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# Lucas numbers of the form $\binom{2^t}{k}$

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*To the memory of Professor József Závoti*

Abstract. Let $L_m$ denote the $m^{th}$ Lucas number. We show that the solutions to the diophantine equation $\binom{2^t}{k} = L_m$, in non-negative integers $t,\ k \leq 2^{t-1}$, and $m$, are $(t,k,m) = (1,1,0),\ (2,1,3)$, and $(a,0,1)$ with non-negative integers $a$.

## 1. Introduction

As usual, the sequence of Lucas numbers is defined by $L_0 = 2,\ L_1 = 1$, and

$$L_m = L_{m-1} + L_{m-2}, \qquad m \geq 2.$$

This sequence is known as the associate of Fibonacci sequence.

Now we present a short historical background related to the title problem. The occurrence of figurate numbers in linear recurrences has had a very extensive literature. The first challenging result is due to Cohn [1, 2], and independently to Wyler [18], who proved that the square Fibonacci numbers are $F_0 = 0,\ F_1 = F_2 = 1$ and $F_{12} = 144$. Focusing only on the occurrence of binomial coefficients in binary recurrences, first we mention that Ming [11] proved a conjecture of Hoggatt [5]. Namely, he showed that $F_0 = 0,\ F_1 = F_2 = 1,\ F_4 = 3,\ F_8 = 21$ and $F_{10} = 55$ are the only triangular Fibonacci numbers, further $L_1 = 1,\ L_2 = 3$ and $L_{18} = 5778$ are the only Lucas triangular numbers [12]. Note that the triangular number $t_{n-1} = (n-1)n/2$ is equal to the binomial coefficient $\binom{n}{2}$. Therefore, it seems natural to search the binomial coefficients $\binom{n}{k}$ in certain recurrences. Special cases of this question were handled by several authors, see, for example, [3].

Consider the binary recurrence $U_m = AU_{m-1} + BU_{m-2}$ with arbitrary initial values $U_0$ and $U_1$. If $\{V_m\}$ is the associate of $\{U_m\}$ (i.e., the two

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sequences have the same recurrence rule, further $V_0 = 2U_1 - AU_0$ and $V_1 = AU_1 + 2BU_0$), then their terms satisfy

$$V_n^2 - DU_n^2 = 4C(-B)^n, \tag{1.1}$$

where $D = A^2 + 4B$ and $C = U_1^2 - AU_0U_1 - BU_0^2$.

Fix $|B| = 1$. Replacing either $V_n$ or $U_n$ by $\binom{n}{2}$, (1.1) leads to the superelliptic equation

$$y^2 = Dn^4 - 2Dn^3 + Dn^2 \pm 16C.$$

The Magma [10] procedure `IntegralQuarticPoints()` may solve this equation. Hence if the lower index $k$ is 2 in $\binom{n}{k}$, then we are able to handle the problem for certain binary recurrences.

For the lower index $k = 3$ an algorithm was given in [16] to solve the equations

$$U_m = \binom{n}{3} \quad \text{and} \quad V_m = \binom{n}{3},$$

with the conditions $D > 0$, and $U_0 = 0$, $U_1 = 1$ (and $|B| = 1$). Illustrating the algorithm, all integer solutions to the equations

$$F_m = \binom{n}{3}, \quad L_m = \binom{n}{3} \quad \text{and} \quad P_m = \binom{n}{3}$$

were given in [16]. Here $P_m$ is a term of the Pell sequence.

Later, Szalay [15] treated the equations $F_m = \binom{n}{4}$, $L_m = \binom{n}{4}$, and Kovács [6] solved the analogous equation $P_m = \binom{n}{4}$. The more complicated problem $L_m = \binom{n}{5}$ was handled by Tengely [17].

In this paper, as a novelty, we do not fix the lower subscript $k$, but on the other hand we prescribe $n = 2^t$ with unknown non-negative integer $t$. Hence, for the Lucas numbers we study the diophantine equation

$$L_m = \binom{2^t}{k}.$$

The complete description of the result is given by the following theorem.

