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A new characterization of symplectic groups $\mathbf{C_2(3^n)}$

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Abstract. We prove that symplectic groups $C_2(3^n)$, where $n = 2^k$ $(k \geq 0)$ and $(3^{2n}+1)/2$ is a prime number, can be uniquely determined by the order of the group and the number of elements with the same order.

1. Introduction

Let G be a finite group, $\pi(G)$ be the set of prime divisors of the order of G and $\pi_e(G)$ be the set of orders of elements in G. If $k \in \pi_e(G)$, then we denote the number of elements of order k in G by $m_k(G)$ and the set of the numbers of elements with the same order in G by $nse(G)$. In other words,

$$nse(G) = \{m_k(G) : k \in \pi_e(G)\}.$$

Also we denote a Sylow p-subgroup of G by G_p and the number of Sylow p-subgroups of G by $n_p(G)$. The prime graph $\Gamma(G)$ of group G is a graph whose vertex set is $\pi(G)$, and two vertices u and v are adjacent if and only if $uv \in \pi_e(G)$. Moreover, assume that $\Gamma(G)$ has $t(G)$ connected components π_i, for $i = 1, 2, \ldots, t(G)$. In the case where G is of even order, we assume that $2 \in \pi_1$.

The characterization of groups by $nse(G)$ pertains to Thompson's problem (see [6]) which Shi posed in [9]. The first time, this type of characterization was studied by Shao and Shi. In [8], they proved that if S is a finite simple group with $|\pi(S)| = 4$, then S is characterizable by $nse(S)$ and $|S|$. Following this result, in [5, 4, 7] it is proved that sporadic simple groups, linear groups $L_2(p)$, where $2^n - 1$ or $2^n + 1$ is a prime number, and Suzuki groups $Sz(q)$,

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where $q-1$ is a prime number, can be uniquely determined by the order of the group and $nse(G)$. In this paper, we prove that symplectic groups $C_2(3^n)$, where $n = 2^k$ $(k \geq 0)$ and $(3^{2n}+1)/2$ is a prime number can be uniquely determined by the order of the group and the number of elements with the same order. In fact, we prove the following theorem.

Main Theorem. *Let G be a group with $|G| = |C_2(3^n)|$ and $nse(G) = nse(C_2(3^n))$, where $n = 2^k$ $(k \geq 0)$ and $p = (3^{2n}+1)/2$ is a prime number. Then G is isomorphic to $C_2(3^n)$.*

2. Notation and preliminaries

Lemma 2.1 (see [3])**.** *Let G be a Frobenius group of even order with kernel K and complement H. Then*

(a) *$t(G) = 2$, $\pi(H)$ and $\pi(K)$ are vertex sets of the connected components of $\Gamma(G)$;*
(b) *$|H|$ divides $|K|-1$;*
(c) *K is nilpotent.*

Definition 2.2. A group G is called a 2-Frobenius group if there is a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that G/H and K are Frobenius groups with kernels K/H and H, respectively.

Lemma 2.3 (see [1])**.** *Let G be a 2-Frobenius group of even order. Then*

(a) *$t(G) = 2$, $\pi(H) \cup \pi(G/K) = \pi_1$ and $\pi(K/H) = \pi_2$;*
(b) *G/K and K/H are cyclic groups satisfying $|G/K|$ divides $|Aut(K/H)|$.*

Lemma 2.4 (see [10])**.** *Let G be a finite group with $t(G) \geq 2$. Then one of the following statements holds:*

(a) *G is a Frobenius group;*
(b) *G is a 2-Frobenius group;*
(c) *G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that H and G/K are π_1-groups, K/H is a non-abelian simple group, H is a nilpotent group and $|G/K|$ divides $|Out(K/H)|$.*

Lemma 2.5 (see [2])**.** *Let G be a finite group and m be a positive integer dividing $|G|$. If $L_m(G) = \{g \in G \mid g^m = 1\}$, then $m \mid |L_m(G)|$.*

Lemma 2.6. *Let G be a finite group. Then for every $i \in \pi_e(G)$, $\varphi(i)$ divides $m_i(G)$, and i divides $\sum_{j|i} m_j(G)$. Moreover, if $i > 2$, then $m_i(G)$ is even.*

Proof. By Lemma 2.5, the proof is straightforward. $\square$

Lemma 2.7 (see [11]). *Let q, k, l be natural numbers. Then*

(1) $(q^k - 1, q^l - 1) = q^{(k,l)} - 1$.

