ACTA ET COMMENTATIONES UNIVERSITATIS TARTUENSIS DE MATHEMATICA
Volume 23, Number 2, December 2019
Available online at http://acutm.math.ut.ee

On singular systems of nonlinear equations involving $3n$-Caputo derivatives

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Abstract. We study singular fractional systems of nonlinear differential equations involving $3n$-Caputo derivatives. We investigate existence and uniqueness results using the contraction mapping principle. We also discuss the existence of at least one solution by means of Schauder fixed point theorem. Moreover, we define and discuss the Ulam–Hyers stability and the generalized Ulam–Hyers stability of solutions for such systems. To illustrate the main results, we present some examples.

1. Introduction and preliminaries

Fractional differential equations play a central role in engineering sciences and applied mathematics to create mathematical modeling of many physical phenomena. For more details, see [8]. Furthermore, many authors have established existence and uniqueness results for fractional differential equations and for singular fractional differential equation (see, for instance, [1, 2, 3, 4, 9, 11, 12, 13]).

On the other hand, Ulam–Hyers stability for fractional differential problems are quite significant in realistic problems, numerical analysis, biology and economics. Some results concerning this stability have been obtained in [5, 6, 7, 10, 12, 13, 14].

Inspired by the above cited works, this paper is devoted to build the existence and uniqueness of solution in addition to the existence of at least one solution, Ulam–Hyers stability and the generalized Ulam–Hyers stability of

Received January 23, 2018.
2010 *Mathematics Subject Classification.* 30C45; 39B72; 39B82.
Key words and phrases. Caputo derivative; fixed point; singular fractional differential equation; existence and uniqueness; Ulam–Hyers stability; generalized Ulam–Hyers stability.
https://doi.org/10.12697/ACUTM.2019.23.16

solutions for the following system of singular fractional nonlinear equations:

$$\begin{cases} D^{\alpha_n^i} u_i(t) = f_i(\nabla_u(t)), \quad i = 1,2,3, \quad 0 < t \le 1, \\ k-1 < \alpha_k^i < k, \quad k = 1,2,\dots,n, \quad i = 1,2,3, \\ \sum_{i=1}^{3}\sum_{j=0}^{n-2} \left| u_i^{(j)}(0) \right| = 0, \quad D^{\mu_i} u_i(1) = J^{\delta_i} u_i(1), \\ n-2 < \mu_i < n-1, \quad \delta_i > 0, \quad i = 1,2,3, \end{cases} \tag{1}$$

where

$$\nabla_u(t) := \begin{pmatrix} t, u_1(t), u_2(t), u_3(t), D^{\alpha_1^1} u_1(t), \dots, D^{\alpha_{n-1}^1} u_1(t), \\ D^{\alpha_1^2} u_2(t), \dots, D^{\alpha_{n-1}^2} u_2(t), D^{\alpha_1^3} u_3(t), \dots, D^{\alpha_{n-1}^3} u_3(t) \end{pmatrix},$$

$n \in \mathbb{N}^* := \mathbb{N} \setminus \{1\} = \{2,3,\dots\}$, and the functions $f_i : (0,1] \times \mathbb{R}^{3n} \to \mathbb{R}$ are continuous, singular at $t = 0$, and $\lim\limits_{t \to 0^+} f_i(t) = \infty$. The operators $D^{\alpha_k^i}$, $k = 1,2,\dots,n$, and D^{μ_i}, $i = 1,2,3$, are the derivatives in the sense of Caputo, defined by

$$D^{\kappa} x(t) = \tfrac{1}{\Gamma(m-\kappa)} \int_0^t (t-s)^{m-\kappa-1} x^{(m)}(s)\, ds = J^{m-\kappa} x^{(m)}(t)$$

with $m-1 < \kappa < m$, $m \in \mathbb{N}$. The Riemann–Liouville fractional integral J^{ϑ} of order $\vartheta \ge 0$ for a continuous function ψ on $[0,\infty)$ is defined by

$$J^{\vartheta}\psi(t) = \begin{cases} \frac{1}{\Gamma(\vartheta)} \int_0^t (t-s)^{\vartheta-1} \psi(s)\, ds, \quad \vartheta > 0, \\ \psi(t), \quad \vartheta = 0, \end{cases}$$

where $t \ge 0$, and $\Gamma(\vartheta) := \int_0^{\infty} e^{-x} x^{\vartheta-1} dx$.

We give some properties of the fractional calculus theory which can be found in [8].

(i) For $\alpha, \beta > 0$, $n-1 < \alpha < n$, we have $D^{\alpha} t^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)} t^{\beta-\alpha-1}$, $\beta > n$, and $D^{\alpha} t^j = 0$, $j = 0,1,\dots,n-1$.

(ii) $D^p J^q f(t) = J^{q-p} f(t)$, where $q > p > 0$ and $f \in L^1([a,b])$.

(iii) Let $n \in \mathbb{N}$, $n-1 < \alpha < n$, and $D^{\alpha} u(t) = 0$. Then, $u(t) = \sum\limits_{j=0}^{n-1} c_j t^j$, and $J^{\alpha} D^{\alpha} u(t) = u(t) + \sum\limits_{j=0}^{n-1} c_j t^j$, $(c_j)_{j=0,1,\dots,n-1} \in \mathbb{R}$.

In the following we use, for fixed numbers γ, q, r, w, the notations

$$Q_n^+(t,s,\gamma) := \tfrac{(t-s)^{\alpha_n^i+\gamma-1}}{\Gamma(\alpha_n^i+\gamma)}, \quad Q_n^-(t,s,\gamma) := \tfrac{(t-s)^{\alpha_n^i-\gamma-1}}{\Gamma(\alpha_n^i-\gamma)},$$

$$\Delta_{n,w}(t,q,r) := \tfrac{\Gamma(n+q)\Gamma(n-r)t^{n-1-w}}{\Gamma(n-w)(\Gamma(n+q)-\Gamma(n-r))}, \quad \Delta_{n,w}^+(t,q,r) := \tfrac{\Gamma(n+q)\Gamma(n-r)t^{n-1-w}}{\Gamma(n-w)|\Gamma(n+q)-\Gamma(n-r)|}.$$

It is clear that in the case $w = 0$ we have

$$\Delta_{n,0}(t,q,r) = \tfrac{\Gamma(n+q)\Gamma(n-r)t^{n-1}}{(n-1)!(\Gamma(n+q)-\Gamma(n-r))}, \quad \Delta^{+}_{n,0}(t,q,r) = \tfrac{\Gamma(n+q)\Gamma(n-r)t^{n-1}}{(n-1)!|\Gamma(n+q)-\Gamma(n-r)|}.$$

We need the following fundamental lemma to prove our existence results.

