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Some new Hermite–Hadamard type inequalities for functions whose n^{th} derivatives are convex

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Abstract. We first create an integral identity for n-times differentiable functions. Relying on this identity, we establish some new Hermite–Hadamard type inequalities for functions whose n^{th} derivatives are convex.

1. Introduction

It is well known that convexity plays a central and important role in modern analysis. We recall that a function $f : I \to \mathbb{R}$ on the interval I of real numbers is said to be convex, if

$$f\left(tx + (1-t)\,y\right) \leq tf\left(x\right) + (1-t)\,f(y)$$

holds for all $x, y \in I$ and $t \in [0,1]$

One of the well-known inequalities in mathematics for convex functions is the Hermite–Hadamard integral inequality, which can be stated as follows: for every convex function f on the finite interval $[a,b]$, $a < b$, we have

$$f\left(\tfrac{a+b}{2}\right) \leq \tfrac{1}{b-a}\int_a^b f(x)dx \leq \tfrac{f(a)+f(b)}{2}. \tag{1.1}$$

If the function f is concave, then (1.1) holds in the reverse direction (see [10]).

The above double inequality has a number of various generalizations, refinements, extensions, and variants. Many papers deal with the estimating of differences

$$\mathcal{A}(f) := \tfrac{f(a)+f(b)}{2} - \tfrac{1}{b-a}\int_a^b f(t)dt \;\text{ and }\; \mathcal{B}(f) := f\left(\tfrac{a+b}{2}\right) - \tfrac{1}{b-a}\int_a^b f(t)dt$$

under some convexity conditions on derivatives of the function f.

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Assume that the function $f\colon I\to\mathbb{R}$ is differentiable on I° and $f'\in L[a,b]$, where $a,b\in I,\ a<b$. Kavurmaci et al. [6] established the inequality

$$|\mathcal{A}(f)|\leq \tfrac{b-a}{12}\left(\left|f'(a)\right|+\left|f'(\tfrac{a+b}{2})\right|+\left|f'(b)\right|\right)$$

if $|f'|$ is convex. Moreover, they also showed that if $|f'|^{q}$ is convex for $q>1$, then

$$|\mathcal{A}(f)|\leq \tfrac{b-a}{4}\left(\tfrac{1}{p+1}\right)^{\frac{1}{p}}\left(\tfrac{1}{2}\right)^{\frac{1}{q}}\left(\left(\left|f'\left(a\right)\right|^{q}+\left|f'(\tfrac{a+b}{2})\right|^{q}\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\left|f'\left(\tfrac{a+b}{2}\right)\right|^{q}+\left|f'\left(b\right)\right|^{q}\right)^{\frac{1}{q}}\right),\quad \tfrac{1}{p}+\tfrac{1}{q}=1,$$

and if $|f'|^{q}$ is convex for $q\geq 1$, then

$$|\mathcal{A}(f)|\leq \tfrac{b-a}{8}\left(\tfrac{1}{3}\right)^{\frac{1}{q}}\left(\left(2\left|f'\left(a\right)\right|^{q}+\left|f'(\tfrac{a+b}{2})\right|^{q}\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\left|f'\left(\tfrac{a+b}{2}\right)\right|^{q}+2\left|f'\left(b\right)\right|^{q}\right)^{\frac{1}{q}}\right).$$

Özdemir et al. [11], obtained the following results in the case $I=[0,\infty)$:

$$|\mathcal{B}(f)|\leq \tfrac{b-a}{24}\left(\left|f'(a)\right|+4\left|f'(\tfrac{a+b}{2})\right|+\left|f'(b)\right|\right)$$

if $|f'|$ is convex,

$$|\mathcal{B}(f)|\leq \tfrac{(b-a)}{4(p+1)^{\frac{1}{p}}}\left(\tfrac{1}{2}\right)^{\frac{1}{q}}\left(\left(\left|f'\left(a\right)\right|^{q}+\left|f'\left(\tfrac{a+b}{2}\right)\right|^{q}\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\left|f'\left(\tfrac{a+b}{2}\right)\right|^{q}+\left|f'(b)\right|^{q}\right)^{\frac{1}{q}}\right)$$

if $|f'|^{q}$ is convex for $q>1$ with $1/p+1/q=1$, and

$$|\mathcal{B}(f)|\leq \tfrac{b-a}{4}\left(\tfrac{1}{2}\right)^{1-\frac{1}{q}}\left(\left(\tfrac{1}{6}\left|f'\left(a\right)\right|^{q}+\tfrac{1}{3}\left|f'\left(\tfrac{a+b}{2}\right)\right|^{q}\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\tfrac{1}{3}\left|f'\left(\tfrac{a+b}{2}\right)\right|^{q}+\tfrac{1}{6}\left|f'(b)\right|^{q}\right)^{\frac{1}{q}}\right)$$

if $|f'|^{q}$ is convex for $q\geq 1$.

Park [12] considered, for $n\in\mathbb{N}$ with $n\geq 2$, the difference

$$\mathcal{B}(f,n):=\tfrac{1}{2}\left[f\left(\tfrac{(n-1)a+b}{n}\right)+f\left(\tfrac{a+(n-1)b}{n}\right)\right]-\tfrac{1}{b-a}\int_{a}^{b}f(t)dt$$

and proved that if $|f'|$ is convex, then

$$\begin{aligned}|\mathcal{B}(f,n)| \leq \tfrac{b-a}{3n^2}\Big\{&\tfrac{|f'(a)|+|f'(b)|}{2} + \left|f'\left(\tfrac{(n-1)a+b}{n}\right)\right| + \left|f'\left(\tfrac{a+(n-1)b}{n}\right)\right| \\ &+ \left(\tfrac{n-2}{2}\right)^2\left(\left|f'(\tfrac{a+b}{2})\right| + \left|f'\left(\tfrac{(n-1)a+b}{n}\right)\right|\right. \\ &\left.+\left|f'\left(\tfrac{a+(n-1)b}{n}\right)\right|\right)\Big\},\end{aligned} \tag{1.2}$$

but if $|f'|^q$ is convex for $q > 1$ with $1/p + 1/q = 1$, then

$$\begin{aligned}|\mathcal{B}(f,n)| \leq \tfrac{b-a}{n^2}\left(\tfrac{1}{p+1}\right)^{\frac{1}{p}}\left(\tfrac{1}{2}\right)^{\frac{1}{q}}\Bigg(&\left(\left|f'\left(a\right)\right|^q + \left|f'\left(\tfrac{(n-1)a+b}{n}\right)\right|^q\right)^{\frac{1}{q}} \\ &+ \left(\left|f'\left(b\right)\right|^q + \left|f'\left(\tfrac{a+(n-1)b}{n}\right)\right|^q\right)^{\frac{1}{q}} \\ &+ \left(\tfrac{n-2}{2}\right)^2\Bigg(\left(\left|f'\left(\tfrac{a+b}{2}\right)\right|^q + \left|f'\left(\tfrac{(n-1)a+b}{n}\right)\right|^q\right)^{\frac{1}{q}} \\ &+ \left(\left|f'\left(\tfrac{a+b}{2}\right)\right|^q + \left|f'\left(\tfrac{a+(n-1)b}{n}\right)\right|^q\right)^{\frac{1}{q}}\Bigg)\Bigg),\end{aligned} \tag{1.3}$$

