ACTA ET COMMENTATIONES UNIVERSITATIS TARTUENSIS DE MATHEMATICA
Volume 23, Number 1, June 2019
Available online at http://acutm.math.ut.ee

Natural vibrations of stepped nanobeams with defects

Jaan Lellep and Artur Lenbaum

Abstract. Exact solutions for the transverse vibration of nanobeams based on the nonlocal theory of elasticity are presented. The nanobeams under consideration have piecewise constant dimensions of cross sections and are weakened with crack-like defects. It is assumed that the stationary cracks occur at the re-entrant corners of steps and that the mechanical behaviour of the nanomaterial can be modelled with the Eringen's nonlocal theory. The influence of cracks on the natural vibration is prescribed with the aid of additional local compliance at the weakened cross section. The local compliance is coupled with the stress intensity factor at the crack tip. A general algorithm for determination of eigenfrequencies is developed. It can be used in the case of an arbitrary finite number of steps and cracks.

1. Introduction

Theoretical and experimental results have revealed significant size effects in mechanical properties of structures and bodies when the dimensions of these bodies become small. Classical theory of elasticity is unable to predict the size effects. During the last decades, several versions of size dependent theories have been developed. One of these developed by Eringen [11, 12] and his co-workers was called non local continuum theory of elasticity and was widely accepted by researchers. Originally Eringen [12] applied his concept for modelling of screw dislocations and surface waves. Peddieson et al. [21] studied the small-scale behaviour of nanobeams as actuators in the small-scale systems and used a simplified version of the theory of Eringen [11]. Reddy [23] has extended various beam theories including Eringen–Bernoulli, Timoshenko, Reddy and Levinson beam theories to the case when the constitutive relations are prescribed with nonlocal theory of elasticity. Particular

Received November 8, 2018.
2010 *Mathematics Subject Classification.* 74B20.
Key words and phrases. Nanobeam; nonlocal elasticity; natural vibrations; crack.
http://dx.doi.org/10.12697/ACUTM.2019.23.14

solutions of problems of vibration of nanobeams are obtained in the papers by Lu et al. [20], Li et al. [19], and Thai [27] making use of the model of nonlocal elasticity. A vibration theory accompanied with the nonlocal elasticity was used in the investigation of vibrations of rotating non-uniform nanocantilevers by Aranda-Ruiz et al. [3]. The equations of Eringen's nonlocal theory are solved by a pseudo-spectral collocation method based on Chebyshev polynomials.

Challamel [7] developed a variational approach to the application of strain gradient elasticity and Eringen's nonlocal elasticity models to the beam mechanics. This approach embraces the Euler–Bernoulli, Timoshenko and higher order shear models.

Several authors including Ghannadpour et al. [13] have demonstrated the applicability of the Ritz method in solving bending, buckling and vibration problems in the case of nonlocal Euler beams. The governing equations are derived with the aid of Hamilton's principle. Other variational principles including the total potential energy, the complementary energy and the mixed Hu–Washizu principles are reported by Polizzotto [22] in connection with nonlocal theories of elasticity. Analytical solutions are obtained for nonlinear vibration problems of nanobeams with different boundary conditions by Bagdath [4]. Vibrations of the hanging tube of variable thickness are treated as vibrations of a nanoscale Euler–Bernoulli beam by Roostai and Haghpanahi [24]. The governing equations based on the nonlocal elasticity are solved by the method of Rayleigh–Ritz. In the present paper natural vibrations of nanobeams with stepped cross sections are investigated. An attempt is made to account for the influence of cracks on the frequencies of free vibrations of nanobeams. The cracks are treated as stationary flaws located at the cross sections where the thickness changes abruptly.

2. Formulation of the problem

Let us consider a nanobeam of length l subjected to the axial tension N. It is assumed that the origin of the system of coordinates is located at the centre of the left-hand edge of the nanobeam and that at $x = a_j$ $(j = 1, \ldots, n)$ the cross-sectional area changes rapidly. Let the cross sections of the nanobeam be of rectangular shape with width $b = const$ and height $h = h_j = const$ for $x \in (a_j, a_{j+1})$, where $j = 0, \ldots, n$.

The quantities a_j, h_j $(j = 0, \ldots, n)$ are expected to be given constants; it is reasonable to denote $a_0 = 0$, $a_{n+1} = l$.

It is assumed that the nanobeam is weakened with cracks or crack-like defects located at $x = a_j$ $(j = 1, \ldots, n)$. The cracks are assumed to be stable surface cracks with lengths c_j, respectively.

The aim of the study is to determine the eigenfrequencies of stepped nanobeams with cracks and to elucidate the sensitivity of these with respect to the location of the crack, dimensions of the nanobeam and other geometrical and physical parameters.

3. Governing equations

It is well known that in the classical theory of elasticity the relationship between stress components σ_{ij}^{c} and strains ε_{kl} is defined by the Hooke's law as

$$\sigma_{ij}^{c} = c_{ijkl}\varepsilon_{kl}, \tag{1}$$

where c_{ijkl} stands for a tensor of fourth order with coefficients of elasticity. It is expected herein that the subscripts i, j, k, l take the values $1, 2$ and 3. However, it is accepted that the mechanical behaviour of nanomaterials can be prescribed with non-local theories of elasticity (see Eringen [11], Reddy [23], Alves and Ribeiro [1]). Within the framework of a non-local theory of elasticity the stress state at a fixed point of the body depends on the strain state at each point of the body. Eringen [11] suggested to present the constitutive equations of the material at the point x as

$$\sigma_{ij}^{n}(x) = \int_{(V)} \mathrm{K}\left(|x-y|, \tau\right) c_{ijkl}\varepsilon_{kl}\, dV, \tag{2}$$

where $y \in V$ is an arbitrary point of the body and τ stands for physical constants. Here the kernel function $\mathrm{K}\left(|x|, \tau\right)$ may have different forms; each of these defines according to (2) the non-local stress components σ_{ij}^{n} at the point x. Taking the kernel function K as the Green's function for a linear differential operator $\mathcal{L}(\mathrm{K})$ one has

$$\mathcal{L}\left[\mathrm{K}\left(|y-x|\right)\right] = \delta\left(|y-x|\right). \tag{3}$$