Lemma 1.1 (Shauder fixed point theorem)**.** *Let (E,d) be a complete metric space, let X be a closed convex subset of E, and let $A : E \to E$ be a mapping such that the set $\{Ax : x \in X\}$ is relatively compact in E. Then A has at least one fixed point.*

Let us now import the integral solution of system (1).

Lemma 1.2. *Let there be given the numbers $n \in \mathbb{N}^*$ and α_n^i $(i = 1,2,3)$ such that $n-1 < \alpha_n^i < n$, and the functions $U_i \in C([0,1],\mathbb{R})$ $(i = 1,2,3)$. Then the system*

$$\begin{cases} D^{\alpha_n^i} u_i(t) = U_i(t), \quad 0 < t \le 1, \quad i = 1,2,3, \\ \sum\limits_{i=1}^{3}\sum\limits_{j=0}^{n-2} \left| u_i^{(j)}(0) \right| = 0, \quad D^{\mu_i} u_i(1) = J^{\delta_i} u_i(1), \\ n-2 < \mu_i < n-1, \quad \delta_i > 0 \end{cases} \tag{2}$$

has a unique solution $(u_1,u_2,u_3)(t)$:

$$\begin{aligned} u_i(t) = {} & \int_0^t Q_n^{+}(t,s,0) U_i(s)\, ds + \Delta_{n,0}(t,\delta_i,\mu_i) \\ & \times \int_0^1 \left(Q_n^{+}(1,s,\delta_i) - Q_n^{-}(1,s,\mu_i) \right) U_i(s)\, ds. \end{aligned} \tag{3}$$

Proof. Thanks to the property (iii), the system (2) can be written as equivalent integral equations

$$u_i(t) = \int_0^t Q_n^{+}(t,s,0) U_i(s)\, ds - \sum_{j=0}^{n-1} c_j^i t^j, \quad i = 1,2,3 \tag{4}$$

with

$$\begin{pmatrix} c_0^1 & c_1^1 & \cdots & c_{n-1}^1 \\ c_0^2 & c_1^2 & \cdots & c_{n-1}^2 \\ c_0^3 & c_1^3 & \cdots & c_{n-1}^n \end{pmatrix} \in M_{3n}(\mathbb{R}).$$

Using the boundary conditions, we observe that

$$\begin{aligned} u_i^{(j)}(0) &= -j! c_j^i = 0, \quad i = 1,2,3, \quad j = 0,1,\ldots,n-2, \\ D^{\mu_i} u_i(1) &= \int_0^1 Q_n^{-}(1,s,\mu_i) U_i(s)\, ds - \tfrac{\Gamma(n)}{\Gamma(n-\mu_i)} c_{n-1}^i, \quad i = 1,2,3, \\ J^{\delta_i} u_i(1) &= \int_0^1 Q_n^{+}(1,s,\delta_i) U_i(s)\, ds - \tfrac{\Gamma(n)}{\Gamma(n+\delta_i)} c_{n-1}^i, \quad i = 1,2,3. \end{aligned}$$

Thus

$$c_j^i = 0, \quad i = 1,2,3, \quad j = 0,1,\dots,n-2,$$
$$c_{n-1}^i = \Delta_{n,0}\left(1,\delta_i,\mu_i\right)\int_0^1 \left(Q_n^+(1,s,\delta_i) - Q_n^-(1,s,\mu_i)\right)U_i\left(s\right)ds. \tag{5}$$

Substituting (5) in (4), we receive (3). □

Now, let us introduce, for $n \in \mathbb{N}^*$, the Banach space

$$B := \left\{\left(u_1,u_2,u_3\right) : u_i,\ D^{\alpha_k^i}u_i \in C\left(\left[0,1\right],\mathbb{R}\right), \quad i = 1,2,3, \quad k = 1,\dots,n-1\right\},$$

endowed with the norm

$$\left\|\left(u_1,u_2,u_3\right)\right\|_B = \max_{\substack{1\leq k\leq n-1\\ 1\leq i\leq 3}}\left\{\left\|u_i\right\|_\infty, \left\|D^{\alpha_k^i}u\right\|_\infty\right\},$$

where

$$\left\|u_i\right\|_\infty = \sup_{t\in[0,1]}\left|u_i(t)\right|, \quad \left\|D^{\alpha_k^i}u_i\right\|_\infty = \sup_{t\in[0,1]}\left|D^{\alpha_k^i}u_i(t)\right|.$$

2. Existence and uniqueness

In the present section we establish sufficient conditions for the existence and uniqueness of solutions to system (1). Then, we will give some examples to illustrate the applications of our theorems.

Define the nonlinear operator $T : B \to B$ by

$$T\left(u_1,u_2,u_3\right)(t) := \left(T_1\left(u_1,u_2,u_3\right)(t), T_2\left(u_1,u_2,u_3\right)(t), T_3\left(u_1,u_2,u_3\right)(t)\right),$$

where $t \in [0,1]$ and, for $i = 1,2,3$,

$$\begin{aligned} T_i\left(u_1,u_2,u_3\right)(t) :&= \int_0^t Q_n^+(t,s,0)f_i\left(\nabla_u\left(s\right)\right)ds + \Delta_{n,0}\left(t,\delta_i,\mu_i\right) \\ &\times \int_0^1 \left(Q_n^+(1,s,\delta_i) - Q_n^-(1,s,\mu_i)\right)f_i\left(\nabla_u\left(s\right)\right)ds. \end{aligned}$$

Lemma 2.1. *For $n \in \mathbb{N}^*$, let $n-1 < \alpha_n^i < n$ $(i = 1,2,3)$. Assume that $G_i : (0,1] \to \mathbb{R}$ $(i = 1,2,3)$ are continuous functions with $\lim_{t\to 0^+} G_i\left(t\right) = \infty$. Let $0 < \eta_i < 1$ $(i = 1,2,3)$ be such that $t^{\eta_i}G_i\left(t\right)$ are continuous for each $t \in [0,1]$. Then the functions*

$$\begin{aligned} u_i(t) :&= \int_0^t Q_n^+(t,s,0)G_i(s)ds + \Delta_{n,0}\left(t,\delta_i,\mu_i\right) \\ &\times \int_0^1 \left(Q_n^+(1,s,\delta_i) - Q_n^-(1,s,\mu_i)\right)G_i(s)ds. \end{aligned}$$

are continuous on $[0,1]$.