and

$$\begin{aligned}|\mathcal{B}(f,n)| \leq \tfrac{b-a}{n^2}\left(\tfrac{1}{2}\right)^{\frac{1}{p}}\left(\tfrac{1}{3}\right)^{\frac{1}{q}}\Bigg(&\left(\tfrac{1}{2}\left|f'\left(a\right)\right|^q + \left|f'\left(\tfrac{(n-1)a+b}{n}\right)\right|^q\right)^{\frac{1}{q}} \\ &+ \left(\tfrac{1}{2}\left|f'\left(b\right)\right|^q + \left|f'\left(\tfrac{a+(n-1)b}{n}\right)\right|^q\right)^{\frac{1}{q}} \\ &+ \left(\tfrac{n-2}{2}\right)^2\Bigg(\left(\tfrac{1}{2}\left|f'\left(\tfrac{a+b}{2}\right)\right|^q + \left|f'\left(\tfrac{(n-1)a+b}{n}\right)\right|^q\right)^{\frac{1}{q}} \\ &+ \left(\tfrac{1}{2}\left|f'\left(\tfrac{a+b}{2}\right)\right|^q + \left|f'\left(\tfrac{a+(n-1)b}{n}\right)\right|^q\right)^{\frac{1}{q}}\Bigg)\Bigg).\end{aligned} \tag{1.4}$$

Some similar inequalities may be found, for example, in the papers [1–5, 7–9, 13–17].

Motivated by the results given above, we first establish an integral identity for n-times differentiable functions, and then we prove some new Hermite–Hadamard type inequalities for functions whose n^{th} derivatives are convex. Several known results are also derived, and applications to special means are given.

2. Main results

Lemma 1. *Let $f : [a, b] \to \mathbb{R}$ be a mapping such that the derivative $f^{(n-1)}$ $(n \geq 1)$ is absolutely continuous on $[a, b]$, $a < b$. Then the sum*

$$\mathcal{C}(f,x,n,\lambda) := \sum_{p=0}^{n-1} \tfrac{1}{(n-p)!(b-a)^p}\left(\left(\left(\lambda - \tfrac{x-a}{b-a}\right)^{n-p} - (-1)^{n-p}(1-\lambda)^{n-p}\right)\right.$$

$$\times f^{(n-1-p)}((1-\lambda)a+\lambda b)-\left(\left(\tfrac{b-x}{b-a}-\lambda\right)^{n-p}-(1-\lambda)^{n-p}\right)$$
$$\times f^{(n-1-p)}(\lambda a+(1-\lambda)b)\Big)+\tfrac{(-1)^n}{(b-a)^n}\int_a^b f(t)dt$$

satisfies the equality

$$\mathcal{C}(f,x,n,\lambda)=\int_a^b k_n(x,t)\,f^{(n)}(t)dt \tag{2.1}$$

for all $x\in[a,b]$ *and* $\lambda\in[\frac{1}{2},1]$*, where* $n\in\mathbb{N}$ *and the kernel* $k_n(x,t)$: $[a,b]^2\to\mathbb{R}$ *is given by*

$$k_n(x,t)=\begin{cases}\frac{1}{n!}\left(\frac{t-a}{b-a}\right)^n & \text{if } t\in[a,\lambda a+(1-\lambda)b),\\ \frac{1}{n!}\left(\frac{t-x}{b-a}\right)^n & \text{if } t\in[\lambda a+(1-\lambda)b,(1-\lambda)a+\lambda b),\\ \frac{1}{n!}\left(\frac{t-b}{b-a}\right)^n & \text{if } t\in[(1-\lambda)a+\lambda b,b].\end{cases}$$

Proof. The proof is given by mathematical induction. For $n=1$ we have

$$\begin{aligned}&\int_a^b k_1(x,t)\,f'(t)dt\\ &=\int_a^{\lambda a+(1-\lambda)b}\tfrac{t-a}{b-a}f'(t)dt+\int_{\lambda a+(1-\lambda)b}^{(1-\lambda)a+\lambda b}\tfrac{t-x}{b-a}f'(t)dt+\int_{(1-\lambda)a+\lambda b}^{b}\tfrac{t-b}{b-a}f'(t)dt\\ &=(1-\lambda)f(\lambda a+(1-\lambda)b)-\tfrac{1}{b-a}\int_a^{\lambda a+(1-\lambda)b}f(t)dt\\ &\quad-\left(\tfrac{x-a}{b-a}-\lambda\right)f((1-\lambda)a+\lambda b)-\left(\tfrac{b-x}{b-a}-\lambda\right)f(\lambda a+(1-\lambda)b)\\ &\quad-\tfrac{1}{b-a}\int_{\lambda a+(1-\lambda)b}^{(1-\lambda)a+\lambda b}f(t)dt+(1-\lambda)f((1-\lambda)a+\lambda b)-\tfrac{1}{b-a}\int_{(1-\lambda)a+\lambda b}^{b}f(t)dt\\ &=\tfrac{x-a}{b-a}f(\lambda a+(1-\lambda)b)+\tfrac{b-x}{b-a}f((1-\lambda)a+\lambda b)-\tfrac{1}{b-a}\int_a^b f(t)dt,\end{aligned}$$

i.e., (2.1) holds for $n=1$.

Assume that (2.1) holds for m, $m\geq1$, and let us prove it for $m+1$. We have

$$\int_a^b k_{m+1}(x,t)\,f^{(m+1)}(t)dt$$

$$= \int_{a}^{\lambda a+(1-\lambda)b} \frac{\left(\frac{t-a}{b-a}\right)^{m+1}}{(m+1)!} f^{(m+1)}(t)dt + \int_{\lambda a+(1-\lambda)b}^{(1-\lambda)a+\lambda b} \frac{\left(\frac{t-x}{b-a}\right)^{m+1}}{(m+1)!} f^{(m+1)}(t)dt$$

$$+ \int_{(1-\lambda)a+\lambda b}^{b} \frac{\left(\frac{t-b}{b-a}\right)^{m+1}}{(m+1)!} f^{(m+1)}(t)dt$$

$$= \frac{(1-\lambda)^{m+1}}{(m+1)!} f^{(m)}(\lambda a + (1-\lambda)\, b) - \frac{1}{b-a} \int_{a}^{\lambda a+(1-\lambda)b} \frac{\left(\frac{t-b}{b-a}\right)^{m}}{m!} f^{(m)}(t)dt$$

$$+ \frac{\left(\lambda - \frac{x-a}{b-a}\right)^{m+1}}{(m+1)!} f^{(m)}((1-\lambda)\, a + \lambda b) - \frac{\left(\frac{b-x}{b-a} - \lambda\right)^{m+1}}{(m+1)!} f^{(m)}(\lambda a + (1-\lambda)\, b)$$