Here δ stands for the Dirac's δ-function. Making use of a two-phase integral model of Eringen [11], Polizzotto [22] introduced an attenuation function which can be expressed in the one-dimensional case as

$$A\left(|x-y|, \tau\right) = \xi_1\delta\left(|x-y|\right) + \xi_2\mathrm{K}\left(|x-y|, \tau\right),$$

where ξ_1, ξ_2 are real numbers satisfying the requirements $\xi_1 + \xi_2 = 1$, $\xi_1 \geq 0$, $\xi_2 \geq 0$. The kernel K in this two-phase model was taken in the exponential form as

$$\mathrm{K} = \frac{1}{2e_0a} exp\left(\frac{-(x-y)}{e_0a}\right). \tag{4}$$

The constants e_0 and a in (4) are material constants (a is the dimension of the lattice of the material). It is reasonable to introduce the notation

$$e_0a = \sqrt{\eta}.$$

It was pointed out by several authors (among others Eringen [11], Lu et al. [20], and Eptaimeros et al. [10]) that a simple two-dimensional kernel function is obtained by taking the differential operator $\mathcal{L}$ as shown by (3) and

$$\mathcal{L}\left[\mathrm{K}\right] = \mathrm{K} - \eta \nabla^2 \mathrm{K}, \tag{5}$$

where ∇ is the Laplacian operator.

In the present paper the governing equations presented by (1) – (3) and (5) will be used. The aforementioned relations yield the constitutive equations

$$\sigma_{ij}^n - \eta \nabla^2 \sigma_{ij}^n = \sigma_{ij}^c, \tag{6}$$

where σ_{ij}^c and σ_{ij}^n are defined by (1) and (2), respectively. According to (6) the relationship between bending moments can be presented as

$$M - \eta \frac{\partial^2 M}{\partial x^2} - M_c = 0. \tag{7}$$

In (7), M is the bending moment engendered by the nonlocal elasticity stress tensor and M_c is the classical bending moment in the Euler–Bernoulli beam theory. It is well known that (see Soedel [25], Wang et al. [28])

$$M_c = -EIw''. \tag{8}$$

In (8), w is the transverse deflection, E stands for the Young modulus and I is the moment of inertia of the cross section. The primes in (8) denote the differentiation with respect to the axial coordinate x. In the case of a beam with stepped cross section

$$I = I_j \tag{9}$$

for $x \in (a_j,\ a_{j+1})\,,\ j = 0, \ldots, n$, while $I_j = bh_j^3/12$ for a rectangular cross section with width b and height h_j.

4. The equation of motion and its solution

Denoting by N the axial force and by Q the shear force one can present the equilibrium equations for a beam element as (see Soedel [25])

$$M' = Q, \quad Q' = \mu \ddot{w} - N w'', \tag{10}$$

where μ stands for the mass of the element per its unit length. Here dots denote the differentiation with respect to time t and primes – with respect to the coordinate. In other words,

$$\ddot{w} = \frac{\partial^2 w}{\partial t^2}, \quad w'' = \frac{\partial^2 w}{\partial x^2}.$$

While the dimensions of the cross section of the nanobeam are piecewise constant, one has

$$\mu = \mu_j$$

for $x \in (a_j,\ a_{j+1})\,,\ j = 0, \ldots, n$.

Evidently, the system (10) can be presented as

$$M'' = -Nw'' + \mu_j \ddot{w} \tag{11}$$

for $x \in (a_j,\ a_{j+1})\,,\ j = 0, \ldots, n.$

On the other hand, according to (7) – (9) one has

$$M = \eta M'' - EI_j w'' \tag{12}$$

for $x \in (a_j,\ a_{j+1})$. Substituting M'' according to (11) in (12) gives

$$M = -\left(\eta N + EI_j\right) w'' + \eta \mu_j \ddot{w}. \tag{13}$$

Eliminating the bending moment from the system (11), (13) leads to the equation

$$\left(\eta N + EI_j\right) w^{IV} + \mu_j \left(\ddot{w} - \eta \ddot{w}''\right) - Nw'' = 0, \tag{14}$$

which is valid in the intervals $(a_j,\ a_{j+1})$ for each $j = 0, \ldots, n.$ The equation of motion (14) is integrated together with appropriate boundary conditions. If the nanobeam is simply supported at both ends, then

$$w\left(0, t\right) = w\left(l, t\right) = 0 \tag{15}$$

$$M\left(0, t\right) = M\left(l, t\right) = 0. \tag{16}$$

The boundary conditions (15) remain valid for a nanobeam clamped at both ends. However, the conditions (16) must be replaced by

$$w'\left(0, t\right) = w'\left(l, t\right) = 0. \tag{17}$$

In the case of a cantilever one has

$$w\left(0, t\right) = w'\left(0, t\right) = 0 \tag{18}$$

$$M\left(l, t\right) = Q\left(l, t\right) = 0, \tag{19}$$

provided the edge at $x = 0$ is clamped and the one at $x = l$ is completely free. Although the free vibrations will be studied underneath, it can be assumed that at the initial instant

$$w\left(x, 0\right) = 0, \quad \dot{w}\left(\frac{l}{2}, 0\right) = v_0, \tag{20}$$

v_0 being a given constant.

