ACTA ET COMMENTATIONES UNIVERSITATIS TARTUENSIS DE MATHEMATICA
Volume 23, Number 1, June 2019
Available online at http://acutm.math.ut.ee

Estimations of Riemann–Liouville k-fractional integrals via convex functions

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Abstract. The k-fractional integrals introduced by S. Mubeen and G. M. Habibullah in 2012 are a generalization of Riemann–Liouville fractional integrals. Some estimations of these fractional integrals via convexity have been established.

1. Introduction

Riemann–Liouville fractional integral operator is the first formulation of an integral operator of non-integral order.

Definition 1. Let $f \in L_1[a,b]$. Then the Riemann–Liouville fractional integrals of f of order $\alpha > 0$ with $a \geq 0$ are defined by

$$I_{a^+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, \; x > a,$$

and

$$I_{b_-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt, \; x < b.$$

In fact these formulations of fractional integral operators have been established by Letnikov [10], Sonin [12], and then by Laurent [9]. A lot of fractional integral inequalities have been established in literature (for more details, see [1, 3, 4, 5, 6, 7, 8, 13]).

In [11], the following generalization of Riemann–Liouville fractional integrals was studied.

Received May 27, 2018.
2010 *Mathematics Subject Classification.* 26A51; 26A33; 26D15.
Key words and phrases. Convex function; Riemann–Liouville k-fractional integral.
http://dx.doi.org/10.12697/ACUTM.2019.23.07

Definition 2. Let $f \in L_1[a,b]$. The Riemann–Liouville k-fractional integrals of f of order α, with $k > 0$ and $a \geq 0$, are defined by

$$I_{a^+}^{\alpha,k} f(x) = \frac{1}{k\Gamma_k(\alpha)} \int_a^x (x-t)^{\frac{\alpha}{k}-1} f(t)dt, \; x > a,$$

and

$$I_{b_-}^{\alpha,k} f(x) = \frac{1}{k\Gamma_k(\alpha)} \int_x^b (t-x)^{\frac{\alpha}{k}-1} f(t)dt, \; x < b,$$

where $\Gamma_k(\alpha)$ is the k-Gamma function defined as $\Gamma_k(\alpha) = \int_0^\infty t^{\alpha-1} e^{-\frac{t^k}{k}} dt$.

Inequalities have always proved to be useful in establishing mathematical models and their solutions in almost all branches of applied sciences, in particular, in physics and engineering. Convexity plays a very important role in the optimization of solutions of mathematical problems. The aim of this paper is to extend some k-fractional inequalities via convexity properties of functions.

2. Main results

The following theorem gives an estimate for the sum of the left and right handed Riemann–Liouville k-fractional integrals.

Theorem 1. *Let $f \colon I \longrightarrow \mathbb{R}$ be a positive convex function. Then, for $a, b \in I$ and $\alpha, \beta \geq k$, the following inequality for the Riemann–Liouville k-fractional integrals holds:*

$$\begin{aligned} I_{a^+}^{\alpha,k} f(x) + I_{b^-}^{\beta,k} f(x) \leq \; & \frac{(x-a)^{\frac{\alpha}{k}} f(a) + (b-x)^{\frac{\beta}{k}} f(b)}{2k\Gamma_k(\alpha)} \\ & + f(x)\left(\frac{(x-a)^{\frac{\alpha}{k}} + (b-x)^{\frac{\beta}{k}}}{2k\Gamma_k(\beta)}\right), \quad x \in (a,b). \end{aligned} \tag{2.1}$$

Proof. It is easy to observe the following inequality for $\alpha > k$ and $x \in [a,b]$:

$$(x-t)^{\frac{\alpha}{k}-1} \leq (x-a)^{\frac{\alpha}{k}-1}, \quad t \in [a,x]. \tag{2.2}$$

The convexity of f provides the inequality

$$f(t) \leq \frac{x-t}{x-a} f(a) + \frac{t-a}{x-a} f(x), \quad t \in [a,x], \; x \in (a,b). \tag{2.3}$$

