{bmatrix} u \\ s \end{bmatrix}_{p,q} \mathcal{B}_s^{[m-1,\alpha]}(0, y; \lambda : p, q) l^{s-u} p^{\binom{u-s}{2}} - \mathcal{B}_u^{[m-1,\alpha]}(0, y; \lambda : p, q) \right)$$

and

$$\mathcal{E}_n^{[m-1,\alpha]}(x, y; \lambda : p, q) = \frac{1}{[2]_{p,q}} \sum_{u=0}^{n} \begin{bmatrix} n \\ u \end{bmatrix}_{p,q} \mathcal{E}_{n-u}(0, ly; \lambda : p, q) l^{u-n}$$

$$\times \left(\lambda \sum_{s=0}^{u} \begin{bmatrix} u \\ s \end{bmatrix}_{p,q} \mathcal{E}_s^{[m-1,\alpha]}(x, 0; \lambda : p, q) l^{s-u} p^{\binom{u-s}{2}} + \mathcal{E}_u^{[m-1,\alpha]}(x, 0; \lambda : p, q) \right)$$

$$= \frac{1}{[2]_{p,q}} \sum_{u=0}^{n} \begin{bmatrix} n \\ u \end{bmatrix}_{p,q} \mathcal{E}_{n-u}(lx, 0; \lambda : p, q) l^{u-n}$$

$$\times \left(\lambda \sum_{s=0}^{u} \begin{bmatrix} u \\ s \end{bmatrix}_{p,q} \mathcal{E}_s^{[m-1,\alpha]}(0, y; \lambda : p, q) l^{s-u} p^{\binom{u-s}{2}} + \mathcal{E}_u^{[m-1,\alpha]}(0, y; \lambda : p, q) \right)$$

where $\mathcal{B}_n(x, y; \lambda : p, q)$ and $\mathcal{E}_n(x, y; \lambda : p, q)$ are, respectively, the Apostol type (p, q)-Bernoulli polynomials and the Apostol type (p, q)-Euler polynomials defined in [2].

Proof. The claim follows from Definition 1 by simple calculations. $\qquad \square$

New connections including $\mathcal{B}_n^{[m-1,\alpha]}(x, y; \lambda : p, q)$ and $\mathcal{E}_n^{[m-1,\alpha]}(x, y; \lambda : p, q)$, which derive from Definition 1 using the Cauchy product, are presented in the following two theorems. We state these theorems without proofs.

Theorem 2. *For $n \in \mathbb{N}_0$, $m \in \mathbb{N}$ and $x, y \in \mathbb{C}$, the following relations hold true:*

$$\mathcal{B}_n^{[m-1,\alpha]}(x, y; \lambda : p, q) = \frac{1}{[2]_{p,q}} \sum_{u=0}^{n} \begin{bmatrix} n \\ u \end{bmatrix}_{p,q} \mathcal{E}_{n-u}(0, ly; \lambda : p, q) l^{u-n}$$

$$\times \left(\lambda \sum_{s=0}^{u} \begin{bmatrix} u \\ s \end{bmatrix}_{p,q} \mathcal{B}_s^{[m-1,\alpha]}(x, 0; \lambda : p, q) l^{s-u} p^{\binom{u-s}{2}} + \mathcal{B}_u^{[m-1,\alpha]}(x, 0; \lambda : p, q) \right)$$

and

$$\mathcal{E}_n^{[m-1,\alpha]}(x,y;\lambda:p,q) = \frac{1}{[n+1]_{p,q}} \sum_{u=0}^{n+1} \begin{bmatrix} n+1 \\ u \end{bmatrix}_{p,q} \mathcal{B}_{n-u+1}(0,ly;\lambda:p,q)\, l^{u-n}$$

$$\times \left(\lambda \sum_{s=0}^{u} \begin{bmatrix} u \\ s \end{bmatrix}_{p,q} \mathcal{E}_s^{[m-1,\alpha]}(x,0;\lambda:p,q)\, l^{s-u} p^{\binom{u-s}{2}} - \mathcal{E}_u^{[m-1,\alpha]}(x,0;\lambda:p,q) \right).$$