Proof. By the continuity of $t^{\eta_i}G_i(t)$, and since

$$u_i(t) = \int_0^t Q_n^+(t,s,0)s^{-\eta_i}s^{\eta_i}G_i(s)ds + \Delta_{n,0}(t,\delta_i,\mu_i) \times \int_0^1 \left(Q_n^+(1,s,\delta_i) - Q_n^-(1,s,\mu_i)s^{-\eta_i}s^{\eta_i}\right)G_i(s)ds,$$

it is clear that $u_i(0) = 0$ $(i = 1,2,3)$. Let us divide the proof into three cases.

Case 1: Since, for $t_0 = 0$ and $t \in (0,1]$, the functions $t^{\eta_i}G_i(t)$ $(i = 1,2,3)$ are continuous, there exist numbers $R_i > 0$ $(i = 1,2,3)$ such that $|t^{\eta_i}G_i(t)| \le R_i$ for all $t \in [0,1]$. Therefore,

$$\begin{aligned}
|u_i(t) - u_i(0)| &\le \tfrac{R_i}{\Gamma(\alpha_n^i)}\int_0^t (t-s)^{\alpha_n^i-1}s^{-\eta_i}ds + R_i\Delta_{n,0}^+(t,\delta_i,\mu_i) \\
&\quad \times \left|\int_0^1 \left(Q_n^+(1,s,\delta_i)s^{-\eta_i} - Q_n^-(1,s,\mu_i)s^{-\eta_i}\right)ds\right| \\
&\le \tfrac{R_i t^{\alpha_n^i-\eta_i}}{\Gamma(\alpha_n^i)}\int_0^1 (1-s)^{\alpha_n^i-1}s^{-\eta_i}ds + R_i\Delta_{n,0}^+(t,\delta_i,\mu_i) \\
&\quad \times \int_0^1 \left(Q_n^+(1,s,\delta_i)s^{-\eta_i} - Q_n^-(1,s,\mu_i)s^{-\eta_i}\right)ds \\
&\le \frac{R_i Be\left(\alpha_n^i,1-\eta_i\right)t^{\alpha_n^i-\eta_i}}{\Gamma(\alpha_n^i)} + R_i\Delta_{n,0}^+(t,\delta_i,\mu_i) \\
&\quad \times \left(\frac{Be\left(\alpha_n^i+\delta_i,1-\eta_i\right)}{\Gamma(\alpha_n^i+\delta_i)} + \frac{Be\left(\alpha_n^i-\mu_i,1-\eta_i\right)}{\Gamma(\alpha_n^i-\mu_i)}\right),
\end{aligned}$$

where Be denotes the Beta function. Then we have that

$$\begin{aligned}
|u_i(t) - u_i(0)| &\le \tfrac{R_i\Gamma(1-\eta_i)t^{\alpha_n^i-\eta_i}}{\Gamma(\alpha_n^i+1-\eta_i)} \\
&+ R_i\Delta_{n,0}^+(t,\delta_i,\mu_i)\left(\tfrac{\Gamma(1-\eta_i)}{\Gamma(\alpha_n^i+\delta_i+1-\eta_i)} + \tfrac{\Gamma(1-\eta_i)}{\Gamma(\alpha_n^i-\mu_i+1-\eta_i)}\right) \to 0, \quad \text{as } t \to 0.
\end{aligned}$$

Case 2: For $t_0 \in (0,1)$ and for all $t \in (t_0,1]$, one has

$$\begin{aligned}
&|u_i(t) - u_i(t_0)| \\
&\le \left|\int_0^t Q_n^+(t,s,0)s^{-\eta_i}s^{\eta_i}G_i(s)\,ds - \int_0^{t_0} Q_n^+(t_0,s,0)s^{-\eta_i}s^{\eta_i}G_i(s)\,ds\right| \\
&\quad + \Delta_{n,0}^+(1,\delta_i,\mu_i)\left(t^{n-1} - t_0^{n-1}\right) \\
&\quad \times \left|\int_0^1 \left(Q_n^+(1,s,\delta_i) - Q_n^-(1,s,\mu_i)\right)s^{-\eta_i}s^{\eta_i}G_i(s)ds\right| \\
&\le \frac{R_i\Gamma(1-\eta_i)\left(t^{\alpha_n^i-\eta_i} - t_0^{\alpha_n^i-\eta_i}\right)}{\Gamma(\alpha_n^i+1-\eta_i)} + R_i\Delta_{n,0}^+(1,\delta_i,\mu_i)\left(t^{n-1} - t_0^{n-1}\right)
\end{aligned}$$

$$\times\left(\frac{\Gamma(1-\eta_i)}{\Gamma(\alpha_n^i+\delta_i+1-\eta_i)}+\frac{\Gamma(1-\eta_i)}{\Gamma(\alpha_n^i-\mu_i+1-\eta_i)}\right)\to 0,\text{ as } t\to t_0.$$

Case 3: For $t_0\in(0,1]$ and for all $t\in[0,t_0)$, the proof is similar to that of Case 2. This completes the proof. □

Lemma 2.2. *For $n\in\mathbb{N}^*$, let $k-1<\alpha_k^i<k$ $(i=1,2,3,\ k=1,2,\dots,n)$. Assume that $f_i:(0,1]\times\mathbb{R}^{3n}\to\mathbb{R}$ $(i=1,2,3)$ are continuous functions with $\lim_{t\to 0^+} f_i(t,\dots)=\infty$, and there exist constants $0<\eta_i<1$ such that the functions $t^{\eta_i}f_i(t,\dots)$ are continuous on $[0,1]\times\mathbb{R}^{3n}$. Then, for all $k=1,2,\dots,n-1$ and $i=1,2,3$, the functions*

$$\begin{aligned}D^{\alpha_k^i}T_i(u_1,u_2,u_3)(t)&=\int_0^t Q_n^-\left(t,s,\alpha_k^i\right)f_i(\nabla_u(s))\,ds+\Delta_{n,\alpha_k^i}(t,\delta_i,\mu_i)\\&\quad\times\int_0^1\left(Q_n^+(1,s,\delta_i)-Q_n^-(1,s,\mu_i)\right)f_i(\nabla_u(s))\,ds\end{aligned}$$

are continuous on $[0,1]\times\mathbb{R}^{3n}$.

Proof. For $(u_1,u_2,u_3)\in B$ we have u_i, $D^{\alpha_k^i}u_i(t)\in C([0,1])$, $i=1,2,3$, $k=1,2,\dots,n-1$. Then, there exist $a_0^i>0$ and $a_k^i>0$, $i=1,2,3$, such that $|u_i(t)|\le a_0^i$ and $\left|D^{\alpha_k^i}u_i(t)\right|\le a_k^i$ for $k=1,2,\dots,n-1$ and for all $t\in[0,1]$.