For the solution of (14) with (15) – (20) the method of separation of variables will be employed. Thus, one can assume that

$$w\left(x, t\right) = W_j\left(x\right) T\left(t\right), \quad x \in (a_j, a_{j+1}), \ \ j = 0, \ldots, n, \tag{21}$$

where the functions $W_j(x)$ and $T(t)$ depend upon the single variable. Making use of (20) and (21), one can state the initial conditions as

$$T\left(0\right) = 0, \quad \dot{T}\left(0\right) = \frac{v_0}{W_r\left(\frac{l}{2}\right)}, \tag{22}$$

where r is an integer such that $l/2 \in [a_r, a_{r+1}]$. Calculating the necessary partial derivatives from (21) and substituting in (14) leads to the ordinary differential equations

$$(\eta N + EI_j)\, W_j^{IV} - N W_j'' = -\omega^2 \mu_j \left(\eta W_j'' - W_j\right), \tag{23}$$

for $x \in (a_j, a_{j+1})$, and

$$\ddot{T} + \omega^2 T = 0, \tag{24}$$

where ω is the frequency of natural vibrations. Thus, the solution of (22), (24) is

$$T = \frac{v_0}{\omega W_r\left(\frac{l}{2}\right)} \sin(\omega t).$$

In order to solve the linear fourth order equation (23), one has to solve the characteristic equations

$$(\eta N + EI_j)\, \lambda_j^4 + \lambda_j^2 \left(\eta \mu_j \omega^2 - N\right) - \mu_j \omega^2 = 0 \tag{25}$$

with respect to λ_j.
The roots of the fourth order algebraic equation (25) can be presented as

$$(\lambda_j)_{1,2} = \pm i \beta_j, \tag{26}$$

where i is the imaginary unit and

$$(\lambda_j)_{3,4} = \pm \delta_j. \tag{27}$$

In (26) and (27) the notation

$$\beta_j = \sqrt{\frac{\mu_j \eta \omega^2 - N + A_j}{2\,(\eta N + EI_j)}}, \quad \delta_j = \sqrt{\frac{N - \mu_j \eta \omega^2 + A_j}{2\,(\eta N + EI_j)}} \tag{28}$$

is used, where

$$A_j = \sqrt{\left(N - \mu_j \eta \omega^2\right)^2 + 4 \mu_j \omega^2\, (\eta N + EI_j)}. \tag{29}$$

The notation (26) – (29) admits to present the general solution of (23) as

$$W_j\,(x) = A_j \cosh \delta_j x + B_j \sinh \delta_j x + C_j \cos \beta_j x + D_j \sin \beta_j x. \tag{30}$$

It is worthwhile to mention that in (30) it is assumed that $x \in (a_j, a_{j+1})$, $j = 0, \ldots, n$. Arbitrary constants A_j, B_j, C_j, D_j in (30) are to be determined according to the boundary conditions and corresponding continuity and jump conditions imposed at $x = a_j$, $j = 1, \ldots, n$.

5. The continuity and jump conditions

It is reasonable to introduce the following notation. Let $y = y(x,t)$ be a continuous function with piecewise continuous partial derivatives and let $y'(x,t)$ be discontinuous at $x = a$. However, the unilateral limits

$$y'(a \pm 0, t) = \lim_{x \to a \pm 0} y'(x,t)$$

are assumed to be finite. In this case it is useful to denote

$$\left[y'(a,t)\right] = y'(a+0,t) - y'(a-0,t).$$

In the case of natural vibrations of a nanobeam according to physical considerations it is evident that

$$[w(x,t)] = 0$$

for each $x \in (0,l)$, and in particular case for $x = a_j$ $(j = 1, \ldots, n)$.

As regards the slope of the deflection $w'(x,t)$ it is assumed that w' is continuous everywhere except $x = a_j$ $(j = 1, \ldots, n)$. Following [8, 9], it is assumed that

$$\left[w'(a_j,t)\right] = C_{0j} M(a_j,t), \tag{31}$$

where C_{0j} stands for the additional compliance due to the crack at $x = a_j$. In the elastic fracture mechanics it is recognized that there exists a relationship between the compliance C_{0j} caused by the crack and the stress intensity factor K_j.

As a matter of fact the energy release rate G_j, compliance C_{0j}, and the stress intensity factor K_j are related as (see Anderson [2], Broek [5])

$$G_j = \frac{M_j^2}{2b} \frac{dC_{0j}}{dc_j}, \tag{32}$$

where c_j stands for the length of the crack engendered at the cross section $x = a_j$, and $M_j = M(a_j,t)$. On the other hand (see Broek [5]),

$$G_j = \frac{K_j^2}{E'}. \tag{33}$$

Here $E' = E$ in the case of plane stress state, and $E' = E/1 - \nu^2$ in the case of plane strain state. The stress intensity factor corresponding to a surface crack in a beam element can be calculated as (see [5, 8, 15])

$$K_j = \sigma_j \sqrt{\pi c_j} F(s_j), \quad s_j = c_j/h_{0j}, \ \ h_{0j} = \min(h_{j-1}, h_j). \tag{34}$$

Here

$$\sigma_j = \frac{6M_j}{bh_{0j}^2} \tag{35}$$

and $F(s_j)$ stands for the shape factor determined experimentally and numerically by many researchers. It can be easily inferred from (32) – (35) that

$$\frac{dC_{0j}}{ds_j} = \frac{72\pi\left(1-\nu^2\right)}{Ebh_{0j}^2} s_j F^2\left(s_j\right).$$

It is reasonable to assume that in the case of no cracks $C_{0j} = 0$.