$$- \frac{1}{b-a} \int_{\lambda a+(1-\lambda)b}^{(1-\lambda)a+\lambda b} \frac{\left(\frac{t-x}{b-a}\right)^{m}}{m!} f^{(m)}(t)dt$$

$$- \frac{(-1)^{m+1}(1-\lambda)^{m+1}}{(m+1)!} f^{(m)}((1-\lambda)\, a + \lambda b) - \frac{1}{b-a} \int_{(1-\lambda)a+\lambda b}^{b} \frac{\left(\frac{t-b}{b-a}\right)^{m}}{m!} f^{(m)}(t)dt$$

$$= \frac{1}{(m+1)!} \left(\left(\lambda - \frac{x-a}{b-a}\right)^{m+1} - (-1)^{m+1}(1-\lambda)^{m+1} \right) f^{(m)}((1-\lambda)\, a + \lambda b)$$

$$- \frac{1}{(m+1)!} \left(\left(\frac{b-x}{b-a} - \lambda\right)^{m+1} - (1-\lambda)^{m+1} \right) f^{(m)}(\lambda a + (1-\lambda)\, b)$$

$$- \frac{1}{b-a} \int_{a}^{b} k_m(x,t)\, f^{(m)}(t)dt$$

$$= \frac{1}{(m+1)!} \left(\left(\lambda - \frac{x-a}{b-a}\right)^{m+1} - (-1)^{m+1}(1-\lambda)^{m+1} \right) f^{(m)}((1-\lambda)\, a + \lambda b)$$

$$- \frac{1}{(m+1)!} \left(\left(\frac{b-x}{b-a} - \lambda\right)^{m+1} - (1-\lambda)^{m+1} \right) f^{(m)}(\lambda a + (1-\lambda)\, b)$$

$$- \frac{1}{b-a} \left(\sum_{p=0}^{m-1} \frac{1}{(m-p)!(b-a)^p} \left(\left(\left(\lambda - \frac{x-a}{b-a}\right)^{m-p} - (-1)^{m-p}(1-\lambda)^{m-p} \right) \right.\right.$$

$$\times f^{(m-1-p)}((1-\lambda)\, a + \lambda b)$$

$$\left.\left. - \left(\left(\frac{b-x}{b-a} - \lambda\right)^{m-p} - (1-\lambda)^{m-p} \right) f^{(m-1-p)}(\lambda a + (1-\lambda)\, b) \right) \right.$$

$$+\left.\frac{(-1)^m}{(b-a)^m}\int\limits_a^b f(t)dt\right)=\mathcal{C}(f,x,m+1,\lambda),$$

which completes the proof. □

We are now ready to present our results.

Theorem 1. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping such that $f^{(n-1)}$ is absolutely continuous on $[a,b]$, $a<b$. If $\left|f^{(n)}\right|$ is convex, then the inequality*

$$\begin{aligned}|\mathcal{C}(f,x,n,\lambda)|&\leq \tfrac{(b-a)(1-\lambda)^{n+1}}{(n+2)!}\left|f^{(n)}(a)\right|+\tfrac{(1-\lambda)^{n+1}(b-a)^{n+1}+(x-(\lambda a+(1-\lambda)b))^{n+1}}{n!(n+2)(b-a)^n}\\&\times\left|f^{(n)}(\lambda a+(1-\lambda)b)\right|+\tfrac{(x-(\lambda a+(1-\lambda)b))^{n+1}+((1-\lambda)a+\lambda b-x)^{n+1}}{(n+2)!(b-a)^n}\left|f^{(n)}(x)\right|\\&+\tfrac{((1-\lambda)a+\lambda b-x)^{n+1}+(1-\lambda)^{n+1}(b-a)^{n+1}}{n!(n+2)(b-a)^n}\left|f^{(n)}((1-\lambda)a+\lambda b)\right|\\&+\tfrac{(1-\lambda)^{n+1}(b-a)}{(n+2)!}\left|f^{(n)}(b)\right|\end{aligned}$$

holds for all $x\in[\lambda a+(1-\lambda)b,(1-\lambda)a+\lambda b]$ and $\lambda\in[\frac{1}{2},1]$.

Proof. Using Lemma 1, we have

$$\begin{aligned}|\mathcal{C}(f,x,n,\lambda)|&\leq\int_a^b|k_n(x,t)|\left|f^{(n)}(t)\right|dt\\&=\tfrac{1}{(b-a)^n}\int\limits_a^{\lambda a+(1-\lambda)b}\tfrac{(t-a)^n}{n!}\left|f^{(n)}(t)\right|dt+\tfrac{1}{(b-a)^n}\int\limits_{\lambda a+(1-\lambda)b}^{x}\tfrac{(x-t)^n}{n!}\left|f^{(n)}(t)\right|dt\\&+\tfrac{1}{(b-a)^n}\int\limits_x^{(1-\lambda)a+\lambda b}\tfrac{(t-x)^n}{n!}\left|f^{(n)}(t)\right|dt+\tfrac{1}{(b-a)^n}\int\limits_{(1-\lambda)a+\lambda b}^{b}\tfrac{(b-t)^n}{n!}\left|f^{(n)}(t)\right|dt\\&=\tfrac{(b-a)(1-\lambda)^{n+1}}{n!}\int_0^1\alpha^n\left|f^{(n)}((1-\alpha)a+\alpha(\lambda a+(1-\lambda)b))\right|d\alpha\\&+\tfrac{(x-(\lambda a+(1-\lambda)b))^{n+1}}{n!(b-a)^n}\int_0^1(1-\alpha)^n\left|f^{(n)}((1-\alpha)(\lambda a+(1-\lambda)b)+\alpha x)\right|d\alpha\\&+\tfrac{((1-\lambda)a+\lambda b-x)^{n+1}}{n!(b-a)^n}\int_0^1\alpha^n\left|f^{(n)}((1-\alpha)x+\alpha((1-\lambda)a+\lambda b))\right|d\alpha\\&+\tfrac{(1-\lambda)^{n+1}(b-a)}{n!}\int_0^1(1-\alpha)^n\left|f^{(n)}((1-\alpha)((1-\lambda)a+\lambda b)+\alpha b)\right|d\alpha.\end{aligned}$$