On the other hand, the functions $t^{\eta_i}f_i(t,\dots)$ $(i=1,2,3)$ are continuous on $[0,1]\times\mathbb{R}^{3n}$. So, denoting

$$C_i=\left\|t^{\eta_i}f_i(\nabla_u(t))\right\|_\infty,\quad i=1,2,3,$$

for $-a_0^i\le u_i\le a_0^i$ and $-a_k^i\le D^{\alpha_k^i}u_i\le a_k^i$, we get

$$\begin{aligned}&\left|D^{\alpha_k^i}T_i(u_1,u_2,u_3)(t)\right|\\&\le\frac{C_i}{\Gamma\left(\alpha_n^i-\alpha_k^i\right)}\int_0^t(t-s)^{\alpha_n^i-\alpha_k^i-1}s^{-\eta_i}ds+C_i\Delta^+_{n,\alpha_k^i}(t,\delta_i,\mu_i)\\&\quad\times\left|\int_0^1\left(Q_n^+(1,s,\delta_i)s^{-\eta_i}-Q_n^-(1,s,\mu_i)s^{-\eta_i}\right)ds\right|\\&\le\frac{C_i\Gamma(1-\eta_i)t^{\alpha_n^i-\alpha_k^i-\eta_i}}{\Gamma\left(\alpha_n^i-\alpha_k^i+1-\eta_i\right)}+C_i\Delta^+_{n,\alpha_k^i}(t,\delta_i,\mu_i)\\&\quad\times\left(\frac{\Gamma(1-\eta_i)}{\Gamma(\alpha_n^i+\delta_i+1-\eta_i)}+\frac{\Gamma(1-\eta_i)}{\Gamma(\alpha_n^i-\mu_i+1-\eta_i)}\right),\end{aligned}\tag{6}$$

where $k=1,2,\dots,n-1$, $i=1,2,3$. From inequality (6), by the same method as in Lemma 2.1, we can show that the functions $D^{\alpha_k^i}T_i(u_1,u_2,u_3)$ are continuous on $[0,1]$. □

Lemma 2.3. *For $n\in\mathbb{N}^*$, let $n-1<\alpha_n^i<n$ $(i=1,2,3)$. Assume that $f_i:(0,1]\times\mathbb{R}^{3n}\to\mathbb{R}$ $(i=1,2,3)$ are continuous functions*

with $\lim_{t\to 0^+} f_i(t,\dots) = \infty$*, and there exist constants* $0 < \eta_i < 1$ *such that the functions* $t^{\eta_i} f_i(t,\dots)$ *are continuous on* $[0,1]\times\mathbb{R}^{3n}$*. Then the operator* $T : B \to B$ *is completely continuous.*

Proof. From Lemma 2.1 and Lemma 2.2 we have that $T : B \to B$. Now, we divide the proof into three steps.

Step 1: We shall show that $T\colon B \to B$ is continuous. For $\left(u_1^0, u_2^0, u_3^0\right) \in B$ with $\left\|\left(u_1^0, u_2^0, u_3^0\right)\right\|_B =: \phi_0$ and for $(u_1,u_2,u_3) \in B$ with $\|(u_1,u_2,u_3) - \left(u_1^0, u_2^0, u_3^0\right)\|_B < 1$, we have $\|(u_1,u_2,u_3)\|_B < 1+\phi_0 =: \phi$.

Since $t^{\eta_i} f_i(t,\dots)$ are continuous on $[0,1]\times\mathbb{R}^{3n}$, it yields that $t^{\eta_i} f_i(t,\dots)$ are uniformly continuous on $[0,1]\times[-\phi,\phi]^{3n}$.

Then, for all $t\in[0,1]$ and $\epsilon>0$, there exists $\varsigma > 0$ ($\varsigma < 1$) such that

$$\left|t^{\eta_i} f_i\left(\nabla_u(t)\right) - t^{\eta_i} f_i\left(\nabla_{u^0}(t)\right)\right| < \epsilon, \tag{7}$$

whenever $(u_1,u_2,u_3)\in B$ and $\left\|(u_1,u_2,u_3) - \left(u_1^0, u_2^0, u_3^0\right)\right\|_B < \varsigma$.

Thanks to (7), we obtain, for $i=1,2,3$,

$$\left\|T_i(u_1,u_2,u_3) - T_i\left(u_1^0,u_2^0,u_3^0\right)\right\|_\infty \le \epsilon F_0^i,$$

and

$$\left\|D^{\alpha_k^i}\left(T_i(u_1,u_2,u_3) - T_i\left(u_1^0,u_2^0,u_3^0\right)\right)\right\|_\infty \le \epsilon F_k^i,$$

where

$$F_0^i := \tfrac{\Gamma(1-\eta_i)}{\Gamma(\alpha_n^i+1-\eta_i)} + \Delta_{n,0}^+(1,\delta_i,\mu_i)\left(\tfrac{\Gamma(1-\eta_i)}{\Gamma(\alpha_n^i+\delta_i+1-\eta_i)} + \tfrac{\Gamma(1-\eta_i)}{\Gamma(\alpha_n^i-\mu_i+1-\eta_i)}\right),$$
$$F_k^i := \tfrac{\Gamma(1-\eta_i)}{\Gamma\left(\alpha_n^i-\alpha_k^i+1-\eta_i\right)} + \Delta_{n,\alpha_k^i}^+(1,\delta_i,\mu_i)\left(\tfrac{\Gamma(1-\eta_i)}{\Gamma(\alpha_n^i+\delta_i+1-\eta_i)} + \tfrac{\Gamma(1-\eta_i)}{\Gamma(\alpha_n^i-\mu_i+1-\eta_i)}\right).$$

Therefore,

$$\left\|T(u_1,u_2,u_3) - T\left(u_1^0,u_2^0,u_3^0\right)\right\|_B \le \epsilon \max_{\substack{1\le k\le n-1\\ 1\le i\le 3}}\left(F_0^i, F_k^i\right).$$

This implies that $\left\|T(u_1,u_2,u_3) - T\left(u_1^0,u_2^0,u_3^0\right)\right\|_B \to 0$ as $\|(u_1,u_2,u_3) - \left(u_1^0,u_2^0,u_3^0\right)\|_B \to 0$. Thus $T : B\to B$ is a continuous operator.