Thus, the finite discontinuity (jump) of the slope of deflection at $x = a_j$ is expressed by (31) and the bending moment by (13). Substituting (13) in (31) leads to the jump conditions

$$\left[w'\left(a_j, t\right)\right] = C_{0j}\left\{\eta\mu_j \ddot{w}\left(a_j+0, t\right) - \left(\eta N + EI_j\right) w''\left(a_j+0, t\right)\right\} \tag{36}$$

for every $j = 1, \ldots, n$. Taking (21) into account, one can present (36) as

$$\begin{aligned} W_j'\left(a_j+0\right) = W_{j-1}'\left(a_j-0\right) - C_{0j}&\left\{\eta\mu_j\omega^2 W_j\left(a_j+0\right)\right. \\ &\left.+\left(\eta N + EI_j\right) W_j''\left(a_j+0\right)\right\}, \quad j = 1, \ldots, n. \end{aligned} \tag{37}$$

According to the present model of the nanobeam the finite jumps of the slope are accepted. However, the deflection itself must be continuous at $x = a_j$. Thus

$$W_j\left(a_j+0\right) = W_{j-1}\left(a_j-0\right), \quad j = 1, \ldots, n. \tag{38}$$

The bending moment M and shear force $Q = M'$ are continuous, as well.

Making use of (13) and (21), one can present these requirements, for $j = 1, \ldots, n$, as

$$\begin{aligned} &\eta\mu_j\omega^2 W_j\left(a_j+0\right) + \left(\eta N + EI_j\right) W_j''\left(a_j+0\right) \\ &\quad = \eta\mu_{j-1}\omega^2 W_{j-1}\left(a_j-0\right) + \left(\eta N + EI_{j-1}\right) W_{j-1}''\left(a_j-0\right) \end{aligned} \tag{39}$$

and

$$\begin{aligned} &\eta\mu_j\omega^2 W_j'\left(a_j+0\right) + \left(\eta N + EI_j\right) W_j'''\left(a_j+0\right) \\ &\quad = \eta\mu_{j-1}\omega^2 W_{j-1}'\left(a_j-0\right) + \left(\eta N + EI_{j-1}\right) W_{j-1}'''\left(a_j-0\right). \end{aligned} \tag{40}$$

The system (37) – (40) complemented with corresponding boundary conditions serves for determination of unknown constants A_j, B_j, C_j, D_j in (30), where $j = 0, \ldots, n$. The total number of equations equals $4n + 4$. However, first of all one has to specify the functions $F\left(s_j\right)$ and $f\left(s_j\right)$. Various approximations of these functions are presented in the handbook by Tada et al. [26]. A comparison of different approximations was undertaken by Caddemi and Calio [6]. In the present paper the version suggested by Dimarogonas [9] will be used.

6. Natural frequencies of a nanobeam simply supported at both edges

The transverse deflection of the vibrating nanobeam is given by (21), where the wave modes are defined by (30). In the case of a simply supported nanobeam boundary conditions (15) and (16) must be taken into account during the solution procedure. Making use of (13) and (21), one can present these requirements as

$$\begin{aligned} &W_0(0)=0,\ \ W_n(l)=0,\\ &W_0''(0)=0\ ,\ W_n''(l)=0. \end{aligned} \tag{41}$$

The conditions (41) together with (30) lead us to the equalities $A_0=C_0=0$ and

$$A_n+B_n th\delta_n l=0,\quad C_n+D_n tan\beta_n l=0. \tag{42}$$

Making use of (30), one can put the jump conditions (37) into the form of recurrent equations

$$\begin{aligned} &A_j\left\{\delta_j sinh\delta_j a_j+C_{0j}\left(\eta\mu_j\omega^2+\delta_j^2\left(\eta N+EI_j\right)cosh\delta_j a_j\right)\right\}\\ &\quad+B_j\left\{\delta_j cosh\delta_j a_j+C_{0j}\left(\eta\mu_j\omega^2+\delta_j^2\left(\eta N+EI_j\right)sinh\delta_j a_j\right)\right\}\\ &\quad+C_j\left\{-\beta_j sin\beta_j a_j+C_{0j}\left(\eta\mu_j\omega^2-\beta_j^2\left(\eta N+EI_j\right)cos\beta_j a_j\right)\right\}\\ &\quad+D_j\left\{\beta_j cos\beta_j a_j+C_{0j}\left(\eta\mu_j\omega^2-\beta_j^2\left(\eta N+EI_j\right)sin\beta_j a_j\right)\right\}\\ &=\left(A_{j-1}sinh\delta_{j-1}a_j+B_{j-1}cosh\delta_{j-1}a_j\right)\delta_{j-1}\\ &\quad+\left(-C_{j-1}sin\beta_{j-1}a_j+D_{j-1}cos\beta_{j-1}a_j\right)\beta_{j-1},\quad j=1,\ldots,n. \end{aligned} \tag{43}$$

The equation (38) can be presented, for $j=1,\ldots,n$, as

$$\begin{aligned} &A_j\cosh\delta_j a_j+B_j\sinh\delta_j a_j+C_j\cos\beta_j a_j+D_j\sin\beta_j a_j\\ &=A_{j-1}\cosh\delta_{j-1}a_j+B_{j-1}\sinh\delta_{j-1}a_j\\ &\quad+C_{j-1}\cos\beta_{j-1}a_j+D_{j-1}\sin\beta_{j-1}a_j. \end{aligned} \tag{44}$$

In a similar way, the equations (39) and (40) can be put, respectively, into the form

$$\begin{aligned} &\left(\eta\mu_j\omega^2+\delta_j^2\left(\eta N+EI_j\right)\right)\left(A_j\cosh\delta_j a_j+B_j\sinh\delta_j a_j\right)\\ &\quad+\left(\eta\mu_j\omega^2-\beta_j^2\left(\eta N+EI_j\right)\right)\left(C_j\cos\beta_j a_j+D_j\sin\beta_j a_j\right)\\ &=\left(\eta\mu_{j-1}\omega^2+\delta_{j-1}^2\left(\eta N+EI_{j-1}\right)\right)\left(A_{j-1}\cosh\delta_{j-1}a_j+B_{j-1}\sinh\delta_{j-1}a_j\right)\\ &\quad\left(\eta\mu_{j-1}\omega^2-\beta_{j-1}^2\left(\eta N+EI_{j-1}\right)\right)\left(C_{j-1}\cos\beta_{j-1}a_j+D_{j-1}\sin\beta_{j-1}a_j\right) \end{aligned} \tag{45}$$