From (2.2) and (2.3), we obtain that

$$\int_a^x (x-t)^{\frac{\alpha}{k}-1} f(t)dt \leq \frac{(x-a)^{\frac{\alpha}{k}-1}}{x-a} \left(f(a) \int_a^x (x-t)dt + f(x) \int_a^x (t-a)dt \right).$$

Therefore, in view of the definition of the Riemann–Liouville k-fractional integrals, we get

$$I_{a^+}^{\alpha,k} f(x) \leq \frac{(x-a)^{\frac{\alpha}{k}}}{2k\Gamma_k(\alpha)} (f(a) + f(x)). \tag{2.4}$$

Now, for $x \in [a,b]$ and $\beta > k$, the following inequality can be observed:

$$(t-x)^{\frac{\beta}{k}-1} \leq (b-x)^{\frac{\beta}{k}-1},\ t \in [x,b]. \tag{2.5}$$

By the convexity of f, we also have

$$f(t) \leq \frac{t-x}{b-x} f(b) + \frac{b-t}{b-x} f(x),\ t \in [x,b]. \tag{2.6}$$

From the inequalities (2.5) and (2.6), one obtains that

$$\int_x^b (t-x)^{\frac{\beta}{k}-1} f(t)dt \leq \frac{(b-x)^{\frac{\beta}{k}-1}}{b-x}\left(f(b)\int_x^b (t-x)dt + f(x)\int_x^b (b-t)dt\right).$$

Therefore, in view of the definition of the Riemann–Liouville k-fractional integrals, we conclude that

$$I_{b^-}^{\beta,k} f(x) \leq \frac{(b-x)^{\frac{\beta}{k}}}{2k\Gamma_k(\beta)}(f(b)+f(x)). \tag{2.7}$$

Adding (2.4) and (2.7), we get the required inequality (2.1). □

Corollary 1. *By setting $\alpha = \beta$ in (2.1), this inequality reduces to the fractional integral inequality*

$$\begin{aligned} I_{a^+}^{\alpha,k} f(x) + I_{b^-}^{\alpha,k} f(x) \leq \frac{1}{2k\Gamma_k(\alpha)} &\Big((x-a)^{\frac{\alpha}{k}} f(a) + (b-x)^{\frac{\alpha}{k}} f(b) \\ &+ f(x)\Big((x-a)^{\frac{\alpha}{k}} + (b-x)^{\frac{\alpha}{k}}\Big)\Big). \end{aligned}$$

Corollary 2 (see [3], Corollary 2)**.** *By setting $\alpha = \beta = k = 1$ and taking $x = b$ or $x = a$ in (2.1), we get the inequality*

$$\frac{1}{b-a}\int_a^b f(t)dt \leq \frac{f(a)+f(b)}{2}.$$

Corollary 3 (see [3], Corollary 3)**.** *By setting $\alpha = \beta = 1$ and taking $x = (a+b)/2$ in (2.1), we have the inequalities*

$$0 \leq \frac{1}{b-a}\int_a^b f(t)dt - f\left(\frac{a+b}{2}\right) \leq \frac{f(a)+f(b)}{2}.$$

Remark 1. It is interesting to see that if, in Theorem 1, the function f is concave and $0 < \alpha,\ \beta \geq k$, then the reverse of inequality (2.1) holds.

In the following, we prove a fractional integral inequality for functions whose derivative in absolute value is convex.