Step 2: Let $\Omega := \{(u_1,u_2,u_3)\in B : \|(u_1,u_2,u_3)\|_B \le \theta\}$, $\theta > 0$. Our aim is to show that $T(\Omega)$ is bounded. The continuity of $t^{\eta_i} f_i(t,\dots)$ on $[0,1]\times[-\theta,\theta]^{3n}$ yields that, for all $t\in[0,1]$ and for all $(u_1,u_2,u_3)\in\Omega$, there exist $P_i > 0$ $(i=1,2,3)$ such that

$$\left|t^{\eta_i} f_i\left(\nabla_u(t)\right)\right| \le P_i, \quad i=1,2,3. \tag{8}$$

By (8) we get

$$\|T_i(u_1,u_2,u_3)\|_\infty \le P_i F_0^i, \quad \left\|D^{\alpha_k^i} T_i(u_1,u_2,u_3)\right\|_\infty \le P_i F_k^i.$$

Hence,

$$\|T(u_1,u_2,u_3)\|_B \leq \max_{\substack{1\leq k\leq n-1\\ 1\leq i\leq 3}} P_i\left(F_0^i, F_k^i\right). \tag{9}$$

That is, $T(\Omega)$ is bounded.

Step 3: We show that $T(\Omega)$ is equicontinuous. For $(u_1,u_2,u_3)\in\Omega$ and $t_1,t_2\in[0,1]$, $t_1<t_2$, we have

$$\begin{aligned}&\|T_i(u_1,u_2,u_3)(t_2)-T_i(u_1,u_2,u_3)(t_1)\|_\infty\\ &\leq \tfrac{P_i\Gamma(1-\eta_i)\left(t_2^{\alpha_n^i-\eta_i}-t_1^{\alpha_n^i-\eta_i}\right)}{\Gamma(\alpha_n^i+1-\eta_i)}+P_i\Delta_{n,0}^+(1,\delta_i,\mu_i)\left(t_2^{n-1}-t_1^{n-1}\right)\\ &\quad\times\left(\tfrac{\Gamma(1-\eta_i)}{\Gamma(\alpha_n^i+\delta_i+1-\eta_i)}+\tfrac{\Gamma(1-\eta_i)}{\Gamma(\alpha_n^i-\mu_i+1-\eta_i)}\right)\end{aligned}\tag{10}$$

and

$$\begin{aligned}&\left\|D^{\alpha_k^i}T_i(u_1,u_2,u_3)(t_2)-D^{\alpha_k^i}T_i(u_1,u_2,u_3)(t_1)\right\|_\infty\\ &\leq \tfrac{P_i\Gamma(1-\eta_i)\left(t_2^{\alpha_n^i-\alpha_k^i-\eta_i}-t_1^{\alpha_n^i-\alpha_k^i-\eta_i}\right)}{\Gamma\left(\alpha_n^i-\alpha_k^i+1-\eta_i\right)}+P_i\Delta_{n,\alpha_k^i}^+(1,\delta_i,\mu_i)\left(t_2^{n-1-\alpha_k^i}-t_1^{n-1-\alpha_k^i}\right)\\ &\quad\times\left(\tfrac{\Gamma(1-\eta_i)}{\Gamma(\alpha_n^i+\delta_i+1-\eta_i)}+\tfrac{\Gamma(1-\eta_i)}{\Gamma(\alpha_n^i-\mu_i+1-\eta_i)}\right).\end{aligned}\tag{11}$$

The right-hand sides of (10) and (11) are independent of (u_1,u_2,u_3) and tend to zero as $t_1\to t_2$. Hence, $T(\Omega)$ is equicontinuous. By Arzela–Ascoli theorem we state that T is a completely continuous operator. □

Theorem 2.4. *Suppose that*

(H1) *there exist nonnegative constants* $\left(\lambda_j^i\right)_{j=1,\ldots,3n}^{i=1,2,3}$ *such that*

$$t^{\eta_i}\left|f_i(t,x_1,\ldots,x_{3n})-f_i(t,y_1,\ldots,y_{3n})\right|\leq\sum_{j=1}^{3n}\lambda_j^i|x_j-y_j|$$

for $i=1,2,3$, $t\in[0,1]$, *and each* $(x_1,\ldots,x_{3n}),(y_1,\ldots,y_{3n})\in\mathbb{R}^{3n}$;

(H2) $\Lambda:=\max_{\substack{1\leq k\leq n-1\\ 1\leq i\leq 3}}\left(\sum_{j=1}^{3n}\lambda_j^iF_0^i,\sum_{j=1}^{3n}\lambda_j^iF_k^i,\right)<1.$

Then the system (1) *has a unique solution on* $[0,1]$.

Proof. We shall show that T is contractive on B. If $(u_1,u_2,u_3),(v_1,v_2,v_3)\in B$ and $t\in[0,1]$, then

$$\begin{aligned}&\|T_i(u_1,u_2,u_3)-T_i(v_1,v_2,v_3)\|_\infty\\ &\leq\sup_{t\in[0,1]}\int_0^t Q_n^+(t,s,0)\,s^{-\eta_i}s^{\eta_i}\left|f_i(\nabla_u(s))-f_i(\nabla_v(s))\right|ds+\Delta_{n,0}^+(1,\delta_i,\mu_i)\end{aligned}$$

$$\times \int_0^1 \left(Q_n^+(1,s,\delta_1)+Q_n^-(1,s,\mu_1)\right) s^{-\eta_1}s^{\eta_1}\left|f_i(\nabla_u(s))-f_i(\nabla_v(s))\right| ds.$$

Thanks to the hypothesis (H1), we get

$$\begin{aligned}\left\|T_i(u_1,u_2,u_3)-T_i(v_1,v_2,v_3)\right\|_\infty &\leq \Bigg(\tfrac{\Gamma(1-\eta_i)}{\Gamma(\alpha_n^i+1-\eta_i)}\sup_{t\in[0,1]} t^{\alpha_n^i-\eta_i} \\ &+\Delta_{n,0}^+(1,\delta_i,\mu_i)\left(\tfrac{\Gamma(1-\eta_i)}{\Gamma(\alpha_n^i+\delta_i+1-\eta_i)}+\tfrac{\Gamma(1-\eta_i)}{\Gamma(\alpha_n^i-\mu_i+1-\eta_i)}\right)\Bigg) \\ &\times\big(\lambda_1^i\left\|u_1-v_1\right\|_\infty+\lambda_2^i\left\|u_2-v_2\right\|_\infty+\lambda_3^i\left\|u_3-v_3\right\|_\infty \\ &+\lambda_4^i\left\|D^{\alpha_1}(u_1-v_1)\right\|_\infty+\cdots+\lambda_{n+2}^i\left\|D^{\alpha_{n-1}^2}(u_2-v_2)\right\|_\infty \\ &+\cdots+\lambda_{3n}^i\left\|D^{\alpha_{n-1}^3}(u_3-v_3)\right\|_\infty\Big).\end{aligned}$$