and

$$\begin{aligned} &\left(\delta_j\eta\mu_j\omega^2+\delta_j^3\left(\eta N+EI_j\right)\right)\left(A_j\sinh\delta_j a_j+B_j\cosh\delta_j a_j\right)\\ &\quad+\left(-\beta_j\eta\mu_j\omega^2+\beta_j^3\left(\eta N+EI_j\right)\right)\left(C_j\sin\beta_j a_j-D_j\cos\beta_j a_j\right)\\ &=\left(\delta_{j-1}\eta\mu_{j-1}\omega^2+\delta_{j-1}^3\left(\eta N+EI_{j-1}\right)\right)\left(A_{j-1}\sinh\delta_{j-1}a_j+B_{j-1}\cosh\delta_{j-1}a_j\right)\\ &\quad+\left(-\beta_{j-1}\eta\mu_{j-1}\omega^2+\beta_{j-1}^3(\eta N+EI_{j-1})\right)\left(C_{j-1}\sin\beta_{j-1}a_j+D_{j-1}\cos\beta_{j-1}a_j\right), \end{aligned} \tag{46}$$

where $j=1,\ldots,n$. Thus, the total number of equations in (42) – (46) is equal to $4n+4$, and the number of unknowns is also $4n+4$.

7. Eigenfrequencies for a clamped nanobeam

In the case of a clamped nanobeam the boundary conditions are presented by (15) and (17), which can be converted into the form

$$W_0\,(0) = 0, \quad W_0'\,(0) = 0, \quad W_n\,(l) = 0, \quad W_n'\,(l) = 0, \tag{47}$$

where (21) is taken into account.

From (30) and (47) it follows immediately that

$$A_0 = -C_0, \quad B_0 = -\frac{\beta_0}{\delta_0} D_0, \tag{48}$$

and

$$\begin{aligned} A_n \mathrm{cosh} \delta_n l + B_n \mathrm{sinh} \delta_n l + C_n \mathrm{cos} \beta_n l + D_n \mathrm{sin} \beta_n l = 0, \\ \delta_n \left(A_n \mathrm{sinh} \delta_n l + B_n \mathrm{cosh} \delta_n l\right) + \beta_n \left(-C_n \mathrm{sin} \beta_n l + D_n \mathrm{cos} \beta_n l\right) = 0. \end{aligned} \tag{49}$$

Evidently, the jump condition for the slope w' and continuity requirements for w, M, and Q presented in the form (37) – (40) for a simply supported beam remain valid in the present case, as well. These conditions, together with (48) and (49) present a linear system of equations with $4n + 4$ unknown constants and $4n + 4$ equations, as in the previous case. It should be mentioned that similar systems of equations can be compiled for nanobeams with arbitrary boundary conditions.

8. Numerical results

Numerical results are obtained for nanobeams with one step and with cracks at the re-entrant corners of the steps. In this paper the case of a clamped at both ends nanobeam is considered. The results of calculations are presented in Figures 1 – 5 for a clamped nanobeam of length $l = 1000\ nm$ with the cross-section dimensions $h_o = 200\ nm$, $b = 50\ nm$. The material constants are (except Figure 5) $E = 117\ GPa$, $\nu = 0.36$, $\rho = 8960\ kg/m^3$, and $e_0 a = 0.15\ nm$.

In Figures 1 and 2 the influence of the axial force N on the natural frequencies of nanobeams is illustrated in the case of stepped nanobeams with a crack located in the middle of the beam. In Figure 1 the ratio of thickness $h_1/h_0 = 1.25$ (denoted by Υ) and different curves correspond to the depth of the crack $c_1 = 0.2\ h_0$, $c_2 = 0.3\ h_0$, $c_3 = 0.4\ h_0$ (denoted by s), respectively. It can be seen from Figure 1 that the smaller is the crack the higher is the natural frequency. The curves presented in Figures 1 and 2 demonstrate the matter that when the tension increases, then the natural frequency also increases, as might be expected.

The influence of the step location on the natural frequency of the nanobeam is portrayed in Figures 3 – 5. Figure 4 corresponds to the crack depth $c = 0.3\ h_0$. Different curves presented in Figure 4 are obtained for nanobeams with thicknesses $h_1 = 1.25\ h_0$, $h_1 = 1.5\ h_0$, $h_1 = 2\ h_0$, respectively.

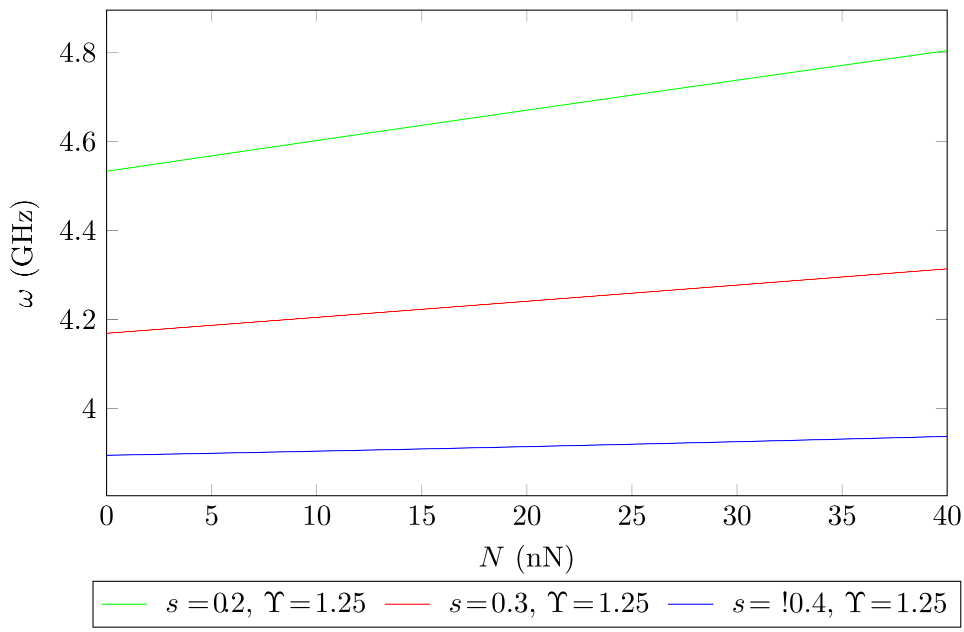

FIGURE 1. Eigenfrequencies depending on axial tension for different crack depths.