**Theorem 1.** *The solutions to the diophantine equation*

$$L_m = \binom{2^t}{k} \tag{1.2}$$

*in non-negative integers $t$, $k \leq 2^{t-1}$, and $m$ are*

$$(t, k, m) = (1, 1, 0), \ (2, 1, 3) \text{ and } (a, 0, 1)$$

*with non-negative integers $a$.*

## 2. Auxiliary results

Assume that $p$ is a prime number. The $p$-adic order of a non-zero integer $n$ is the largest positive integer exponent $\nu$ of $p$ such that $p^\nu$ divides $n$. As usual, let $\nu$ be denoted by $\nu_p(n)$. For the integer $n = a_0 + a_1 p + a_2 p^2 + \cdots + a_d p^d$, $(0 \le a_i < p)$, the digit sum function (in base $p$) is defined by

$$s_p(n) = a_0 + a_1 + \cdots + a_d.$$

In particular, Legendre [8] showed that

$$\nu_p(n!) = \frac{n - s_p(n)}{p-1}. \tag{2.1}$$

**Lemma 1.** *Assume that $n$ and $k \le 2^n - 1$ are positive integers. Then*

$$\nu_2\left(\binom{2^n}{k}\right) = n - \nu_2(k).$$

*Proof.* It is clear that $\nu_2(2^n - j) = \nu_2(j)$ holds if $1 \le j \le 2^n - 1$. Expanding the binomial coefficient we get

$$\begin{aligned}\nu_2\left(\binom{2^n}{k}\right) &= \nu_2\left(\frac{2^n(2^n-1)\dots(2^n-k+1)}{k!}\right)\\ &= \nu_2(2^n) + \nu_2(2^n-1) + \ldots + \nu_2(2^n-(k-1)) - \nu_2(k!)\\ &= n + \nu_2((k-1)!) - \nu_2(k!) =\\ &= n - \nu_2(k).\end{aligned}$$

□

We note that Kummer [7] derived a result from Legendre's theorem, which also proves the statement of the above lemma. Kummer's theorem says that the $p$-adic valuation of the binomial coefficient $\binom{a}{b}$ is equal to the number of carries when $a-b$ is added to $b$ in base $p$.

Citing [9], here we present the 2-adic order of the Lucas numbers.

**Lemma 2.** *If $n \ge 0$ is an integer, then*

$$\nu_2(L_n) = \begin{cases} 0, & \text{if } n \equiv 1, 2 \pmod 3,\\ 1, & \text{if } n \equiv 0 \pmod 6,\\ 2, & \text{if } n \equiv 3 \pmod 6.\end{cases}$$

**Lemma 3.** *For any integer $n \ge 0$ we have $L_n \not\equiv 6 \pmod 8$.*

*Proof.* Consider the Lucas numbers modulo 8. The sequence becomes periodic with length 12, and looking at the period, it leads immediately to the statement. □

**Lemma 4.** *A Lucas number $L_n$ with odd subscript $n$ is composed only of primes $p$ satisfying $p = 2$ or $p \equiv \pm 1 \pmod 5$.*

*Proof.* Although the proof comes straightaway from the well-know identity $L_n^2 - 5F_n^2 = 4(-1)^n$, we simply refer to [13], last row of page 280. □

**Lemma 5.** *Suppose that $a$, $b$ and $n$ are positive integers. Then*

$$\binom{an+bn}{an} \equiv 0 \mod \left(\frac{bn+1}{\gcd(a, bn+1)}\right).$$

*Proof.* See Theorem 1.1 in [14]. □

**Lemma 6.** *For $n \geq 1$ we have*

$$\binom{2^{n+1}}{2^n} \equiv 6 \pmod 8.$$

*Proof.* It is obvious when $n = 1$. Therefore we may assume $n \geq 2$.

In case of $p = 2$ the Legendre formula (2.1) implies $\nu_2(2^a!) = 2^a - 1$. Subsequently,

$$\nu_2\left(\binom{2^{n+1}}{2^n}\right) = \nu_2\left(\frac{2^{n+1}!}{(2^n!)^2}\right) = \nu_2(2^{n+1}!) - 2\nu_2(2^n!) = 1,$$

hence it is sufficient to consider the odd ingredients of the binomial coefficient in the lemma. To do that, observe that $h(a) := 2^a!/2^{2^a-1}$ is an odd integer, and we need to see that $h(a) \equiv 3 \pmod 8$ for $a \geq 2$. It is a direct consequence of Lemma 3.3 in the paper [4] by fixing there $p = 2$, $b = 3$, $t = 1$, $i = 0$, and $j = 1$. Finally, $h(n+1)/h^2(n) \equiv 3/3^2 \equiv 3 \pmod 8$ proves the lemma. □

## 3. Proof of Theorem 1

The statement is trivial for $k = 0$, and we obtain the infinite family of solutions $(t, k, m) = (a, 0, 1)$, $a \geq 0$.