So,

$$\begin{aligned}&\left\|T_i(u_1,u_2,u_3)-T_i(v_1,v_2,v_3)\right\|_\infty \\ &\qquad\leq \sum_{j=1}^{3n}\lambda_j^iF_0^i\left\|(u_1-v_1,u_2-v_2,u_3-v_3)\right\|_B.\end{aligned}\tag{12}$$

Similarly, using anew (H1), we obtain

$$\begin{aligned}&\left\|D^{\alpha_k^i}\left(T_i(u_1,u_2,u_3)-T_i(v_1,v_2,v_3)\right)\right\|_\infty \\ &\qquad\leq \sum_{j=1}^{3n}\lambda_j^iF_k^i\left\|(u_1-v_1,u_2-v_2,u_3-v_3)\right\|_B.\end{aligned}\tag{13}$$

It follows from (12) and (13) that

$$\begin{aligned}&\left\|T(u_1,u_2,u_3)-T(v_1,v_2,v_3)\right\|_B \\ &\leq \max_{\substack{1\leq k\leq n-1\\1\leq i\leq 3}}\left(\sum_{j=1}^{3n}\lambda_j^iF_0^i,\sum_{j=1}^{3n}\lambda_j^iF_k^i,\right)\left\|(u_1-v_1,u_2-v_2,u_3-v_3)\right\|_B.\end{aligned}$$

By the hypothesis (H2) we deduce that T is a contractive operator. Consequently, from Banach fixed point theorem we conclude that T has a fixed point which is the unique solution of system (1). This completes the proof. □

Example 2.5. Consider, for $0<t\leq 1$, the singular fractional system of equations

$$\begin{aligned}D^{\frac{7}{3}}u_1(t) = (&\sin u_1(t)+\sin u_2(t)+\sin u_3(t) \\ &+\cos D^{\frac{2}{3}}u_1(t)+\cos D^{\frac{3}{2}}u_1(t)+\cos D^{\frac{1}{3}}u_2(t)+\cos D^{\frac{4}{3}}u_2(t) \\ &+\cos D^{\frac{1}{2}}u_3(t)+\cos D^{\frac{6}{5}}u_3(t)\Big)\Big/\left(180\pi\sqrt{t}\right),\end{aligned}$$

$$
\begin{aligned}
D^{\frac{5}{2}}u_2(t) &= \Big(\cos u_1(t) + \cos u_2(t) + \cos u_3(t) \\
&\quad - \sin D^{\frac{2}{3}}u_1(t) - \sin D^{\frac{3}{2}}u_1(t) - \sin D^{\frac{1}{3}}u_2(t) - \sin D^{\frac{4}{3}}u_2(t) \\
&\quad - \sin D^{\frac{1}{2}}u_3(t) - \sin D^{\frac{6}{5}}u_3(t)\Big) \Big/ \Big(144\pi t^{\frac{1}{3}}\Big), \\
D^{\frac{9}{4}}u_3(t) &= \Big|u_1(t) + u_2(t) + u_3(t) + D^{\frac{2}{3}}u_1(t) + D^{\frac{3}{2}}u_1(t) \\
&\quad + D^{\frac{1}{3}}u_2(t) + D^{\frac{4}{3}}u_2(t) + D^{\frac{1}{2}}u_3(t) + D^{\frac{6}{5}}u_3(t)\Big| \Big/ \Big(72\pi^2 t^{\frac{2}{5}} \\
&\quad \times \Big(1 + \Big|D^{\frac{2}{3}}u_1(t) + \cos D^{\frac{3}{2}}u_1(t) + \cos D^{\frac{1}{3}}u_2(t) \\
&\quad + \cos D^{\frac{4}{3}}u_2(t) + \cos D^{\frac{1}{2}}u_3(t) + \cos D^{\frac{6}{5}}u_3(t)\Big|\Big)\Big),
\end{aligned}
$$

where $\sum_{i=1}^{3}\sum_{j=0}^{1}\left|u_i^{(j)}(0)\right| = 0$ and $D^{\frac{5}{3}}u_1(1) = J^{\frac{10}{3}}u_1(1)$, $D^{\frac{3}{2}}u_2(1) = J^{\frac{9}{2}}u_2(1)$, $D^{\frac{7}{4}}u_3(1) = J^{\frac{11}{4}}u_3(1)$.

We have $n = 3$, $\alpha_3^1 = 7/3$, $\alpha_3^2 = 5/2$, $\alpha_3^3 = 9/4$, $\alpha_1^1 = 2/3$, $\alpha_2^1 = 3/2$, $\alpha_1^2 = 1/3$, $\alpha_2^2 = 4/3$, $\alpha_1^3 = 1/2$, $\alpha_2^3 = 6/5$, $\mu_1 = 5/3$, $\mu_2 = 3/2$, $\mu_3 = 7/4$, $\delta_1 = 10/3$, $\delta_2 = 9/2$, $\delta_3 = 11/4$.

Then, for each $t \in [0,1]$ and $(x_1, \dots, x_9)$, $(y_1, \dots, y_9) \in \mathbb{R}^9$, we get

$$
\begin{aligned}
t^{\frac{3}{4}}\left|f_1(t, x_1, \dots, x_9) - f_1(t, y_1, \dots, y_9)\right| &\le \tfrac{t^{\frac{1}{4}}}{180\pi}\sum_{j=1}^{9}|x_j - y_j|, \\
t^{\frac{2}{3}}\left|f_2(t, x_1, \dots, x_9) - f_2(t, y_1, \dots, y_9)\right| &\le \tfrac{t^{\frac{1}{3}}}{144\pi}\sum_{j=1}^{9}|x_j - y_j|, \\
t^{\frac{4}{5}}\left|f_3(t, x_1, \dots, x_9) - f_3(t, y_1, \dots, y_9)\right| &\le \tfrac{t^{\frac{2}{5}}}{72\pi^2}\sum_{j=1}^{9}|x_j - y_j|,
\end{aligned}
$$

where $\eta_1 = 3/4$, $\eta_2 = 2/3$ and $\eta_3 = 4/5$. So, we can take

$$
\begin{gathered}
\left(\lambda_j^1\right)_{j=1,2,\dots,9} = \tfrac{1}{180\pi}, \quad \left(\lambda_j^2\right)_{j=1,2,\dots,9} = t\frac{1}{144\pi}, \quad \left(\lambda_j^3\right)_{j=1,2,\dots,9} = \tfrac{1}{72\pi^2}, \\
\sum_{j=1}^{9}\lambda_j^1 = \tfrac{1}{20\pi}, \quad \sum_{j=1}^{9}\lambda_j^2 = \tfrac{1}{16\pi}, \quad \sum_{j=1}^{9}\lambda_j^3 = \tfrac{1}{8\pi^2}.
\end{gathered}
$$

