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Simpson's type inequalities for η-convex functions via k-Riemann–Liouville fractional integrals

Seth Kermausuor

Abstract. We introduce some Simpson's type integral inequalities via k-Riemann–Liouville fractional integrals for functions whose derivatives are η-convex. These results generalize some results in the literature.

1. Introduction

The well known Simpson's inequality states as follows.

Theorem 1.1. *Let $f : [a, b] \to \mathbb{R}$ be a four times differentiable function on (a, b). Then*

$$\left| \int_a^b (t)dt - \frac{b-a}{6}\left[f(a) + 4f\Big(\frac{a+b}{2}\Big) + f(b)\right] \right| \leq \frac{(b-a)^5}{2880} \left\| f^{(4)} \right\|_\infty,$$

where $\left\| f^{(4)} \right\|_\infty = \sup_{t\in(a,b)} \left| f^{(4)}(t) \right| < \infty$.

Many authors have studied and provided several generalizations of this inequality over the years. For some results related to the Simpson's inequality, we refer the interested reader to the papers [1, 2, 5, 9, 10, 15–17].

Gordji et al. [7] introduced the concept of η-convexity which generalizes the classical concept of convexity.

Definition 1.2. A function $f : I \to \mathbb{R}$ is said to be η-convex on I with respect to the bifunction $\eta : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ if

$$f(tx + (1-t)y) \leq f(y) + t\eta(f(x), f(y)) \text{ for all } x, y \in I \text{ and } t \in [0, 1].$$

Remark 1.3. If $\eta(x, y) = x - y$, then we recover the classical notion of convexity.

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For more information on η-convex functions and some related results, we refer the interested reader to the papers [4,7,12] and the references therein. We complete this section with the definition of the k-Riemann–Liouville fractional integrals.

Definition 1.4 (see [11])**.** The Riemann–Liouville k-fractional integrals of order $\alpha > 0$, for a real-valued continuous function f, are defined as

$$ {}_kJ^{\alpha}_{a^+}f(x) = \frac{1}{k\Gamma_k(\alpha)}\int_a^x (x-t)^{\frac{\alpha}{k}-1}f(t)dt,\ x > a, $$

and

$$ {}_kJ^{\alpha}_{b^-}f(x) = \frac{1}{k\Gamma_k(\alpha)}\int_x^b (t-x)^{\frac{\alpha}{k}-1}f(t)dt,\ x < b, $$

where $k > 0$, and Γ_k is the k-gamma function given by

$$ \Gamma_k(x) = \int_0^{\infty} t^{x-1}e^{-\frac{t^k}{k}}dt,\ Re(x) > 0 $$

with the properties that $\Gamma_k(x+k) = x\Gamma_k(x)$ and $\Gamma_k(k) = 1$.

Remark 1.5. If $k = 1$, then we have the Riemann–Liouville fractional integral of order $\alpha > 0$. If $\alpha = k = 1$, then we get the classical Riemann integral.

For more information and some results related to this integral operator, we refer the interested reader to [8,11–13] and the references therein.

Our goal in this paper is to provide some Simpson's type integral inequalities involving the k-Riemann–Liouville fractional integral for functions whose first derivatives in absolute value at some powers are η-convex. Our results generalizes some results in the literature.

2. Main results

To prove our main results, we need the following integral identity via the k-Riemann–Liouville fractional integrals.

Lemma 2.1. *Let $f\colon [a,b] \to \mathbb{R}$ be a differentiable on (a,b), $a < b$, function. If $f' \in L_1([a,b])$, $n \geq 0$, $\alpha > 0$, and $k > 0$, then the following identity holds:*

$$ \begin{aligned} \mathcal{S}(\alpha,n,k) := \frac{1}{6}&\left[f(a) + f(b) + 2f\left(\frac{a+nb}{n+1}\right) + 2f\left(\frac{na+b}{n+1}\right)\right] \\ &- \frac{\Gamma_k(\alpha+k)}{3}\left(\frac{n+1}{b-a}\right)^{\frac{\alpha}{k}}\left[{}_kJ^{\alpha}_{\left(\frac{na+b}{n+1}\right)^-}f(a) + {}_kJ^{\alpha}_{\left(\frac{a+nb}{n+1}\right)^+}f(b)\right] \\ &- \frac{\Gamma_k(\alpha+k)}{6}\left(\frac{n+1}{b-a}\right)^{\frac{\alpha}{k}}\left[{}_kJ^{\alpha}_{a^+}f\left(\frac{na+b}{n+1}\right) + {}_kJ^{\alpha}_{b^-}f\left(\frac{a+nb}{n+1}\right)\right] \end{aligned} $$

$$= \frac{b-a}{2(n+1)}\left\{ \int_0^1 \left[\frac{2(1-t)^{\frac{\alpha}{k}} - t^{\frac{\alpha}{k}}}{3}\right] f'\left(\frac{n+t}{n+1}a + \frac{1-t}{n+1}b\right) dt + \int_0^1 \left[\frac{t^{\frac{\alpha}{k}} - 2(1-t)^{\frac{\alpha}{k}}}{3}\right] f'\left(\frac{1-t}{n+1}a + \frac{n+t}{n+1}b\right) dt \right\}.$$