In the sequel, we assume $1 \leq k \leq 2^{t-1}$. Combining (1.2), Lemma 2, and Lemma 1, it provides

$$j = t - \nu_2(k),$$

where $j = 0, 1, 2$. Thus, $t - j = \nu_2(k)$, and, consequently, $k = 2^{t-j}s$ holds with some positive odd integer $s$. The condition $k = 2^{t-j}s \leq 2^{t-1}$ is fulfilled only if $j = 1$ or 2, and in these cases $s = 1$ necessarily holds. Hence $k = 2^{t-j}$ ($j = 1, 2$). For our convenience put $a = t - j$. Then $k = 2^a$, and we distinguish two cases.

First let $j = 1$. Now we have the equation

$$L_m = \binom{2^{a+1}}{2^a}$$

to solve. Taking both sides of this equation modulo 8, Lemma 3 contradicts to Lemma 6 if $a \geq 1$. The remaining value $a = 0$ leads to the solution $(t, k, m) = (1, 1, 0)$.

Now let $j = 2$. Clearly, by Lemma 2 we know that $m = 6\kappa + 3$. We have

$$L_m = \binom{2^{a+2}}{2^a},$$

and first assume that $a$ is even. The case $a = 0$ provides the solution $(t, k, m) = (2, 1, 3)$. Then we may suppose $a \geq 1$. Applying Lemma 5, it yields that

$$\binom{2^{a+2}}{2^a} = \binom{3 \cdot 2^a + 2^a}{3 \cdot 2^a} \equiv 0 \mod \left(\frac{2^a + 1}{\gcd(3, 2^a + 1)}\right).$$

The parity of $a$ guarantees that the denominator of the modulus is 1, i.e., the modulus is $2^a + 1$. Put $a = 2\ell$. Note that $\ell \geq 1$. Then we obtain

$$L_{6\kappa+3} = \binom{4^{\ell+1}}{4^\ell} \equiv 0 \pmod{4^\ell + 1}.$$

This gives that $4^\ell + 1 \mid L_{6\kappa+3}$. By Lemma 4 we have

$$L_{6\kappa+3} = p_1 p_2 \cdots p_n,$$

where $p_i$ are primes with $p_i = 2$ or $p_i \equiv \pm 1 \pmod 5$ for $1 \leq i \leq n$. Hence every prime factor $p_j$ of $4^\ell + 1$ ($\ell \geq 1$) has the form $p_j \equiv \pm 1 \pmod 5$. Thus,

$$4^\ell + 1 = p_{i_1} p_{i_2} \cdots p_{i_t} \tag{3.1}$$

follows with $t \leq n$. Now reduce (3.1) modulo 5, and we arrive at a contradiction since $4^\ell + 1 \equiv 0$ or $2 \pmod 5$, and at the same time $p_1 p_2 \cdots p_t \equiv 1$ or $4 \pmod 5$.

Assume that $a$ is odd, and let $a = 2\ell + 1$ with a non-negative integer $\ell$. The case $\ell = 0$ does not provide a solution to (1.2). So we may assume $\ell \geq 1$. Now we get

$$\binom{2^{a+2}}{2^a} = \binom{2^a + 3 \cdot 2^a}{2^a} \equiv 0 \pmod{3 \cdot 2^a + 1}$$

because trivially $\gcd(1, 3 \cdot 2^a + 1) = 1$. Thus,

$$3 \cdot 2^a + 1 = 6 \cdot 4^\ell + 1 \mid L_{6\kappa+3},$$

where the prime factors $p_j$ of $L_{6\kappa+3}$ again satisfy $p_j \equiv \pm 1 \pmod 5$. A modulo 5 consideration of $6 \cdot 4^\ell + 1$, similarly to the previous case, leads to a contradiction.

The proof of Theorem 1 is complete.

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