By the convexity of $\left|f^{(n)}\right|$, this gives that

$$|\mathcal{C}(f,x,n,\lambda)|\leq\tfrac{(b-a)(1-\lambda)^{n+1}}{n!}$$

$$\times \left(\left| f^{(n)}(a) \right| \int_0^1 (1-\alpha)\,\alpha^n d\alpha + \left| f^{(n)}(\lambda a + (1-\lambda)\,b) \right| \int_0^1 \alpha^{n+1} d\alpha \right)$$
$$+ \tfrac{(x-(\lambda a+(1-\lambda)b))^{n+1}}{n!(b-a)^n}$$
$$\times \left(\left| f^{(n)}(\lambda a + (1-\lambda)\,b) \right| \int_0^1 (1-\alpha)^{n+1} d\alpha + \left| f^{(n)}(x) \right| \int_0^1 \alpha\,(1-\alpha)^n d\alpha \right)$$
$$+ \tfrac{((1-\lambda)a+\lambda b-x)^{n+1}}{n!(b-a)^n}$$
$$\times \left(\left| f^{(n)}(x) \right| \int_0^1 (1-\alpha)\,\alpha^n d\alpha + \left| f^{(n)}((1-\lambda)\,a + \lambda b) \right| \int_0^1 \alpha^{n+1} d\alpha \right)$$
$$+ \tfrac{(1-\lambda)^{n+1}(b-a)}{n!}$$
$$\times \left(\left| f^{(n)}((1-\lambda)\,a + \lambda b) \right| \int_0^1 (1-\alpha)^{n+1} d\alpha + \left| f^{(n)}(b) \right| \int_0^1 \alpha\,(1-\alpha)^n d\alpha \right)$$
$$= \tfrac{(b-a)(1-\lambda)^{n+1}}{(n+2)!} \left| f^{(n)}(a) \right| + \tfrac{(1-\lambda)^{n+1}(b-a)}{(n+2)!} \left| f^{(n)}(b) \right|$$
$$+ \tfrac{(1-\lambda)^{n+1}(b-a)^{n+1}+(x-(\lambda a+(1-\lambda)b))^{n+1}}{n!(n+2)(b-a)^n} \left| f^{(n)}(\lambda a + (1-\lambda)\,b) \right|$$
$$+ \tfrac{(x-(\lambda a+(1-\lambda)b))^{n+1}+((1-\lambda)a+\lambda b-x)^{n+1}}{(n+2)!(b-a)^n} \left| f^{(n)}(x) \right|$$
$$+ \tfrac{((1-\lambda)a+\lambda b-x)^{n+1}+(1-\lambda)^{n+1}(b-a)^{n+1}}{n!(n+2)(b-a)^n} \left| f^{(n)}((1-\lambda)\,a + \lambda b) \right|,$$

which is the desired result. □

Corollary 1. *If we put $\lambda = 1$ in Theorem 1, then we obtain the following generalized trapezoid inequality for n-times convex functions:*

$$\left| \sum_{p=0}^{n-1} \tfrac{(b-x)^{n-p} f^{(n-1-p)}(b)}{(n-p)!} - (-1)^{n-p}\, \tfrac{(x-a)^{n-p} f^{(n-1-p)}(a)}{(n-p)!} + (-1)^n \int_a^b f(t)dt \right|$$
$$\leq \tfrac{(x-a)^{n+1}}{n!(n+2)} \left| f^{(n)}(a) \right| + \tfrac{(x-a)^{n+1}+(b-x)^{n+1}}{(n+2)!} \left| f^{(n)}(x) \right| + \tfrac{(b-x)^{n+1}}{n!(n+2)} \left| f^{(n)}(b) \right|.$$

Moreover, by choosing $x = \frac{a+b}{2}$, we get

$$\left| \sum_{p=0}^{n-1} \tfrac{(b-a)^{n-p}}{(n-p)!2^{n-p}} \left(f^{(n-1-p)}(b) - (-1)^{n-p}\, f^{(n-1-p)}(a) \right) + (-1)^n \int_a^b f(t)dt \right|$$
$$\leq \tfrac{(b-a)^{n+1}}{n!(n+2)2^{n+1}} \left| f^{(n)}(a) \right| + \tfrac{(b-a)^{n+1}}{(n+2)!2^n} \left| f^{(n)}(\tfrac{a+b}{2}) \right| + \tfrac{(b-a)^{n+1}}{n!(n+2)2^{n+1}} \left| f^{(n)}(b) \right|.$$

Remark 1. Corollary 1 reduces, for $n = 1$, to Theorem 4 from [5]. Moreover, if we put $x = \frac{a+b}{2}$, then we obtain Corollary 2 from [5].

Corollary 2. *If we choose $\lambda=\frac{1}{2}$ in Theorem 1, then we obtain the following midpoint inequality for n-times convex functions:*

$$\left|\sum_{p=0}^{n-1}\tfrac{1-(-1)^{n-p}}{(n-p)!(b-a)^{p}2^{n-p}}f^{(n-1-p)}(\tfrac{a+b}{2})+\tfrac{(-1)^{n}}{(b-a)^{n}}\int_{a}^{b}f(t)dt\right|$$
$$\leq\tfrac{b-a}{(n+2)!2^{n+1}}\left(\left|f^{(n)}(a)\right|+2\left(n+1\right)\left|f^{(n)}(\tfrac{a+b}{2})\right|+\left|f^{(n)}(b)\right|\right).$$

Remark 2. Corollary 2 reduces, for $n=1$, to Corollary 1 from [8]. We also recapture the inequality (1.2) for $n=2$.

Corollary 3. *If we put $x=\frac{a+b}{2}$ and $\lambda=\frac{2}{3}$ in Theorem 1, then we obtain the following two-point open Newton–Cotes inequality for n-times convex functions:*

$$\left|\sum_{p=0}^{n-1}\tfrac{\left(1-(-1)^{n-p}2^{n-p}\right)}{(n-p)!(b-a)^{p}6^{n-p}}\left(f^{(n-1-p)}(\tfrac{a+2b}{3})-(-1)^{n-p}f^{(n-1-p)}(\tfrac{2a+b}{3})\right)\right.$$
$$\left.+\tfrac{(-1)^{n}}{(b-a)^{n}}\int_{a}^{b}f(t)dt\right|$$
$$\leq\tfrac{b-a}{(n+2)!\times6^{n+1}}\left(2^{n+1}\left|f^{(n)}(a)\right|+(n+1)\left(2^{n+1}+1\right)\left|f^{(n)}(\tfrac{2a+b}{3})\right|\right.$$
$$\left.++2\left|f^{(n)}(\tfrac{a+b}{2})\right|(n+1)\left(2^{n+1}+1\right)\left|f^{(n)}(\tfrac{a+2b}{3})\right|+2^{n+1}\left|f^{(n)}(b)\right|\right).$$

Remark 3. Taking $n=1$ in Corollary 3, we obtain the inequality (1.2) for $n=3$.

Theorem 2. *Let $f:[a,b]\rightarrow\mathbb{R}$ be a differentiable mapping such that $f^{(n-1)}$ is absolutely continuous on $[a,b]$, $a<b$. If $\left|f^{(n)}\right|^{q}$ with $q>1$ is convex, then*

$$\left|\mathcal{C}(f,x,n,\lambda)\right|\leq\tfrac{(b-a)(1-\lambda)^{n+1}}{n!(np+1)^{\frac{1}{p}}2^{\frac{1}{q}}}\left(\left|f^{(n)}\left(a\right)\right|^{q}+\left|f^{(n)}\left(\lambda a+\left(1-\lambda\right)b\right)\right|^{q}\right)^{\frac{1}{q}}$$
$$+\tfrac{(x-(\lambda a+(1-\lambda)b))^{n+1}}{n!(b-a)^{n}(np+1)^{\frac{1}{p}}2^{\frac{1}{q}}}\left(\left|f^{(n)}\left(\lambda a+\left(1-\lambda\right)b\right)\right|^{q}+\left|f^{(n)}(x)\right|^{q}\right)^{\frac{1}{q}}$$
$$+\tfrac{((1-\lambda)a+\lambda b-x)^{n+1}}{n!(b-a)^{n}(np+1)^{\frac{1}{p}}2^{\frac{1}{q}}}\left(\left|f^{(n)}\left(x\right)\right|^{q}+\left|f^{(n)}\left(\left(1-\lambda\right)a+\lambda b\right)\right|^{q}\right)^{\frac{1}{q}}$$
$$+\tfrac{(1-\lambda)^{n+1}(b-a)}{n!(np+1)^{\frac{1}{p}}2^{\frac{1}{q}}}\left(\left|f^{(n)}\left(\left(1-\lambda\right)a+\lambda b\right)\right|^{q}+\left|f^{(n)}(b)\right|^{q}\right)^{\frac{1}{q}}$$