Theorem 2. *Let $f\colon I \longrightarrow \mathbb{R}$ be a differentiable function. If $|f'|$ is convex, then, for $a, b \in I$, $a < b$, and $\alpha, \beta > 0$, the following inequality for the Riemann–Liouville k-fractional integrals holds:*

$$\begin{aligned}
&\Big|\Gamma_k(\alpha+k)I_{a^+}^{\alpha,k}f(x)+\Gamma_k(\beta+k)I_{b^-}^{\beta,k}f(x)\\
&\qquad -\left((x-a)^{\frac{\alpha}{k}}f(a)+(b-x)^{\frac{\beta}{k}}f(b)\right)\Big|\\
&\qquad \le \frac{1}{2}\left((x-a)^{\frac{\alpha}{k}+1}|f'(a)|+(b-x)^{\frac{\beta}{k}+1}|f'(b)|\right.\\
&\qquad \left. +|f'(x)|\left((x-a)^{\frac{\alpha}{k}+1}+(b-x)^{\frac{\beta}{k}+1}\right)\right), \quad x\in(a,b).
\end{aligned} \tag{2.8}$$

Proof. By the convexity of $|f'|$, we have

$$|f'(t)| \le \frac{x-t}{x-a}|f'(a)|+\frac{t-a}{x-a}|f'(x)|,\ t\in[a,x], \quad x\in(a,b),$$

from which it follows that

$$-\left(\frac{x-t}{x-a}|f'(a)|+\frac{t-a}{x-a}|f'(x)|\right) \le f'(t) \le \frac{x-t}{x-a}|f'(a)|+\frac{t-a}{x-a}|f'(x)|. \tag{2.9}$$

We firstly consider the right hand side of (2.9):

$$f'(t) \le \frac{x-t}{x-a}|f'(a)|+\frac{t-a}{x-a}|f'(x)|. \tag{2.10}$$

Now, using the inequality

$$(x-t)^{\frac{\alpha}{k}} \le (x-a)^{\frac{\alpha}{k}}, \quad t\in[a,x],\ \alpha,k>0, \tag{2.11}$$

from (2.10) we get

$$\begin{aligned}
&\int_a^x (x-t)^{\frac{\alpha}{k}}f'(t)dt\\
&\le (x-a)^{\frac{\alpha}{k}-1}\left(|f'(a)|\int_a^x (x-t)dt+|f'(x)|\int_a^x (t-a)dt\right)\\
&= (x-a)^{\frac{\alpha}{k}+1}\left(\frac{|f'(a)|+|f'(x)|}{2}\right).
\end{aligned} \tag{2.12}$$

Since

$$\begin{aligned}
\int_a^x (x-t)^{\frac{\alpha}{k}}f'(t)dt &= f(t)(x-t)^{\frac{\alpha}{k}}\big|_a^x+\frac{\alpha}{k}\int_a^x (x-t)^{\frac{\alpha}{k}-1}f(t)dt\\
&= -f(a)(x-a)^{\frac{\alpha}{k}}+\Gamma_k(\alpha+k)I_{a^+}^{\alpha,k}f(x),
\end{aligned}$$

by the definition of the Riemann–Liouville fractional integral, from (2.12), we have

$$\Gamma_k(\alpha+k)I_{a^+}^{\alpha,k}f(x)-f(a)(x-a)^{\frac{\alpha}{k}} \le (x-a)^{\frac{\alpha}{k}+1}\left(\frac{|f'(a)|+|f'(x)|}{2}\right). \tag{2.13}$$

Now, considering the left hand side of (2.9) and proceeding as we did for (2.10), we get

$$f(a)(x-a)^{\frac{\alpha}{k}} - \Gamma_k(\alpha+k) I_{a^+}^{\alpha,k} f(x) \leq (x-a)^{\frac{\alpha}{k}+1} \left(\frac{|f'(a)| + |f'(x)|}{2} \right). \quad (2.14)$$

From (2.13) and (2.14), we conclude that

$$\left| \Gamma_k(\alpha+k) I_{a^+}^{\alpha,k} f(x) - f(a)(x-a)^{\frac{\alpha}{k}} \right| \leq (x-a)^{\frac{\alpha}{k}+1} \left(\frac{|f'(a)| + |f'(x)|}{2} \right). \quad (2.15)$$