Theorem 3. *We have*

$$\mathcal{B}_n^{[m-1,\alpha]}(x,y;\lambda:p,q) = \frac{1}{[2]_{p,q}} \sum_{u=0}^{n} \begin{bmatrix} n \\ u \end{bmatrix}_{p,q} \mathcal{E}_{n-u}(lx,0;\lambda:p,q)\, l^{u-n}$$

$$\times \left(\lambda \sum_{s=0}^{u} \begin{bmatrix} u \\ s \end{bmatrix}_{p,q} \mathcal{B}_s^{[m-1,\alpha]}(0,y:p,q)\, l^{s-u} p^{\binom{u-s}{2}} + \mathcal{B}_u^{[m-1,\alpha]}(0,y;\lambda:p,q) \right)$$

and

$$\mathcal{E}_n^{[m-1,\alpha]}(x,y;\lambda:p,q) = \frac{1}{[n+1]_{p,q}} \sum_{u=0}^{n+1} \begin{bmatrix} n+1 \\ u \end{bmatrix}_{p,q} \mathcal{B}_{n-u+1}(lx,0;\lambda:p,q)\, l^{u-n}$$

$$\times \left(\lambda \sum_{s=0}^{u} \begin{bmatrix} u \\ s \end{bmatrix}_{p,q} \mathcal{E}_s^{[m-1,\alpha]}(0,y:p,q)\, l^{s-u} p^{\binom{u-s}{2}} - \mathcal{E}_u^{[m-1,\alpha]}(0,y;\lambda:p,q) \right).$$

From Proposition 1 and Theorem 3, we deduce the following corollary.

Corollary 4. *The following relations hold true:*

$$\mathcal{B}_n^{[m-1,\alpha]}(x,y;\lambda:p,q) = \frac{1}{[2]_{p,q}} \sum_{u=0}^{n} \begin{bmatrix} n \\ u \end{bmatrix}_{p,q} l^{u-n} \mathcal{E}_{n-u}(lx,0;\lambda:p,q)$$

$$\times \left(\lambda \mathcal{B}_u^{[m-1,\alpha]}\left(\frac{1}{l},y;\lambda:p,q\right) + \mathcal{B}_u^{[m-1,\alpha]}(0,y;\lambda:p,q) \right) \tag{2.10}$$

and

$$\mathcal{E}_n^{[m-1,\alpha]}(x,y;\lambda:p,q) = \frac{1}{[n+1]_{p,q}} \sum_{u=0}^{n+1} \begin{bmatrix} n+1 \\ u \end{bmatrix}_{p,q} l^{u-n} \mathcal{B}_{n-u+1}(lx,0;\lambda:p,q)$$

$$\times \left(\lambda \mathcal{E}_u^{[m-1,\alpha]}\left(\frac{1}{l},y;\lambda:p,q\right) - \mathcal{E}_u^{[m-1,\alpha]}(0,y;\lambda:p,q) \right).$$

Proposition 6 and Corollary 4 yield the following result.

Theorem 4. *The following formulae are valid for $n \in \mathbb{N}_0$:*

$$\mathcal{B}_n^{[m-1,\alpha]}(x,y;\lambda:p,q) = \frac{\lambda}{[2]_{p,q}} \sum_{u=0}^{n} \begin{bmatrix} n \\ u \end{bmatrix}_{p,q}$$

$$\times \left[[u]_{p,q} \sum_{k=0}^{u-1} \begin{bmatrix} u-1 \\ k \end{bmatrix}_{p,q} \mathcal{B}_k^{[m-1,\alpha]}(0,y;\lambda:p,q) \mathcal{B}_{u-1-k}^{[0,-1]}(\lambda:p,q) \right.$$

$$+ \sum_{k=0}^{\min(u,m-1)} \begin{bmatrix} u \\ k \end{bmatrix}_{p,q} \mathcal{B}_{u-k}^{[m-1,\alpha]}(0,y;\lambda:p,q) \tag{2.11}$$

$$\left. + \frac{1}{\lambda} \mathcal{B}_u^{[m-1,\alpha]}(0,y;\lambda:p,q) \right] \mathcal{E}_{n-u}(x,0;\lambda:p,q)$$

and

$$\mathcal{E}_n^{[m-1,\alpha]}(x,y;\lambda:p,q) = \frac{\lambda}{[n+1]_{p,q}} \sum_{u=0}^{n+1} \left[\begin{bmatrix} n+1 \\ u \end{bmatrix} \right]_{p,q}$$

$$\times \left[2 \sum_{k=0}^{u} \begin{bmatrix} u \\ k \end{bmatrix} \mathcal{E}_k^{[m-1,\alpha-1]}(0,y;\lambda:p,q) \mathcal{E}_{u-k}^{[0,-1]}(\lambda:p,q) \right.$$

$$- \sum_{k=0}^{\min(u,m-1)} \begin{bmatrix} u \\ k \end{bmatrix}_{p,q} \mathcal{E}_{u-k}^{[m-1,\alpha]}(0,y;\lambda:p,q)$$

$$\left. - \frac{1}{\lambda} \mathcal{E}_u^{[m-1,\alpha]}(0,y;\lambda:p,q) \right] \mathcal{B}_{n-u+1}(x,0;\lambda:p,q).$$