Indeed,

$$
\begin{aligned}
&F_0^1 = 24.2284, \quad F_1^1 = 40.1320, \quad F_2^1 = 52.6663, \\
&F_0^2 = 2.8842, \quad F_1^2 = 3.7843, \quad F_2^2 = 5.9714, \\
&F_0^3 = 5.2610, \quad F_1^3 = 7.2219, \quad F_2^3 = 8.6856,
\end{aligned}
$$

$$\sum_{j=1}^{9}\lambda_j^1F_0^1=0.3856,\quad \sum_{j=1}^{9}\lambda_j^1F_1^1=0.6387,\quad \sum_{j=1}^{9}\lambda_j^1F_2^1=0.8382,$$
$$\sum_{j=1}^{9}\lambda_j^2F_0^2=0.0574,\quad \sum_{j=1}^{9}\lambda_j^2F_1^2=0.0753,\quad \sum_{j=1}^{9}\lambda_j^2F_2^2=0.1188,$$
$$\sum_{j=1}^{9}\lambda_j^3F_0^3=0.0666,\quad \sum_{j=1}^{9}\lambda_j^3F_1^3=0.0915,\quad \sum_{j=1}^{9}\lambda_j^3F_2^3=0.1100.$$

It is clear that $\Lambda<1$. Thus, our system has a unique solution on $[0,1]$.

Theorem 2.6. *For $n\in\mathbb{N}^*$, let $n-1<\alpha_n^i<n$ $(i=1,2,3)$. Assume that $f_i:(0,1]\times\mathbb{R}^{3n}\to\mathbb{R}$ $(i=1,2,3)$ are continuous functions with $\lim_{t\to0^+}f_i(t,\dots)=\infty$, and there exist constants $0<\eta_i<1$ such that the functions $t^{\eta_i}f_i(t,\dots)$ are continuous on $[0,1]\times\mathbb{R}^{3n}$. Then the system (1) has at least one solution on $[0,1]$.*

Proof. Let
$$L_i=\sup_{t\in[0,1]}t^{\eta_i}\left|f_i\left(\nabla_u(t)\right)\right|, \tag{14}$$
and consider $B_r:=\left\{(u_1,u_2,u_3)\in B:\|(u_1,u_2,u_3)\|_B\le r\right\}$, where
$$r=\max_{\substack{1\le k\le n-1\\1\le i\le3}}L_i\left(F_0^i,F_k^i\right).$$
We will show that $T:B_r\to B_r$. Let $(u_1,u_2,u_3)\in B_r$ and $t\in[0,1]$. By (14), taking into account (9), we get
$$\|T(u_1,u_2,u_3)\|_B\le r. \tag{15}$$
Then, by Lemmas 2.1 and 2.2, we have $T_i(u_1,u_2,u_3),\ D^{\alpha_k^i}T_i(u_1,u_2,u_3)\in C([0,1])$. Thus, $T:B_r\to B_r$. From Lemma 2.3 it follows that T is completely continuous. Consequently, by Lemma 1.1 we deduce that the system (1) has at least one solution on $[0,1]$. This completes the proof. ☐

Example 2.7. Consider the system of equations
$$\begin{aligned}D^{\frac72}u_1(t)&=t^{-\frac47}\Big(\cos\left(u_1(t)u_2(t)u_3(t)\right)+\cos\Big(D^{\frac12}u_1(t)+D^{\frac32}u_1(t)\\&\quad+D^{\frac73}u_1(t)\Big)\Big)\Big/\Big(\pi e^t+\sin\Big(D^{\frac13}u_2(t)+D^{\frac43}u_2(t)+D^{\frac{11}{5}}u_2(t)\Big)\\&\quad\times\sin\Big(D^{\frac14}u_3(t)+D^{\frac53}u_3(t)+D^{\frac{15}{7}}u_3(t)\Big)\Big),\\D^{\frac{11}{3}}u_2(t)&=t^{-\frac13}\Big(\cos\Big(u_1(t)+D^{\frac12}u_1(t)+D^{\frac32}u_1(t)+D^{\frac73}u_1(t)\Big)\\&\quad\times\sin\Big(u_2(t)+D^{\frac13}u_2(t)+D^{\frac43}u_2(t)+D^{\frac{11}{5}}u_2(t)\Big)\Big)\Big/\big(\pi-\sin(u_3(t)\end{aligned}$$

$$+D^{\frac{1}{4}}u_3\left(t\right)+D^{\frac{5}{3}}u_3\left(t\right)+D^{\frac{15}{7}}u_3\left(t\right)\Big)\Big),$$

$$D^{\frac{15}{4}}u_3\left(t\right)=t^{-\frac{1}{9}}\left(\sin\left(D^{\frac{1}{2}}u_1\left(t\right)+D^{\frac{3}{2}}u_1\left(t\right)+D^{\frac{7}{3}}u_1\left(t\right)\right)\cos\Big(D^{\frac{1}{3}}u_2\left(t\right)\right.$$
$$+D^{\frac{4}{3}}u_2\left(t\right)+D^{\frac{11}{5}}u_2\left(t\right)\Big)\Big)\Big/\left(2\pi^2-\sin\left(u_1\left(t\right)+u_2\left(t\right)\right.\right.$$
$$\left.+w\left(t\right)\right)+\cos\left(u_3\left(t\right)+D^{\frac{1}{4}}u_3\left(t\right)+D^{\frac{5}{3}}u_3\left(t\right)+D^{\frac{15}{7}}u_3\left(t\right)\right)\Big),$$

where $0<t\leq 1$, $\sum\limits_{i=0}^{3}\sum\limits_{j=0}^{2}\left|u_i^{(j)}\left(0\right)\right|=0$, and $D^{\frac{7}{3}}u_1\left(1\right)=J^{\frac{5}{2}}u_1\left(1\right),\ D^{\frac{9}{4}}u_2\left(1\right)=J^{\frac{5}{2}}u_2\left(1\right),\ D^{\frac{5}{2}}u_3\left(1\right)=J^{\frac{11}{4}}u_3\left(1\right)$.

We have $n=4$, $\alpha_4^1=7/2$, $\alpha_4^2=11/3$, $\alpha_4^3=15/4$, $\alpha_1^1=1/2$, $\alpha_2^1=3/2$, $\alpha_3^1=7/3$, $\alpha_1^2=1/3$, $\alpha_2^2=4/3$, $\alpha_3^2=11/5$, $\alpha_1^3=1/4$, $\alpha_2^3=5/3$, $\alpha_3^3=15/7$.