(2) $(q^k + 1, q^l + 1) = \begin{cases} q^{(k,l)} + 1 & \text{if both } k/(k,l) \text{ and } l/(k,l) \text{ are odd,} \\ (2, q+1) & \text{otherwise.} \end{cases}$

(3) $(q^k - 1, q^l + 1) = \begin{cases} q^{(k,l)} + 1 & \text{if } k/(k,l) \text{ is even and } l/(k,l) \text{ is odd,} \\ (2, q+1) & \text{otherwise.} \end{cases}$

In particular, for every $q \geq 2$ and $k \geq 1$, the inequality $(q^k - 1, q^k + 1) \leq 2$ holds.

Lemma 2.8. *Let G be a symplectic group $C_2(3^n)$, where $p = (3^{2n} + 1)/2$ is a prime number. Then $m_p(G) = (p-1)|G|/(8p)$ and, for every $i \in \pi_e(G) - \{1, p\}$, p divides $m_i(G)$.*

Proof. Since $|G_p| = p$, we deduce that G_p is a cyclic group of order p. Thus

$$m_p(G) = \varphi(p) n_p(G) = (p-1) n_p(G).$$

Now it is enough to show that $n_p(G) = |G|/(8p)$. By [10], p is an isolated vertex of $\Gamma(G)$. Hence $|C_G(G_p)| = p$ and $|N_G(G_p)| = xp$ for a natural number x. We know that $N_G(G_p)/C_G(G_p)$ embeds in $Aut(G_p)$, which implies $x \mid p - 1$. Furthermore, by Sylow's theorem, $n_p(G) = |G : N_G(G_p)|$ and $n_p(G) \equiv 1 \pmod{p}$. Therefore p divides $|G|/(xp) - 1$. Thus $q^2 + 1/2$ divides $q^4(q^4 - 1)(q^2 - 1)/2/(xp) - 1$. It follows that $q^2 + 1$ divides $(2q^8 - 4q^6 + 2q^4 - x)$, hence $q^2 + 1$ divides $(q^2+1)(2q^6 - 6q^4 + 8q^2 - 8) + (8 - x)$, and since $x \mid p - 1$, we obtain that $x = 8$. Let $i \in \pi_e(G) - \{1, p\}$. Since p is an isolated vertex of $\Gamma(G)$, we conclude that $p \nmid i$ and $pi \notin \pi_e(G)$. Thus G_p acts fixed point freely on the set of elements of order i by conjugation and hence $|G_p| \mid m_i(G)$. So we conclude that $p \mid m_i(G)$. □

3. Proof of the Main Theorem

In this section, we prove the main theorem by the following lemmas. We denote by C the symplectic group $C_2(3^n)$, where $n = 2^k$ $(k \geq 0)$ and $p := (3^{2n} + 1)/2$ is a prime number. Recall that G is a group with $|G| = |C|$ and $nse(G) = nse(C)$.

Lemma 3.1. *We have*

$$m_2(G) = m_2(C), \quad m_p(G) = m_p(C), \quad n_p(G) = n_p(C),$$

p is an isolate vertex of $\Gamma(G)$, and $p \mid m_k(G)$ for every $k \in \pi_e(G) - \{1, p\}$.