The equation (2.11) is a (p,q)-extension of the Srivastava–Pintér addition theorem (cf. [14]) for the generalized Apostol type Bernoulli and Apostol type Euler polynomials of order α given by (see [15])

$$\mathcal{B}_n^{[m-1,\alpha]}(x+y;\lambda)$$

$$= \frac{1}{2} \sum_{u=0}^{n} \binom{n}{u} \left[\mathcal{B}_u^{[m-1,\alpha]}(y;\lambda) + \lambda \sum_{k=0}^{\min(u,m-1)} \binom{u}{k} \mathcal{B}_{u-k}^{[m-1,\alpha]}(y;\lambda) \right.$$

$$\left. \times u\lambda \sum_{k=0}^{u-1} \binom{u-1}{k} \mathcal{B}_k^{[m-1,\alpha]}(y;\lambda) \mathcal{B}_{u-1-k}^{[0,-1]}(\lambda) \right] \mathcal{E}_{n-u}(x;\lambda).$$

We define the generalized (p,q)-Stirling polynomials $S_{p,q}^{[m-1]}(n,v;\lambda)$ of the second kind of order v by means of the generating function

$$\sum_{n=0}^{\infty} S_{p,q}^{[m-1]}(n,v;\lambda) \frac{z^n}{[n]_{p,q}!} = \frac{\left(\lambda e_{p,q}(z) - \sum_{h=0}^{m-1} \frac{z^h}{[h]_{p,q}!} \right)^v}{[v]_{p,q}!}. \tag{2.12}$$

Remark 4. When $q \to p = m = \lambda = 1$, the above polynomials reduce to the usual Stirling numbers of second kind given by (see [5])

$$\sum_{n=0}^{\infty} S(n, v) \frac{z^n}{n!} = \frac{(e^z - 1)^v}{v!}.$$

The following theorem includes a relationship between the generalized Apostol type (p, q)-Bernoulli polynomials of order v and the generalized (p, q)-Stirling polynomials of the second kind of order v.

Theorem 5. *The following formula holds true:*

$$\sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_{p,q} S_{p,q}^{[m-1]} (n - j, v; \lambda) (x \oplus y)_{p,q}^j$$

$$= \frac{[n]_{p,q}!}{[v]_{p,q}! [n - vm]_{p,q}!} \mathcal{B}_{n-vm}^{[m-1,-v]} (x, y; \lambda : p, q).$$

Proof. In view of Definition 1 and (2.12), we get

$$\sum_{n=0}^{\infty} S_{p,q}^{[m-1]} (n, v; \lambda) \frac{z^n}{[n]_{p,q}!} \sum_{n=0}^{\infty} (x \oplus y)_{p,q}^n \frac{z^n}{[n]_{p,q}!}$$

$$= \sum_{n=0}^{\infty} S_{p,q}^{[m-1]} (n, v; \lambda) \frac{z^n}{[n]_{p,q}!} e_{p,q} (xz) E_{p,q} (yz)$$

$$= \left(\frac{\lambda e_{p,q} (z) - \sum_{h=0}^{m-1} \frac{z^h}{[h]_{p,q}!}}{z^m} \right)^v e_{p,q} (xz) E_{p,q} (yz) \frac{z^{vm}}{[v]_{p,q}!}$$

$$= \sum_{n=0}^{\infty} \mathcal{B}_n^{[m-1,-v]} (x, y; \lambda : p, q) \frac{z^{n+vm}}{[n]_{p,q}!} \frac{1}{[v]_{p,q}!}.$$

By matching the coefficients z^n on both sides above, we obtain the desired result. □

Corollary 5. *We have*

$$\sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_{p,q} S_{p,q}^{[m-1]} (n-j, v; \lambda) \mathcal{B}_j^{[m-1,v]}(x, y; \lambda : p, q) = (x \oplus y)_{p,q}^{n-vm} \frac{[n - vm]_{p,q}!}{[v]_{p,q}! [n]_{p,q}!}.$$

Acknowledgement

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