Taking $\eta_1=5/7$, $\eta_2=2/3$ and $\eta_3=5/9$, we satisfy all assumptions of Theorem 2.6. Thus, the considered system has at least one solution on $[0,1]$.

3. Ulam–Hyers stability

In this section, we define and discuss the Ulam-Hyers stability and the generalized Ulam-Hyers stability for system (1).

Definition 3.1 (cf. [12, 13, 14])**.** System (1) has Ulam–Hyers stablity, if there exists $\sigma>0$, such that for all $\epsilon_1,\epsilon_2,\epsilon_3>0$, and for all solutions $(v_1,v_2,v_3)\in B$ of

$$\begin{cases}\left|D^{\alpha_n^i}v_i\left(t\right)-f_i\left(\nabla_v\left(t\right)\right)\right|<\epsilon_i, \quad i=1,2,3,\quad 0<t\leq 1,\\ k-1<\alpha_k^i<k,\quad k=1,2,...,n,\ i=1,2,3,\\ \sum\limits_{i=0}^{3}\sum\limits_{j=0}^{n-2}\left|v_i^{(j)}\left(0\right)\right|=0,\ \ D^{\mu_i}v_i\left(1\right)=\ J^{\delta_i}v_i\left(1\right),\\ n-2<\mu_i<n-1,\quad \delta_i>0,\quad i=1,2,3,\end{cases}\tag{16}$$

there exists a solution $(u_1,u_2,u_3)\in B$ of system (1)) with

$$\|(v_1-u_1,v_2-u_2,v_3-u_3)\|_B<\sigma\epsilon,\quad \epsilon>0.$$

Definition 3.2 (cf. [12, 13, 14])**.** System (1) has generalized Ulam–Hyers stablity if there exists $\Upsilon\in C\left(\mathbb{R}^+,\mathbb{R}^+\right)$, $\Upsilon\left(0\right)=0$, such that for all $\epsilon>0$, and for each solution $(v_1,v_2,v_3)\in B$ of system (16), there exists a solution $(u_1,u_2,u_3)\in B$ of system (1), where

$$\|(v_1-u_1,v_2-u_2,v_3-u_3)\|_B<\Upsilon\left(\epsilon\right).$$

Theorem 3.3. *For $n\in\mathbb{N}^*$, let $n-1<\alpha_n^i<n$ and $0<\eta_i<1$, $i=1,2,3$. Assume that*

(S1) $f_i : (0,1] \times \mathbb{R}^{3n} \to \mathbb{R}$, $i = 1,2,3$, *are continuous functions with* $\lim_{t\to 0^+} f_i(t,\dots) = \infty$, *and* $t^{\eta_i} f_i(t,\dots)$ *are continuous on* $[0,1] \times \mathbb{R}^{3n}$;

(S2) *the following inequality holds:*

$$\left\|t^{\eta_i} D^{\alpha_n^i} v_i\right\|_\infty \geq \max_{\substack{1\leq k\leq n-1\\ 1\leq i\leq 3}} L_i\left(F_0^i, F_k^i\right);$$

(S3) *the hypotheses* (Hi), $i = 1,2$, *of Theorem 2.4 hold;*

(S4) $\max\limits_{1\leq i\leq 3} \sum\limits_{j=1}^{3n} \lambda_j^i < 1$.

Then the system (1) *is generalized Ulam–Hyers stable in* B.

Proof. Using (S1), we receive (15), and for each solution $(v_1, v_2, v_3) \in B$ of (16), we can write

$$\|(v_1, v_2, v_3)\|_B \leq \max_{\substack{1\leq k\leq n-1\\ 1\leq i\leq 3}} L_i\left(F_0^i, F_k^i\right). \tag{17}$$

By (17) and (S2) we get

$$\|(v_1, v_2, v_3)\|_B \leq \max_{1\leq i\leq 3} \left\|t^{\eta_i} D^{\alpha_n^i} v_i\right\|_\infty. \tag{18}$$

Thanks to (S3), there exists a solution $(u_1, u_2, u_3) \in B$ of system (1). So, the inequality (18) implies

$$\begin{aligned}\|(v_1-u_1, v_2-u_2, v_3-u_3)\|_B &\leq \max_{1\leq i\leq 3} \left\|t^{\eta_i} D^{\alpha_n^i}\left(v_i-u_i\right)\right\|_\infty \\ \leq \max_{1\leq i\leq 3} &\left\|t^{\eta_i}\left(D^{\alpha_n^i} v_i - f_i\left(\nabla_u(t)\right)\right) - t^{\eta_i}\left(D^{\alpha_n^i} u_i - f_i\left(\nabla_v(t)\right)\right)\right. \\ &\left.+ t^{\eta_i}\left(f_i\left(\nabla_u(t)\right) - f_i\left(\nabla_v(t)\right)\right)\right\|_\infty.\end{aligned}$$

Then

$$\begin{aligned}&\|(v_1-u_1, v_2-u_2, v_3-u_3)\|_B \\ &\quad\leq \max_{1\leq i\leq 3}\Big(\|t^{\eta_i}\|_\infty \left\|D^{\alpha_n^i} v_i - f_i\left(\nabla_u(t)\right)\right\|_\infty \\ &\qquad+ \|t^{\eta_i}\|_\infty \left\|D^{\alpha_n^i} u_i - f_i\left(\nabla_v(t)\right)\right\|_\infty + \left\|t^{\eta_i}\left(f_i\left(\nabla(t)\right) - f_i\left(\nabla_v(t)\right)\right)\right\|_\infty\Big).\end{aligned}$$

By (1), (16), and (S3) we obtain

$$\begin{aligned}&\|(v_1-u_1, v_2-u_2, v_3-u_3)\|_B \\ &\qquad\leq \max_{1\leq i\leq 3}\epsilon_i + \max_{1\leq i\leq 3}\sum_{j=1}^{3n}\lambda_j^i \|(v_1-u_1, v_2-u_2, v_3-u_3)\|_B.\end{aligned}$$

Thus,

$$\|(v_1 - u_1, v_2 - u_2, v_3 - u_3)\|_B \leq \frac{\epsilon}{1 - \max\limits_{1\leq i\leq 3} \sum\limits_{j=1}^{3n} \lambda_j^i} := \sigma\epsilon, \quad \epsilon = \max_{1\leq i\leq 3} \epsilon_i.$$

It follows by (S4) that $\sigma > 0$. Hence, the system (1) has Ulam–Hyers stablity. Putting $\Upsilon(\epsilon) = \sigma\epsilon$, we see that the system (1) is generalized Ulam–Hyers stable. This completes the proof. □

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