Proof. By Lemma 2.6, for every $1 \neq r \in \pi_e(G)$, $r = 2$ if and only if $m_r(G)$ is odd. Thus we deduce that $m_2(G) = m_2(C)$. According to Lemma 2.6, $(m_p(G), p) = 1$. Thus $p \nmid m_p(G)$ and hence Lemma 2.8 implies that $m_p(G) \in \{m_1(C), m_2(C), m_p(C)\}$. Moreover, $m_p(G)$ is even, so we conclude that $m_p(G) = m_p(C)$. Since G_p and C_p are cyclic groups of order p and $m_p(G) = m_p(C)$, we deduce that $m_p(G) = \varphi(p)n_p(G) = \varphi(p)n_p(C) = m_p(C)$, so $n_p(G) = n_p(C)$.

Now we prove that p is an isolated vertex of Γ(G). Assume the contrary. Then there is $t \in \pi(G) - \{p\}$ such that $tp \in \pi_e(G)$. So $m_{tp}(G) = \varphi(tp)n_p(G)k$, where k is the number of cyclic subgroups of order t in $C_G(G_p)$ and since $n_p(G) = n_p(C)$, it follows that

$$m_{tp}(G) = (t-1)(p-1)|C|k/(8p).$$

If $m_{tp}(G) = m_p(C)$, then $t = 2$ and $k = 1$. Furthermore, Lemma 2.5 yields $p \mid m_2(G) + m_{2p}(G)$ and since $m_2(G) = m_2(C)$ and $p \mid m_2(C)$, we have $p \mid m_{2p}(G)$, which is a contradiction. So Lemma 2.8 implies that $p \mid m_{tp}(G)$. Hence $p \mid t-1$, and since $m_{tp}(G) < |G|$, we have that $t - 1 \leq 8$. In conclusion we deduce that $t \in \{3, 4, 5, 6, 7, 8, 9\}$. Now, since $p \nmid m_{tp}(G)$, this is a contradiction.

Let $k \in \pi_e(G) - \{1, p\}$. Since p is an isolated vertex of $\Gamma(G)$, we have that $p \nmid k$ and $pk \notin \pi_e(G)$. Thus G_p acts fixed point freely on the set of elements of order k by conjugation and hence $|G_p| \mid m_k(G)$. So we conclude that $p \mid m_k(G)$. □

Lemma 3.2. *The group G is neither a Frobenius group nor a 2-Frobenius group.*

Proof. Let G be a Frobenius group with kernel K and complement H. Then by Lemma 2.1, $t(G) = 2$ and $\pi(H)$ and $\pi(K)$ are vertex sets of the connected components of $\Gamma(G)$, and $|H|$ divides $|K| - 1$. Now by Lemma 3.1, p is an isolated vertex of $\Gamma(G)$. Thus we deduce that (i) $|H| = p$ and $|K| = |G|/p$, or (ii) $|H| = |G|/p$ and $|K| = p$. Since $|H|$ divides $|K| - 1$, we conclude that the last case can not occur. So $|H| = p$ and $|K| = |G|/p$, hence

$$(q^2+1)/2 \mid \frac{q^4(q^4-1)(q^2-1)/2}{(q^2+1)/2} - 1.$$

We conclude that

$$(q^2+1) \mid ((q^2+1)(2q^6 - 6q^4 + 8q^2 - 8) + 7.$$

Thus $q^2 + 1 \mid 7$ which is impossible.

We now show that G is not a 2-Frobenius group. Let G be a 2-Frobenius group. Then G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that G/H and K are Frobenius groups by kernels K/H and H, respectively. Set $|G/K| = x$. Since p is an isolated vertex of $\Gamma(G)$, we have $|K/H| = p$ and $|H| = |G|/(xp)$. By Lemma 2.3, $|G/K|$ divides $|Aut(K/H)|$. Thus $x \mid p-1$ and since, by Lemma 2.7, $(p-1, q-1) = 1$, we have $(q^2 - 1/2, q^2 + 1/2) = 1$. Now, since $|G/K||(p-1)$, we deduce that $q^2 + 1/2|H$. The group H is nilpotent. Therefore, $H_t \rtimes K/H$ is a Frobenius group with kernel H_t and complement