holds for all $x\in[\lambda a+\left(1-\lambda\right)b,\left(1-\lambda\right)a+\lambda b]$, $\lambda\in[\frac{1}{2},1]$ and $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Using Lemma 1 and Hölder's inequality, we get

$$\begin{aligned}
|\mathcal{C}(f,x,n,\lambda)| &\leq \tfrac{(b-a)(1-\lambda)^{n+1}}{n!}\left(\int_0^1 \alpha^{np}d\alpha\right)^{\frac{1}{p}} \\
&\times\left(\int_0^1 \left|f^{(n)}((1-\alpha)\,a+\alpha\,(\lambda a+(1-\lambda)\,b))\right|^q d\alpha\right)^{\frac{1}{q}} \\
&+\tfrac{(x-(\lambda a+(1-\lambda)b))^{n+1}}{n!(b-a)^n}\left(\int_0^1 (1-\alpha)^{np}\,d\alpha\right)^{\frac{1}{p}} \\
&\times\left(\int_0^1 \left|f^{(n)}((1-\alpha)\,(\lambda a+(1-\lambda)\,b)+\alpha x)\right|^q d\alpha\right)^{\frac{1}{q}} \\
&+\tfrac{((1-\lambda)a+\lambda b-x)^{n+1}}{n!(b-a)^n}\left(\int_0^1 \alpha^{np}d\alpha\right)^{\frac{1}{p}} \\
&\times\left(\int_0^1 \left|f^{(n)}((1-\alpha)\,x+\alpha\,((1-\lambda)\,a+\lambda b))\right|^q d\alpha\right)^{\frac{1}{q}} \\
&+\tfrac{(1-\lambda)^{n+1}(b-a)}{n!}\left(\int_0^1 (1-\alpha)^{np}\,d\alpha\right)^{\frac{1}{p}} \\
&\times\left(\int_0^1 \left|f^{(n)}((1-\alpha)\,((1-\lambda)\,a+\lambda b)+\alpha b)\right|^q d\alpha\right)^{\frac{1}{q}} \\
&=\tfrac{(b-a)(1-\lambda)^{n+1}}{n!(np+1)^{\frac{1}{p}}}\left(\int_0^1 \left|f^{(n)}((1-\alpha)\,a+\alpha\,(\lambda a+(1-\lambda)\,b))\right|^q d\alpha\right)^{\frac{1}{q}} \\
&+\tfrac{(x-(\lambda a+(1-\lambda)b))^{n+1}}{n!(b-a)^n(np+1)^{\frac{1}{p}}}\left(\int_0^1 \left|f^{(n)}((1-\alpha)\,(\lambda a+(1-\lambda)\,b)+\alpha x)\right|^q d\alpha\right)^{\frac{1}{q}} \\
&+\tfrac{((1-\lambda)a+\lambda b-x)^{n+1}}{n!(b-a)^n(np+1)^{\frac{1}{p}}}\left(\int_0^1 \left|f^{(n)}((1-\alpha)\,x+\alpha\,((1-\lambda)\,a+\lambda b))\right|^q d\alpha\right)^{\frac{1}{q}} \\
&+\tfrac{(1-\lambda)^{n+1}(b-a)}{n!(np+1)^{\frac{1}{p}}}\left(\int_0^1 \left|f^{(n)}((1-\alpha)\,((1-\lambda)\,a+\lambda b)+\alpha b)\right|^q d\alpha\right)^{\frac{1}{q}}.
\end{aligned}$$

Since $\left|f^{(n)}\right|^q$ is convex, we deduce that

$$\begin{aligned}
|\mathcal{C}(f,x,n,\lambda)| &\leq \tfrac{(b-a)(1-\lambda)^{n+1}}{n!(np+1)^{\frac{1}{p}}} \\
&\times\left(\left|f^{(n)}\,(a)\right|^q\int_0^1 (1-\alpha)\,d\alpha+\left|f^{(n)}\,(\lambda a+(1-\lambda)\,b)\right|^q\int_0^1 \alpha d\alpha\right)^{\frac{1}{q}} \\
&+\tfrac{(x-(\lambda a+(1-\lambda)b))^{n+1}}{n!(b-a)^n(np+1)^{\frac{1}{p}}}
\end{aligned}$$

$$\times\left(\left|f^{(n)}\left(\lambda a+(1-\lambda)b\right)\right|^{q}\int_{0}^{1}(1-\alpha)\,d\alpha+\left|f^{(n)}(x)\right|^{q}\int_{0}^{1}\alpha d\alpha\right)^{\frac{1}{q}}$$
$$+\frac{((1-\lambda)a+\lambda b-x)^{n+1}}{n!(b-a)^{n}(np+1)^{\frac{1}{p}}}$$
$$\times\left(\left|f^{(n)}(x)\right|^{q}\int_{0}^{1}(1-\alpha)\,d\alpha+\left|f^{(n)}\left((1-\lambda)a+\lambda b\right)\right|^{q}\int_{0}^{1}\alpha d\alpha\right)^{\frac{1}{q}}$$
$$+\frac{(1-\lambda)^{n+1}(b-a)}{n!(np+1)^{\frac{1}{p}}}$$
$$\times\left(\left|f^{(n)}\left((1-\lambda)a+\lambda b\right)\right|^{q}\int_{0}^{1}(1-\alpha)\,d\alpha+\left|f^{(n)}(b)\right|^{q}\int_{0}^{1}\alpha d\alpha\right)^{\frac{1}{q}},$$

which gives the desired result after simple calculations. □

Corollary 4. *If we put $\lambda=1$ in Theorem 2, then we obtain the following generalized trapezoid inequality for n-times convex functions:*

$$\left|\sum_{p=0}^{n-1}\frac{\left(\frac{b-x}{b-a}\right)^{n-p}f^{(n-1-p)}(b)-(-1)^{n-p}\left(\frac{x-a}{b-a}\right)^{n-p}f^{(n-1-p)}(a)}{(n-p)!(b-a)^{p}}+\frac{(-1)^{n}}{(b-a)^{n}}\int_{a}^{b}f(t)dt\right|$$
$$\leq\frac{(x-a)^{n+1}}{n!(b-a)^{n}(np+1)^{\frac{1}{p}}2^{\frac{1}{q}}}\left(\left|f^{(n)}(a)\right|^{q}+\left|f^{(n)}(x)\right|^{q}\right)^{\frac{1}{q}}$$
$$+\frac{(b-x)^{n+1}}{n!(b-a)^{n}(np+1)^{\frac{1}{p}}2^{\frac{1}{q}}}\left(\left|f^{(n)}(x)\right|^{q}+\left|f^{(n)}(b)\right|^{q}\right)^{\frac{1}{q}}.$$