On the other hand, using the convexity of $|f'|$, for $t \in [x, b]$ we have

$$|f'(t)| \leq \frac{t-x}{b-x} |f'(b)| + \frac{b-t}{b-x} |f'(x)|. \quad (2.16)$$

Also, since, for $t \in [x, b]$ and $\beta, k > 0$, one has

$$(t-x)^{\frac{\beta}{k}} \leq (b-x)^{\frac{\beta}{k}}, \quad (2.17)$$

by adapting the same approach as we did for (2.10) and (2.11), from (2.16) and (2.17) we obtain the inequality

$$\left| \Gamma_k(\beta+k) I_{b^-}^{\beta,k} f(a) - f(b)(b-x)^{\frac{\beta}{k}} \right| \leq (b-x)^{\frac{\beta}{k}+1} \left(\frac{|f'(b)| + |f'(x)|}{2} \right). \quad (2.18)$$

Combining (2.15) and (2.18) via the triangular inequality, we get the required inequality. □

Corollary 4. *By setting $\alpha = \beta$ in* (2.8)*, this inequality reduces to the fractional integral inequality*

$$\begin{aligned} &\left| \Gamma_k(\alpha+k)[I_{a^+}^{\alpha,k} f(x) + I_{b^-}^{\alpha,k} f(x)] - \left((x-a)^{\frac{\alpha}{k}} f(a) + (b-x)^{\frac{\alpha}{k}} f(b) \right) \right| \\ &\leq \frac{1}{2} \Big((x-a)^{\frac{\alpha}{k}+1} |f'(a)| + (b-x)^{\frac{\alpha}{k}+1} |f'(b)| \\ &\quad + |f'(x)| \left((x-a)^{\frac{\alpha}{k}+1} + (b-x)^{\frac{\alpha}{k}+1} \right) \Big). \end{aligned}$$

Corollary 5 (see [3], Corollary 5)**.** *By setting $\alpha = \beta = k = 1$ and $x = (a+b)/2$ in* (2.8)*, we get the inequality*

$$\left| \frac{1}{b-a} \int_a^b f(t)dt - \frac{f(a)+f(b)}{2} \right| \leq \frac{b-a}{8} \left(|f'(a)| + |f'(b)| + 2f'\left(\frac{a+b}{2} \right) \right).$$

We use the following lemma to prove our next theorem.

Lemma 1 (see [3], Lemma 1)**.** *Let $f\colon [a,b] \longrightarrow \mathbb{R}$, be a convex function. If f is symmetric with respect to $(a+b)/2$, then*

$$f\left(\frac{a+b}{2} \right) \leq f(x), \quad x \in [a, b]. \quad (2.19)$$

Theorem 3. *Let $f\colon I \longrightarrow \mathbb{R}$ be a positive convex function. If f is symmetric with respect to $(a+b)/2$, then the following inequalities for fractional integrals hold:*

$$\begin{aligned}
&\frac{1}{2k}\left(\frac{1}{\frac{\alpha}{k}+1}+\frac{1}{\frac{\beta}{k}+1}\right) f\left(\frac{a+b}{2}\right)\\
&\leq \frac{\Gamma_k(\beta+k) I_{b^-}^{\beta+k,k} f(a)}{2(b-a)^{\frac{\beta}{k}+1}}+\frac{\Gamma_k(\alpha+k) I_{a^+}^{\alpha+k,k} f(b)}{2(b-a)^{\frac{\alpha}{k}+1}} \qquad (2.20)\\
&\leq \frac{f(a)+f(b)}{2k}.
\end{aligned}$$

Proof. For $x \in [a,b]$ and $\beta, k > 0$, we have

$$(x-a)^{\frac{\beta}{k}} \leq (b-a)^{\frac{\beta}{k}}. \qquad (2.21)$$

By the convexity of f, we have

$$f(x) \leq \frac{x-a}{b-a} f(b)+\frac{b-x}{b-a} f(a), \quad x \in [a,b]. \qquad (2.22)$$