K/H, where $t = q^2 + 1/2$. So $|K/H|$ divides $|H_t| - 1$. It implies that $q^2 + 1/2 \leq (q^2 + 1)/2 - 1$, but this is a contradiction. □

Lemma 3.3. *The group G is isomorphic to the group C.*

Proof. By Lemma 3.1, p is an isolated vertex of $\Gamma(G)$. Thus $t(G) > 1$ and G satisfies one of the cases of Lemma 2.4. Now, Lemma 3.2 implies that G is neither a Frobenius group nor a 2-Frobenius group. Thus only the case (c) of Lemma 2.4 occurs. So G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that H and G/K are π_1-groups, and K/H is a non-abelian simple group. Since p is an isolated vertex of $\Gamma(G)$, we have $p \mid |K/H|$. According to the classification of the finite simple groups we know that the possibilities are: alternating groups A_n, where $n > 5$; 26 sporadic finite simple groups; simple groups of Lie type. We deal with the above cases separately.

Step 1. Let $K/H \cong A_n$, where $n \geq 5$, $n = p', p'+1, p'+2$. For this purpose, we consider $(q^2+1)/2 = p'$. Then we deduce $p' + 1 = (q^2 + 3)/2$. Now $p' + 1 \mid |A_n| \mid |G|$, but we can see easily that $\frac{q^2+3}{2} \nmid |G|$, which is a contradiction. Now we consider $(q^2+1)/2 = p' - 2$, so $p' = (q^2+4)/2$. Since $p' \mid |A_n \mid |G|$, but we can see easily that $\frac{q^2+4}{2} \nmid |G|$, we obtain a contradiction.

Step 2. If K/H is a sporadic group, then we consider $(q^2 + 1)/2 = 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 59, 71$, where this number is the order of components of sporadic groups. If, for example, $(q^2 + 1)/2 = 5$, then we deduce $q^2 = 9$, and hence $q = 3$. Now, since $M_{11} \nmid |G|$, this is a contradiction. For the other groups we have a contradiction, similarily.

Step 3. Here we suppose that K/H is isomorphic to a group of Lie type. For this purpose, we consider the following cases.

Case 1. Let $K/H \cong B_n(q')$, where $n \geq 2$, or $C_n(q')$, where $n \geq 3$. If $K/H \cong B_n(q')$, where $n \geq 2$, then we consider $(q^2 + 1)/2 = (q'^n + 1)/2$ so $q^2 = q'^n$. On the other hand,

$$|B_n(q')| = \frac{1}{(2, q'-1)} q'^{n^2} \prod_{i=1}^{n} (q'^{2i} - 1) \mid q^4(q^4 - 1)(q^2 - 1)$$

and also we know that each p-part of K/H divides p-part of G. Since $p = (q^2 + 1)/2$, we have $q^2 = 2p - 1$. Now, since $|B_n(q')| \mid |G|$, we conclude that

$$4p(p-1)(p-2)(2p-1)^2 = q'^{n^2} \prod_{i=1}^{n} (q'^{2i} - 1).$$

Thus we have $p \mid q'^{n^2}$ or $p \mid \prod_{i=1}^{n}(q'^{2i} - 1)$. On the other hand, $q'^{n^2} \mid p - 1$ or $p - 2$ or $(2p-1)$. If $p \mid q'^{n^2}$, then $p \nmid q'$, and so $p \mid \prod_{i=1}^{n}(q'^{2i} - 1)$. In other words, $p \mid q'^{2t} - 1$, where $1 \leq t \leq n$. From $q'^{n^2} \mid p - 1$ it follows that

$$q'^{n^2} \leq p - 1 \leq p \leq q'^{2t} - 1 \leq q'^{2n} - 1 \leq q'^{2n}.$$

As a result $q'^{n^2} \leq q'^{2n}$, so $n^2 \leq 2n$, $n \leq 2$, but this is a contradiction. Similarily, there is a contradiction for other cases.