Moreover, if we choose $x=\frac{a+b}{2}$, then we obtain that

$$\left|\sum_{p=0}^{n-1}\frac{f^{(n-1-p)}(b)-(-1)^{n-p}f^{(n-1-p)}(a)}{(n-p)!(b-a)^{p}2^{n-p}}+\frac{(-1)^{n}}{(b-a)^{n}}\int_{a}^{b}f(t)dt\right|$$
$$\leq\frac{b-a}{n!(np+1)^{\frac{1}{p}}2^{\frac{1}{q}+n+1}}\left(\left(\left|f^{(n)}(a)\right|^{q}+\left|f^{(n)}\left(\frac{a+b}{2}\right)\right|^{q}\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\left|f^{(n)}\left(\frac{a+b}{2}\right)\right|^{q}+\left|f^{(n)}(b)\right|^{q}\right)^{\frac{1}{q}}\right).$$

Remark 4. Corollary 4 reduces to Theorem 5 from [5] if we take $n=1$. Moreover, if we put $x=\frac{a+b}{2}$, then we obtain Corollary 3 from [5].

Corollary 5. *If we put $\lambda=\frac{1}{2}$ in Theorem 2, then we obtain the following midpoint inequality for n-times convex functions:*

$$\left|\sum_{p=0}^{n-1}\frac{\left(1-(-1)^{n-p}\right)}{(n-p)!(b-a)^{p}2^{n-p}}f^{(n-1-p)}\left(\frac{a+b}{2}\right)+\frac{(-1)^{n}}{(b-a)^{n}}\int_{a}^{b}f(t)dt\right|\leq\frac{(b-a)}{n!(np+1)^{\frac{1}{p}}2^{n+1+\frac{1}{q}}}$$

$$\times\left(\left(\left|f^{(n)}(a)\right|^q+\left|f^{(n)}\left(\tfrac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}}+\left(\left|f^{(n)}\left(\tfrac{a+b}{2}\right)\right|^q+\left|f^{(n)}(b)\right|^q\right)^{\frac{1}{q}}\right).$$

Remark 5. Corollary 5 reduces to Corollary 3 from [8] if we take $n=1$. Also we recapture inequality (1.3) for $n=2$.

Corollary 6. *If we choose $x=\frac{a+b}{2}$ and $\lambda=\frac{2}{3}$ in Theorem 2, then we obtain the following two-point open Newton–Cotes inequality for n-times convex functions:*

$$\left|\sum_{p=0}^{n-1}\frac{\left(\left(\frac{1}{2}\right)^{n-p}-(-1)^{n-p}\right)\left(f^{(n-1-p)}\left(\frac{a+2b}{3}\right)-(-1)^{n-p}f^{(n-1-p)}\left(\frac{2a+b}{3}\right)\right)}{(n-p)!(b-a)^p3^{n-p}}+\frac{(-1)^n}{(b-a)^n}\int_a^b f(t)dt\right|$$

$$\leq\frac{b-a}{2^{\frac{1}{q}}\times 3^{n+1}\times n!(np+1)^{\frac{1}{p}}2^{\frac{1}{q}}}\left(\left|f^{(n)}(a)\right|^q+\left|f^{(n)}\left(\tfrac{2a+b}{3}\right)\right|^q\right)^{\frac{1}{q}}$$

$$+\frac{b-a}{2^{\frac{1}{q}}\times 6^{n+1}\times n!(np+1)^{\frac{1}{p}}}\left(\left|f^{(n)}\left(\tfrac{2a+b}{3}\right)\right|^q+\left|f^{(n)}\left(\tfrac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}}$$

$$+\frac{b-a}{2^{\frac{1}{q}}\times 6^{n+1}\times n!(np+1)^{\frac{1}{p}}}\left(\left|f^{(n)}\left(\tfrac{a+b}{2}\right)\right|^q+\left|f^{(n)}\left(\tfrac{a+2b}{3}\right)\right|^q\right)^{\frac{1}{q}}$$

$$+\frac{b-a}{2^{\frac{1}{q}}\times 3^{n+1}\times n!(np+1)^{\frac{1}{p}}}\left(\left|f^{(n)}\left(\tfrac{a+2b}{3}\right)\right|^q+\left|f^{(n)}(b)\right|^q\right)^{\frac{1}{q}}.$$

Remark 6. In Corollary 6 if we take $n=1$, we recapture the inequality (1.3) for $n=3$.

Theorem 3. *By the assumptions of Theorem 2, the inequality*

$$|\mathcal{C}(f,x,n,\lambda)|\leq\frac{(b-a)(1-\lambda)^{n+1}}{n!(n+1)^{1-\frac{1}{q}}}\left(\frac{1}{(n+1)(n+2)}\left|f^{(n)}(a)\right|^q\right.$$

$$\left.+\frac{1}{n+2}\left|f^{(n)}(\lambda a+(1-\lambda)b)\right|^q\right)^{\frac{1}{q}}+\frac{(x-(\lambda a+(1-\lambda)b))^{n+1}}{n!(b-a)^n(n+1)^{1-\frac{1}{q}}}$$

$$\times\left(\frac{1}{n+2}\left|f^{(n)}(\lambda a+(1-\lambda)b)\right|^q+\frac{1}{(n+1)(n+2)}\left|f^{(n)}(x)\right|^q\right)^{\frac{1}{q}}$$

$$+\frac{((1-\lambda)a+\lambda b-x)^{n+1}}{n!(b-a)^n(n+1)^{1-\frac{1}{q}}}\left(\frac{1}{(n+1)(n+2)}\left|f^{(n)}(x)\right|^q+\frac{1}{n+2}\left|f^{(n)}((1-\lambda)a+\lambda b)\right|^q\right)^{\frac{1}{q}}$$

$$+\frac{(1-\lambda)^{n+1}(b-a)}{n!(n+1)^{1-\frac{1}{q}}}\left(\frac{1}{n+2}\left|f^{(n)}((1-\lambda)a+\lambda b)\right|^q+\frac{1}{(n+1)(n+2)}\left|f^{(n)}(b)\right|^q\right)^{\frac{1}{q}}$$

holds for all $x\in[\lambda a+(1-\lambda)b,(1-\lambda)a+\lambda b]$ with $\lambda\in[\frac{1}{2},1]$.