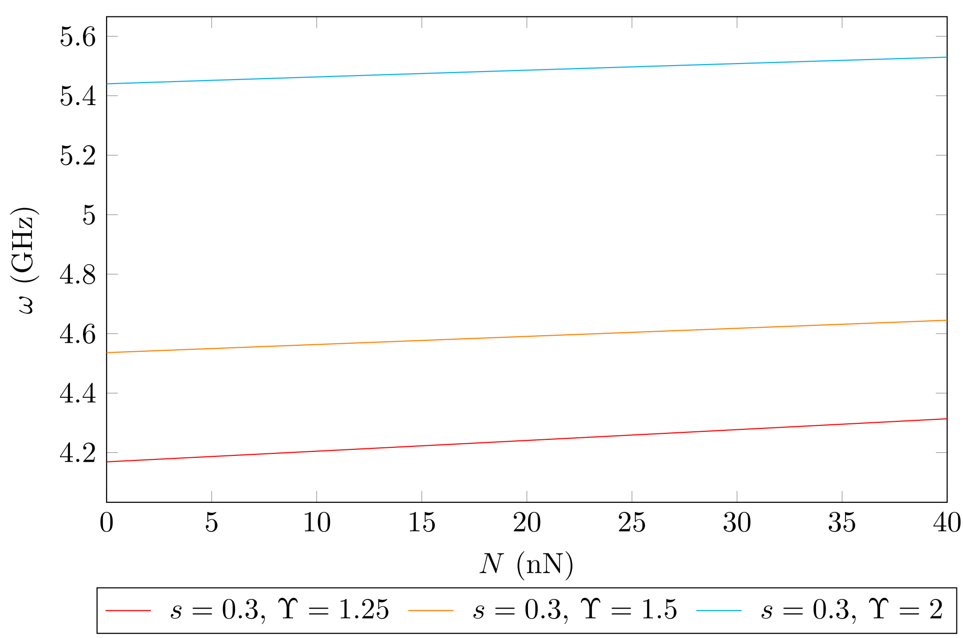

FIGURE 2. Eigenfrequencies depending on axial tension for different ratios of thickness.

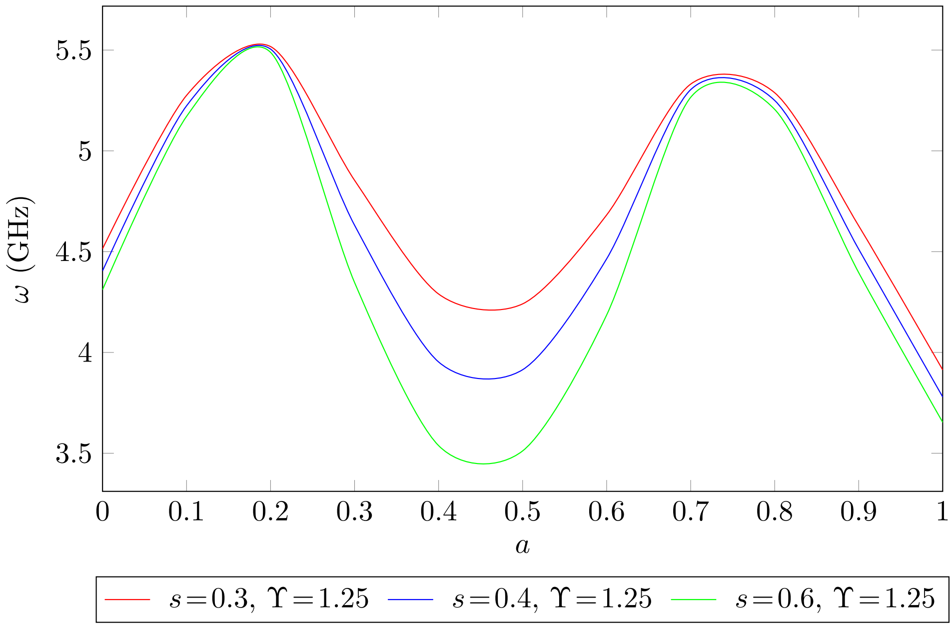

Figure 3. Eigenfrequencies depending on the crack location for different crack depths.