Case 2. Let $K/H \cong D_n(q')$, where $n \geq 4$ or ${}^2D_n(q')$ with $n \geq 4$. Then we consider $(q^2+1)/2 = (q'^n - 1)/(q-1)$. On the other hand,

$$|D_n(q')| = \frac{1}{(4, q'^n - 1)} q'^{n(n-1)}(q'^n - 1)\prod_{i=1}^{n-1}(q'^{2i} - 1) \mid q^4(q^4-1)(q^2-1)$$

and also we know that each p-part of K/H divides p-part of G. Since $|D_n(q')| \mid |G|$, it follows that

$$4p(p-1)(p-2)(2p-1)^2 = \frac{1}{(4, q'^n - 1)} q'^{n(n-1)}(q'^n - 1)\prod_{i=1}^{n-1}(q'^{2i} - 1).$$

Now we have

$$p \mid q'^{n(n-1)} \ \text{ or } \ p \mid q'^n - 1 \ \text{ or } \ p \mid \prod_{i=1}^{n-1}(q'^{2i} - 1).$$

On the other hand, $q'^{n(n-1)} \mid p-1$ or $p-2$ or $(2p-1)$. If $p \mid q'^{n(n-1)}$, then $p \nmid q'$, so $p \mid \prod_{i=1}^{n-1}(q'^{2i}-1)$. In other words, $p \mid q'^{2t} - 1$, where $1 \leq t \leq n-1$. Since $q'^{n(n-1)} \mid p-1$, we have

$$q'^{n(n-1)} \leq p-1 \leq p \leq q'^{2t} - 1 \leq q'^{2n} - 1 \leq q'^{2n}.$$

As a result $q'^{n(n-1)} \leq q'^{2n}$, so $n(n-1) \leq 2n$, $n \leq 3$, but this is a contradiction $(n \geq 4)$. There is a contradiction for other cases. Similarily, $K/H \ncong {}^2D_n(q)$.

Case 3. Let $K/H \cong {}^2A_n(q')$, where $n \geq 2$. Then we consider

$$\frac{q^2+1}{2} = \frac{q'^{n+1}+1}{(q'+1)(q+1, n+1)},$$

and so

$$q^2 + 1 = \frac{q'^{n+1}+1}{(q'+1)(q+1, n+1)} < q'^{n+1} + 1, \quad q^2 < q'^{n+1}.$$

Since $n \geq 2$, we get

$$q'^{n(n+1)/2} > q'^{4n/2} \geq q'^{2n} > q'^n > 2^{4n}.$$

But, on the other hand, we have

$$q'^{n(n+1)/2} = |K/H|_r \leq |G|_r \leq 2^{3n},$$

which is a contradiction.

Case 4. Let $K/H \cong E_6(q), E_7(q), E_8(q)$, where $n \geq 2$, or let $K/H \cong F_4(q)$. If $K/H \cong E_8(q')$, then we consider $(q^2+1)/2 = q'^8 - q'^4 + 1$. On the other hand,

$$|E_8(q')| = q'^{120}(q'^{30}-1)(q'^{24}-1)(q'^{20}-1)(q'^{18}-1)(q'^{14}-1)$$

$$\times (q'^{12}-1)(q'^{8}-1)(q'^{2}-1) \mid q^4(q^4-1)(q^2-1).$$

Since $|E_8(q')| \nmid |G|$, this is a contradiction. For other cases we have similarily a contradiction.

Case 5. Let $K/H \cong G_2(q')$, where $q' \equiv \pm 2 \pmod 5$. Then we consider $(q^2+1)/2 = q'(q' \mp 1)$. Since $(q', q' \mp 1) = 1$, we obtain $q^2 - 1 = 2q'(q' \pm 1)$, Now $|G_2(q')| \nmid |G|$, which is a contradiction.