From the inequalities (2.21) and (2.22), it follows that

$$\int_a^b (x-a)^{\frac{\beta}{k}} f(x) dx \leq \frac{(b-a)^{\frac{\beta}{k}}}{b-a}\left(f(b) \int_a^b (x-a) dx+f(a) \int_a^b (b-x) dx\right).$$

Thus, by the definition of the k-fractional integral, we have

$$\frac{\Gamma_k(\beta+k) I_{b^-}^{\beta+k,k} f(a)}{(b-a)^{\frac{\beta}{k}+1}} \leq \frac{f(a)+f(b)}{2k}. \qquad (2.23)$$

On the other hand, since

$$(b-x)^{\frac{\alpha}{k}} \leq (b-a)^{\frac{\alpha}{k}}, \quad x \in [a,b], \ \alpha, k>0,$$

from (2.22) we get

$$\int_a^b (b-x)^{\frac{\alpha}{k}} f(x) dx \leq (b-a)^{\frac{\alpha}{k}+1} \frac{f(a)+f(b)}{2}.$$

Thus, by the definition of the k-fractional integral, we have

$$\frac{\Gamma_k(\alpha+k) I_{a^+}^{\alpha+k,k} f(b)}{(b-a)^{\frac{\alpha}{k}+1}} \leq \frac{f(a)+f(b)}{2k}. \qquad (2.24)$$

Adding (2.23) and (2.24), we get

$$\frac{\Gamma_k(\beta+k) I_{b^-}^{\beta+k,k} f(a)}{2(b-a)^{\frac{\beta}{k}+1}}+\frac{\Gamma_k(\alpha+k) I_{a^+}^{\alpha+k,k} f(b)}{2(b-a)^{\frac{\alpha}{k}+1}} \leq \frac{f(a)+f(b)}{2k}.$$

Using Lemma 1 and multiplying (2.19) by $(x-a)^{\frac{\beta}{k}}$, integrating over $[a,b]$ gives

$$f\left(\frac{a+b}{2}\right)\int_a^b (x-a)^{\frac{\beta}{k}}dx \leq \int_a^b (x-a)^{\frac{\beta}{k}} f(x)dx, \tag{2.25}$$

$$f\left(\frac{a+b}{2}\right)\frac{1}{2k(\frac{\beta}{k}+1)} \leq \frac{\Gamma_k(\beta+k)I_{b^-}^{\beta+k,k}f(a)}{2(b-a)^{\frac{\beta}{k}+1}}. \tag{2.26}$$

Using Lemma 1 and multiplying (2.19) by $(b-x)^{\alpha}$, integrating over $[a,b]$, gives

$$f\left(\frac{a+b}{2}\right)\frac{1}{2k(\frac{\alpha}{k}+1)} \leq \frac{\Gamma_k(\alpha+k)I_{a^+}^{\alpha+k,k}f(b)}{2(b-a)^{\frac{\alpha}{k}+1}}. \tag{2.27}$$

Adding (2.26) and (2.27), and then combining with (2.25), we obtain the required inequality. □

Corollary 6. *If we put $\alpha=\beta$ in* (2.20)*, then this inequality reduces to the fractional integral inequalities*

$$\begin{aligned} f\left(\frac{a+b}{2}\right)\frac{1}{k(\frac{\alpha}{k}+1)} &\leq \frac{\Gamma_k(\alpha+k)}{2(b-a)^{\frac{\alpha}{k}+1}}\left(I_{b^-}^{\alpha+k,k}f(a)+I_{a^+}^{\alpha+k,k}f(b)\right) \\ &\leq \frac{f(a)+f(b)}{2k}. \end{aligned}$$

3. Concluding remarks

If we take $k=1$ in Theorem 1, Theorem 2, and Theorem 3, then we obtain the results for the Riemann–Liouville fractional integrals (cf. [3]).

Acknowledgments

We dedicate this work to late Professor G. M. Habibullah passed away on February 10, 2018. This research work is supported by Higher Education Commission of Pakistan under NRPU 2016, Project No. 5421.

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