Proof. Using Lemma 1 and power mean inequality, we get

$$|\mathcal{C}(f,x,n,\lambda)|\leq\frac{(b-a)(1-\lambda)^{n+1}}{n!}\left(\int_0^1\alpha^n d\alpha\right)^{1-\frac{1}{q}}$$

$$\times\left(\int_0^1 \alpha^n \left|f^{(n)}((1-\alpha)a+\alpha(\lambda a+(1-\lambda)b))\right|^q d\alpha\right)^{\frac{1}{q}}$$
$$+\tfrac{(x-(\lambda a+(1-\lambda)b))^{n+1}}{n!(b-a)^n}\left(\int_0^1 (1-\alpha)^n d\alpha\right)^{1-\frac{1}{q}}$$
$$\times\left(\int_0^1 (1-\alpha)^n \left|f^{(n)}((1-\alpha)(\lambda a+(1-\lambda)b)+\alpha x)\right|^q d\alpha\right)^{\frac{1}{q}}$$
$$+\tfrac{((1-\lambda)a+\lambda b-x)^{n+1}}{n!(b-a)^n}\left(\int_0^1 \alpha^n d\alpha\right)^{1-\frac{1}{q}}$$
$$\times\left(\int_0^1 \alpha^n \left|f^{(n)}((1-\alpha)x+\alpha((1-\lambda)a+\lambda b))\right|^q d\alpha\right)^{\frac{1}{q}}$$
$$+\tfrac{(1-\lambda)^{n+1}(b-a)}{n!}\left(\int_0^1 (1-\alpha)^n d\alpha\right)^{1-\frac{1}{q}}$$
$$\times\left(\int_0^1 (1-\alpha)^n \left|f^{(n)}((1-\alpha)((1-\lambda)a+\lambda b)+\alpha b)\right|^q d\alpha\right)^{\frac{1}{q}}$$
$$=\tfrac{(b-a)(1-\lambda)^{n+1}}{n!(n+1)^{1-\frac{1}{q}}}\left(\int_0^1 \alpha^n \left|f^{(n)}((1-\alpha)a+\alpha(\lambda a+(1-\lambda)b))\right|^q d\alpha\right)^{\frac{1}{q}}$$
$$+\tfrac{(x-(\lambda a+(1-\lambda)b))^{n+1}}{n!(b-a)^n(n+1)^{1-\frac{1}{q}}}$$
$$\times\left(\int_0^1 (1-\alpha)^n \left|f^{(n)}((1-\alpha)(\lambda a+(1-\lambda)b)+\alpha x)\right|^q d\alpha\right)^{\frac{1}{q}}$$
$$+\tfrac{((1-\lambda)a+\lambda b-x)^{n+1}}{n!(b-a)^n(n+1)^{1-\frac{1}{q}}}\left(\int_0^1 \alpha^n\left|f^{(n)}((1-\alpha)x+\alpha((1-\lambda)a+\lambda b))\right|^q d\alpha\right)^{\frac{1}{q}}$$
$$+\tfrac{(1-\lambda)^{n+1}(b-a)}{n!(n+1)^{1-\frac{1}{q}}}$$
$$\times\left(\int_0^1 (1-\alpha)^n \left|f^{(n)}((1-\alpha)((1-\lambda)a+\lambda b)+\alpha b)\right|^q d\alpha\right)^{\frac{1}{q}}.$$

Taking into account the convexity of $\left|f^{(n)}\right|^q$, we get

$$|\mathcal{C}(f,x,n,\lambda)|\leq\tfrac{(b-a)(1-\lambda)^{n+1}}{n!(n+1)^{1-\frac{1}{q}}}$$
$$\times\left(\left|f^{(n)}(a)\right|^q\int_0^1(1-\alpha)\alpha^n d\alpha+\left|f^{(n)}(\lambda a+(1-\lambda)b)\right|^q\int_0^1\alpha^{n+1}d\alpha\right)^{\frac{1}{q}}$$
$$+\tfrac{(x-(\lambda a+(1-\lambda)b))^{n+1}}{n!(b-a)^n(n+1)^{1-\frac{1}{q}}}\left(\left|f^{(n)}(\lambda a+(1-\lambda)b)\right|^q\int_0^1(1-\alpha)^{n+1}d\alpha\right.$$

$$+\left|f^{(n)}(x)\right|^q\int_0^1(1-\alpha)^n\alpha d\alpha\Big)^{\frac{1}{q}}+\frac{((1-\lambda)a+\lambda b-x)^{n+1}}{n!(b-a)^n(n+1)^{1-\frac{1}{q}}}$$

$$\times\left(\left|f^{(n)}(x)\right|^q\int_0^1(1-\alpha)\alpha^n d\alpha+\left|f^{(n)}((1-\lambda)a+\lambda b)\right|^q\int_0^1\alpha^{n+1}d\alpha\right)^{\frac{1}{q}}$$

$$+\frac{(1-\lambda)^{n+1}(b-a)}{n!(n+1)^{1-\frac{1}{q}}}\Big(\left|f^{(n)}((1-\lambda)a+\lambda b)\right|^q\int_0^1(1-\alpha)^{n+1}d\alpha$$

$$+\left|f^{(n)}(b)\right|^q\int_0^1\alpha(1-\alpha)^n d\alpha\Big)^{\frac{1}{q}},$$

which gives the desired result after simple transformations. □

Corollary 7. *If we put $\lambda=1$ in Theorem 3, then we obtain the following generalized trapezoid inequality for n-times convex functions:*

$$\left|\sum_{p=0}^{n-1}\frac{\left(\frac{b-x}{b-a}\right)^{n-p}f^{(n-1-p)}(b)-(-1)^{n-p}\left(\frac{x-a}{b-a}\right)^{n-p}f^{(n-1-p)}(a)}{(n-p)!(b-a)^p}+\frac{(-1)^n}{(b-a)^n}\int_a^b f(t)dt\right|$$

$$\leq\frac{(x-a)^{n+1}}{n!(b-a)^n(n+1)^{1-\frac{1}{q}}}\left(\frac{1}{n+2}\left|f^{(n)}(a)\right|^q+\frac{1}{(n+1)(n+2)}\left|f^{(n)}(x)\right|^q\right)^{\frac{1}{q}}$$

$$+\frac{(b-x)^{n+1}}{n!(b-a)^n(n+1)^{1-\frac{1}{q}}}\left(\frac{1}{(n+1)(n+2)}\left|f^{(n)}(x)\right|^q+\frac{1}{n+2}\left|f^{(n)}(b)\right|^q\right)^{\frac{1}{q}}.$$

Moreover, if we choose $x=\frac{a+b}{2}$, then we obtain

$$\left|\sum_{p=0}^{n-1}\frac{f^{(n-1-p)}(b)-(-1)^{n-p}f^{(n-1-p)}(a)}{(n-p)!(b-a)^p2^{n-p}}+\frac{(-1)^n}{(b-a)^n}\int_a^b f(t)dt\right|$$

$$\leq\frac{b-a}{n!(n+1)^{1-\frac{1}{q}}2^{n+1}}\left(\left(\frac{1}{n+2}\left|f^{(n)}(a)\right|^q+\frac{1}{(n+1)(n+2)}\left|f^{(n)}(\tfrac{a+b}{2})\right|^q\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\frac{1}{(n+1)(n+2)}\left|f^{(n)}(\tfrac{a+b}{2})\right|^q+\frac{1}{n+2}\left|f^{(n)}(b)\right|^q\right)^{\frac{1}{q}}\right).$$

Remark 7. Corollary 7 reduces to Theorem 7 from [5] if we take $n=1$. Moreover, if we put $x=\frac{a+b}{2}$, then we obtain Corollary 4 from [5].