Case 6. Let $K/H \cong^2 B_2(q')$, where $q' = 2^{2r+1}$, $r \geq 1$. Then we consider $(q^2+1)/2 = q' \pm \sqrt{2q'} + 1$. As a result $q^2 - 1 = 2^{m+1}(2^m \pm 1)$. Since $(2^{m+1}, 2^m \pm 1) = 1$, we deduce that

$$(q-1)(q+1) = 2^{m+1}(2^m \pm 1).$$

In other words,

$$(3^n-1)(3^n+1) = 2^{m+1}(2^m \pm 1).$$

Consequently, $3^n - 1 = 2^m \pm 1$ and $3^n + 1 = 2^{m+1}$, which is a contradiction.

Case 7. Let $K/H \cong^2 F_4(q')$, where $q' = 2^{2s+1}$, $s \geq 1$. Then we consider

$$\frac{q^2+1}{2} = q'^2 \pm \sqrt{2q'^3} + q' \pm \sqrt{2q'} + 1.$$

As a result $q^2 - 1 = q'^2 \pm \sqrt{2q'^3} + q' \pm \sqrt{2q'}$, so we deduce

$$(q-1)(q+1) = 2^{s+1}(2^{3s+1} \pm 2^{2s+1} + 2^s + 1).$$

It follows that $3^n - 1 = 2^{s+1}$ and $3^n + 1 = (2^{3s+1} \pm 2^{2s+1} + 2^s + 1)$, where we can see easily a contradiction.

Case 8. Let $K/H \cong^2 E_6(q)$, where $q = 2^{2t+1}$, $t \geq 1$. Then we consider

$$\frac{q^2+1}{2} = \frac{q'^6 - q'^3 + 1}{(3, q'+1)},$$

so $q^2 < q'^6 - q'^3 + 1 < q'^6$. Hence $q'^{36} > 3^{12n}$.

On the other hand,

$$|^2E_6(q')| = \frac{1}{(3, q'+1)} q'^{36}(q'^{12}-1)(q'^9+1)$$
$$\times (q'^8-1)(q'^6-1)(q'^5+1)(q'^2-1).$$

Now, we obtain $q = r^s$. Therefore, by Lemma 2.4,

$$q^{36} = r^{36s} = |K/H|_r \leq |G|_r \leq 2^{3m},$$

which is a contradiction.

Case 9. Let $K/H \cong^3 D_4(q')$. Then we consider $(q^2+1)/2 = q'^4 - q'^2 + 1$, as a result

$$(3^n-1)(3^n+1) = 2q'^2(q'^2-1).$$

Hence $2q'^2 = 3^n + 1$ and $q'^2 - 1 = 3^n - 1$, which is a contradiction.

Case 10. Let $K/H \cong L_{n+1}(q')$. Then we consider

$$\frac{q^2+1}{2} = \frac{q'^{n+1}-1}{(q'-1)(q'-1,n+1)}$$

so

$$q'^{n+1}-1 > \frac{q'^{n+1}-1}{(q-1)(q-1,n+1)} = q^2+1.$$

As a result $q^2 < q'^{n+1}$, so

$$q'^{n(n+1)/2} > q^{4(n+1)/2} > q^{2n} > 3^{8n}.$$

On the other hand, by Lemma 2.4 we have

$$q^{n(n+1)/2} = |K/H|_r \leq |G|_r \leq 2^{3n},$$

which is a contradiction.

Hence $K/H \cong C_2(3^m)$, in conclusion $|K/H| = |C_2(3^m)|$. We know that $H \trianglelefteq K \trianglelefteq G$. Since p is an isolated vertex of $\Gamma(G)$, we deduce that $p \mid |K/H|$. Hence $(q^2+1)/2 = (q'^2+1)/2$. As a result $q = q'$, so $n = m$. Now, since $|K/H| = |C|$ and $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, we conclude that $H = 1$ and $G = K \cong C$. □

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