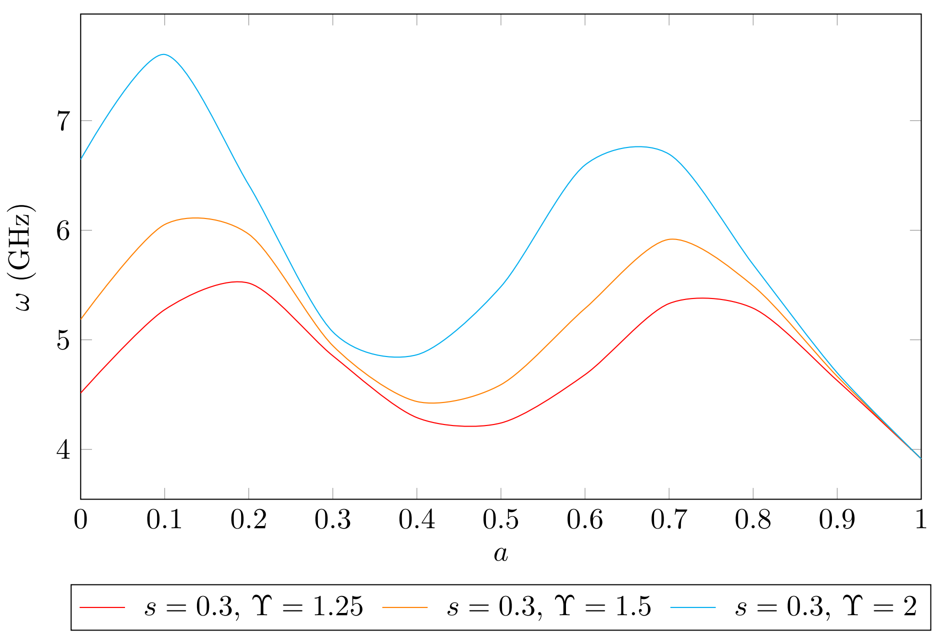

Figure 4. Eigenfrequencies depending on the crack location for different ratios of thickness.

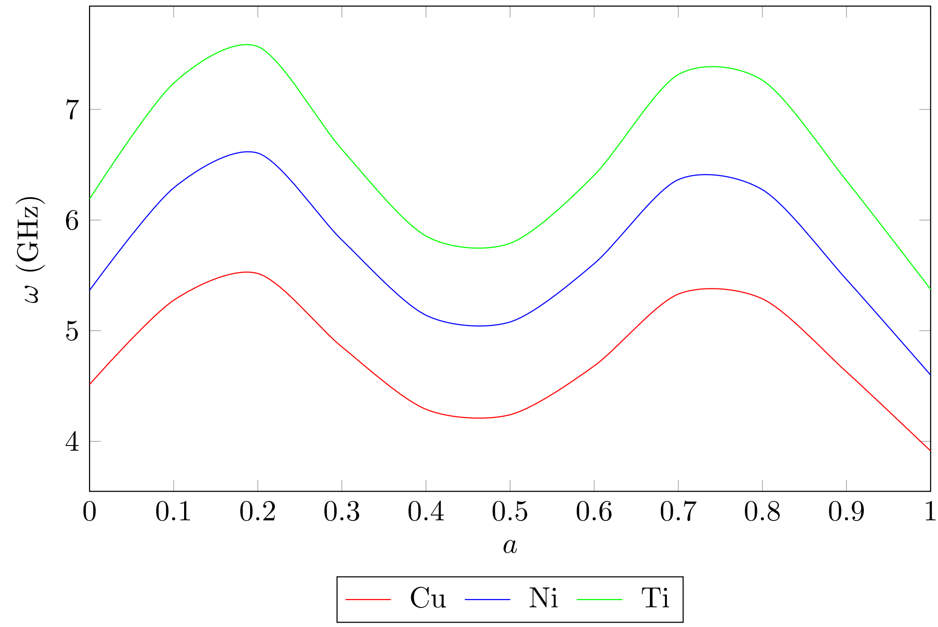

FIGURE 5. Eigenfrequencies depending on the crack location for different materials.

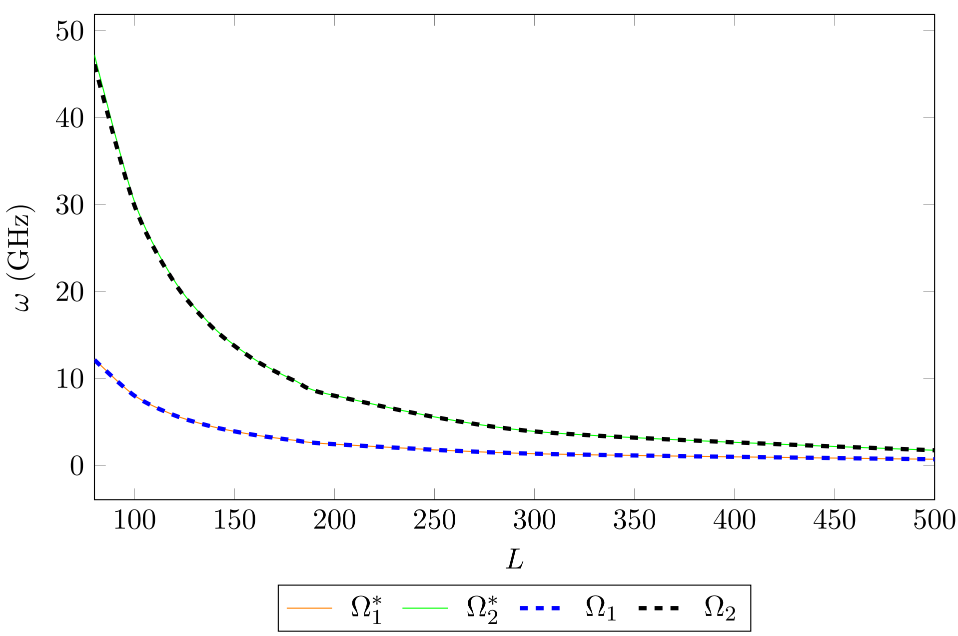

FIGURE 6. Eigenfrequencies of the first and second mode depending on the length of the beam L.

In Figure 5 the dependence of the natural frequency on the step location is depicted for nanobeams made of different materials. It can be seen from Figure 5 that the lowest values of the eigenfrequency correspond to nanobeam made of copper and the highest ones correspond to the nanobeam made of titanium.

Additionally, a comparison with the works of Li et al. [19] for simply supported uniform beams was done using the same dimensional and physical parameters. The results for the value of eigenfrequency of the first two modes Ω_1, Ω_2 were found to be similar as can be seen from Figure 6. Here Ω_1^*, Ω_2^* denote the results found by Li et al. [19], and Ω_1, Ω_2 denote the results of the current paper.

9. Concluding remarks

A method of vibration analysis of nanobeams with various end conditions was developed in the frameworks of Eringen's nonlocal theory of elasticity.