Corollary 8. *If we put $\lambda=\frac{1}{2}$ in Theorem 3, then we obtain the following midpoint inequality for n-times convex functions:*

$$\left|\sum_{p=0}^{n-1}\frac{\left(1-(-1)^{n-p}\right)}{(n-p)!(b-a)^p2^{n-p}}f^{(n-1-p)}(\tfrac{a+b}{2})+\frac{(-1)^n}{(b-a)^n}n\int_a^b f(t)dt\right|$$

$$\leq\frac{b-a}{n!(n+1)^{1-\frac{1}{q}}\times2^{n+1}}\left(\left(\frac{1}{(n+1)(n+2)}\left|f^{(n)}(a)\right|^q+\frac{1}{n+2}\left|f^{(n)}(\tfrac{a+b}{2})\right|^q\right)^{\frac{1}{q}}\right.$$

$$+\left(\tfrac{1}{n+2}\left|f^{(n)}\left(\tfrac{a+b}{2}\right)\right|^{q}+\tfrac{1}{(n+1)(n+2)}\left|f^{(n)}(b)\right|^{q}\right)^{\frac{1}{q}}\right).$$

Remark 8. Corollary 8 reduces to Corollary 5 from [8] if we take $n=1$. We also recapture the inequality (1.4) for $n=2$.

Corollary 9. *If we choose $x=\frac{a+b}{2}$ and $\lambda=\frac{2}{3}$ in Theorem 3, then we obtain the following two-point open Newton–Cotes inequality for n-times convex functions:*

$$\left|\sum_{p=0}^{n-1}\frac{\left(\left(\frac{1}{2}\right)^{n-p}-(-1)^{n-p}\right)\left(f^{(n-1-p)}\left(\frac{a+2b}{3}\right)-(-1)^{n-p}f^{(n-1-p)}\left(\frac{2a+b}{3}\right)\right)}{(n-p)!(b-a)^{p}3^{n-p}}+\frac{(-1)^{n}}{(b-a)^{n}}\int_{a}^{b}f(t)dt\right|$$
$$\leq\frac{b-a}{3^{n+1}\times n!(n+1)^{1-\frac{1}{q}}}\left(\frac{1}{(n+1)(n+2)}\left|f^{(n)}(a)\right|^{q}+\frac{1}{n+2}\left|f^{(n)}\left(\frac{2a+b}{3}\right)\right|^{q}\right)^{\frac{1}{q}}$$
$$+\frac{b-a}{6^{n+1}\times n!(n+1)^{1-\frac{1}{q}}}\left(\frac{1}{n+2}\left|f^{(n)}\left(\frac{2a+b}{3}\right)\right|^{q}+\frac{1}{(n+1)(n+2)}\left|f^{(n)}\left(\frac{a+b}{2}\right)\right|^{q}\right)^{\frac{1}{q}}$$
$$+\frac{b-a}{6^{n+1}\times n!(n+1)^{1-\frac{1}{q}}}\left(\frac{1}{(n+1)(n+2)}\left|f^{(n)}\left(\frac{a+b}{2}\right)\right|^{q}+\frac{1}{n+2}\left|f^{(n)}\left(\frac{a+2b}{3}\right)\right|^{q}\right)^{\frac{1}{q}}$$
$$+\frac{b-a}{3^{n+1}\times n!(n+1)^{1-\frac{1}{q}}}\left(\frac{1}{n+2}\left|f^{(n)}\left(\frac{a+2b}{3}\right)\right|^{q}+\frac{1}{(n+1)(n+2)}\left|f^{(n)}(b)\right|^{q}\right)^{\frac{1}{q}}.$$

Remark 9. Corollary 9 with $n=1$ gives the inequality (1.4) for $n=3$.

3. Applications to special means

We shall consider the arithmetic mean $A(a,b)=\frac{a+b}{2}$ and the p-logarithmic mean

$$L_p(a,b)=\begin{cases}\left(\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right)^{\frac{1}{p}} & \text{if } a\neq b,\\ a & \text{if } a=b,\end{cases}$$

$p\in\mathbb{R}\setminus\{-1,0\}$, $a,b>0$.

Proposition 1. *If $a,b\in\mathbb{R}$ with $0<a<b$, then we have*

$$\left|\sum_{p=0}^{n-1}\frac{b^{3+p}-(-1)^{n-p}a^{3+p}}{2^{n-p-1}(n-p)!(3+p)!(b-a)^{p}}+\frac{2(-1)^{n}}{(n+2)!(b-a)^{n-1}}L_{n+2}^{n+2}(a,b)\right|$$
$$\leq\frac{b-a}{2^{n+1}(n+2)!}\left(2(n+1)A\left(a^{2},b^{2}\right)+2A^{2}(a,b)\right).$$

Proof. The proof is immediate from Theorem 1 with $\lambda=1$ and $x=\frac{a+b}{2}$, when it is applied to the function $f(x)=\frac{2x^{n+2}}{(n+2)!}$, $n\in\mathbb{N}$. Clearly we have $f^{(k)}(x)=\frac{2}{(n+2-k)!}x^{n+2-k}$, and $f^{(n)}(x)=x^{2}$ which is a convex function. □

Proposition 2. *Let $a, b \in \mathbb{R}$, with $0 < a < b$. Then,*

$$\Bigg| \frac{1-(-1)^n}{n!2^n} A^{n+\frac{1}{3}}(a,b) + \sum_{p=1}^{n-1} \frac{\left(1-(-1)^{n-p}\right)}{(n-p)!(b-a)^p 2^{n-p}} \prod_{i=1}^{n-1-p} \left(n + \tfrac{4-3i}{3}\right) A^{\frac{4+3p}{3}}(a,b)$$

$$+ \frac{(-1)^n}{(b-a)^{n-1}} L_{n+\frac{1}{3}}^{n+\frac{1}{3}}(a,b) \Bigg|$$

$$\leq \frac{(b-a) \prod_{i=1}^{n-1}(3n+4-3i)}{3^n \times n!(np+1)^{\frac{1}{p}} 2^{n-1+\frac{1}{q}}} \left(\left(a^2 + A^2(a,b)\right)^{\frac{1}{q}} + \left(A^2(a,b) + b^2\right)^{\frac{1}{q}} \right).$$

Proof. The result follows immediately from Theorem 2 with $\lambda = \frac{1}{2}$ and $q = 6$, applying to the function $f(x) = x^{n+\frac{1}{3}}$, $n \in \mathbb{N}$. Clearly, $f^{(n-1-p)}(x) = \prod_{i=1}^{n-1-p} \left(n + \frac{4-3i}{3}\right) x^{\frac{4+3p}{3}}$ with $f^{(0)}(x) = f(x)$, and $f^{(n)}(x) = \frac{4}{3} \prod_{i=1}^{n-1} \left(n + \frac{4-3i}{3}\right) x^{\frac{1}{3}}$. □

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