The nanobeams under consideration have stepped cross sections and are weakened by stable cracks or crack-like defects. It is assumed that the cracks are located at the re-entrant corners of steps. These are the positions where the occurrence of stress concentration is most probable.

The additional compliance produced by the defect is calculated according to the method of Dimarogonas. Calculations carried out revealed the matter that defects affect the eigenfrequencies of nanobeams. It was shown that the maximal values of eigenfrequencies have the nanobeams without defects. The matter that cracks reduce natural frequencies of beams are recognized at the macro-level, as well.

Similarly, it was shown that when the axial tension of the nanobeam increases then the natural frequency also increases, as might be expected.

Acknowledgement

The authors are grateful to the anonymous referees for their constructive comments and suggestions. The partial support from the Institutional Research Funding IUT 20-57 of Estonian Ministry of Education and Research is gratefully acknowledged.

References

[1] M. Alves and P. Ribeiro, *Non-linear modes of vibration of Timoshenko nanobeams under electrostatic actuation*, Int. J. Mech. Sci. **130** (2017), 188–202.

[2] T. L. Anderson, *Fracture Mechanics: Fundamentals and Applications*, CRC Press, Boca Raton, 2017.

[3] J. Aranda-Ruiz, J. Loya, and J. Fernandez-Saez, *Bending vibrations of rotating nonuniform nanocantilevers using Eringen nonlocal elasticity theory*, Compos. Struct. **94** (2012), 2990–3001.
[4] S. M. Bagdath, *Non-linear vibration of nanobeams with various boundary conditions based on nonlocal elasticity theory*, Composites B **80** (2015), 43–52.
[5] D. Broek, *The Practical Use of Fracture Mechanics*, Kluwer, 1989.
[6] S. Caddemi and I. Calio, *Exact solution of multi-cracked Euler-Bernoulli column*, Int. J. Solids Struct. **45** (2008), 1332–1351.
[7] N. Challamel, *Variational formulation of gradient or/and nonlocal higher-order shear elasticity beams*, Compos. Struct. **105** (2013), 351–368.
[8] T. J. Chondros, A. D. Dimarogonas, and J. Yao, *A continuous cracked beam vibration theory*, J. Sound Vibration bf 25 (1998), 17–34.
[9] A. D. Dimarogonas, *Vibration of cracked structures: A state of the art review.* Engin. Fracture Mech. **55** (1996), 831–857.
[10] K. G. Eptaimeros, C. Koutsoumaris, and G. Tsamasphyros, *Nonlocal integral approach to the dynamical response of nanobeams*, Int. J. Mech. Sci. **115-116** (2016), 68–80.
[11] A. C. Eringen, *On differential equations of nonlocal elasticity and solutions of screw dislocation and surface waves*, J. Appl. Phys. **54** (1983), 4703–4710.
[12] A. C. Eringen, *Nonlocal Continuum Field Theories*, Springer, New York, 2002.
[13] S. A. M Ghannadpour, B. Mohammadi, and J. Fazilati, *Bending, buckling and vibration problems of nonlocal Euler beams using Ritz method*, Compos. Struct. **96** (2013) 584–589.
[14] J. Lellep and E. Kägo, *Vibrations of elastic stretched strips with cracks*, Int. J. Mech. **5** (2011), 27–34.
[15] J. Lellep and T. Kraav, *Buckling of beams and columns with defects*, Int. J. Struct. Stability Dynamics, **16** (2016), 10.1142/S0219455415500480.
[16] J. Lellep and A. Lenbaum, *Natural vibrations of a nano-beam with cracks*, Int. J. Theor. Appl. Mech. **1** (2016), 247–252.
[17] J. Lellep and A. Lenbaum, *Free vibrations of stepped nano-beams*, Int. J. Comp. Meth. Exp. Meas. **6** (2018), 716–725.
[18] J. Lellep and A. Liyvapuu, *Natural vibrations of stepped arches with cracks*, Agron. Res. **14** (2016), 821–830.
[19] C. Li, C. W. Lim, J. Yu, and Q. C. Zeng, *Analytical solutions for vibration of simply supported nonlocal nanobeams with an axial force*, Int. J. Struct. Stability Dynamics **11** (2011), 257–271.
[20] P. Lu, H. P. Lee, C. Lu, and P. Q. Zhang, *Dynamic properties of flexural beams using a nonlocal elasticity model*, J. Appl. Phys. **99** (2006), 073510, 1–9.
[21] J. Peddieson, G. R. Buchanan, and R. P. McNitt, *Application of nonlocal continuum models to nanotechnology*, Int. J. Eng. Sci. **41** (2003), 305–312.
[22] C. Polizzotto, *Nonlocal elasticity and related variational principles*, Int. J. Solids Struct. **38** (2001), 7359–7380.
[23] J. N. Reddy, *Nonlocal theories for bending, buckling and vibration of beams*, Int. J. Eng. Sci. **45** (2007), 288–307.
[24] H. Roostai and M. Haghpanahi, *Vibration of nanobeams of different boundary conditions with multiple cracks based on nonlocal elasticity theory*, Appl. Math. Modell. **38** (2014), 1159–1169
[25] W. Soedel, *Vibration of Shells and Plates*, Marcel Dekker, New York, 2004.
[26] H. Tada, P. C. Paris, and G. R. Irwin, *The Stress Analysis of Cracks Handbook*, ASME Press, 2000.

[27] T. Thai, *A nonlocal beam theory for bending, buckling and vibration of nanobeams*, Int. J. Eng. Sci. **52** (2012), 56–64.
[28] C. M. Wang, J. N. Reddy, and K. H. Lee, *Shear Deformable Beams and Plates*, Elsevier, Amsterdam, 2000.

Institute of Mathematics and Statistics, University of Tartu, Liivi 2, 50409 Tartu, Estonia
E-mail address: jaan.lellep@ut.ee
E-mail address: artur.lenbaum@ut.ee