Proof. Integrating by parts, using change of variables, and the definition of the k-Riemann–Liouville integral, we have

$$\begin{aligned} I_1 :&= \int_0^1 \left[\frac{2(1-t)^{\frac{\alpha}{k}} - t^{\frac{\alpha}{k}}}{3}\right] f'\left(\frac{n+t}{n+1}a + \frac{1-t}{n+1}b\right) dt \\ &= \frac{n+1}{3(b-a)}\left[f(a) + 2f\left(\frac{na+b}{n+1}\right)\right] - \frac{2\Gamma_k(\alpha+k)}{3}\left(\frac{n+1}{b-a}\right)^{\frac{\alpha}{k}+1} \\ &\quad \times {}_kJ^{\alpha}_{\left(\frac{na+b}{n+1}\right)^-} f(a) - \frac{\Gamma_k(\alpha+k)}{3}\left(\frac{n+1}{b-a}\right)^{\frac{\alpha}{k}+1} {}_kJ^{\alpha}_{a^+} f\left(\frac{na+b}{n+1}\right) \end{aligned}$$

and

$$\begin{aligned} I_2 :&= \int_0^1 \left[\frac{t^{\frac{\alpha}{k}} - 2(1-t)^{\frac{\alpha}{k}}}{3}\right] f'\left(\frac{1-t}{n+1}a + \frac{n+t}{n+1}b\right) dt \\ &= \frac{n+1}{3(b-a)}\left[f(b) + 2f\left(\frac{a+nb}{n+1}\right)\right] - \left(\frac{n+1}{b-a}\right)^{\frac{\alpha}{k}+1} \frac{\Gamma_k(\alpha+k)}{3} \\ &\quad \times {}_kJ^{\alpha}_{b^-} f\left(\frac{a+nb}{n+1}\right) - \frac{2\Gamma_k(\alpha+k)}{3}\left(\frac{n+1}{b-a}\right)^{\frac{\alpha}{k}+1} {}_kJ^{\alpha}_{\left(\frac{a+nb}{n+1}\right)^+} f(b). \end{aligned}$$

Using these equalities, we have that

$$I_1 + I_2 = \frac{2(n+1)}{b-a}\mathcal{S}(\alpha, n, k),$$

which gives the desired identity. □

Remark 2.2. If $k = 1$, then we obtain the identity in Lemma 2.1 of [15].

Theorem 2.3. *Under the conditions of Lemma 2.1, suppose that $|f'|$ is η-convex on $[a, b]$. Then*

$$\begin{aligned} |\mathcal{S}(\alpha, n, k)| \le \frac{b-a}{6(n+1)}\Big\{ & \mathcal{A}(\alpha, k)\left(|f'(a)| + |f'(b)|\right) \\ & + \frac{\mathcal{B}(\alpha, n, k)}{n+1}\left(\eta\left(|f'(a)|, |f'(b)|\right) + \eta\left(|f'(b)|, |f'(a)|\right)\right)\Big\}, \end{aligned}$$

where

$$\mathcal{Q}(\alpha, k) := \frac{2^{\frac{k}{\alpha}}}{2^{\frac{k}{\alpha}} + 1}, \quad \mathcal{A}(\alpha, k) := \frac{3 - 4\left(1 - \mathcal{Q}(\alpha, k)\right)^{\frac{\alpha}{k}+1} - 2\left(\mathcal{Q}(\alpha, k)\right)^{\frac{\alpha}{k}+1}}{\frac{\alpha}{k} + 1},$$

and

$$\mathcal{B}(\alpha,n,k) := \frac{3n+1-(n+\mathcal{Q}(\alpha,k))\left(4\left(1-\mathcal{Q}(\alpha,k)\right)^{\frac{\alpha}{k}+1}+2\left(\mathcal{Q}(\alpha,k)\right)^{\frac{\alpha}{k}+1}\right)}{\frac{\alpha}{k}+1} + \frac{1-4\left(1-\mathcal{Q}(\alpha,k)\right)^{\frac{\alpha}{k}+2}+2\left(\mathcal{Q}(\alpha,k)\right)^{\frac{\alpha}{k}+2}}{\left(\frac{\alpha}{k}+1\right)\left(\frac{\alpha}{k}+2\right)}.$$

Proof. Using Lemma 2.1 and the η-convexity of $|f'|$, we get

$$\begin{aligned}
|\mathcal{S}(\alpha,n,k)| &\le \frac{b-a}{6(n+1)}\Bigg\{\int_0^1 \left|2(1-t)^{\frac{\alpha}{k}}-t^{\frac{\alpha}{k}}\right|\left|f'\left(\frac{n+t}{n+1}a+\frac{1-t}{n+1}b\right)\right|dt \\
&\quad + \int_0^1 \left|t^{\frac{\alpha}{k}}-2(1-t)^{\frac{\alpha}{k}}\right|\left|f'\left(\frac{1-t}{n+1}a+\frac{n+t}{n+1}b\right)\right|dt\Bigg\} \\
&\le \frac{b-a}{6(n+1)}\Bigg\{\int_0^1 \left|2(1-t)^{\frac{\alpha}{k}}-t^{\frac{\alpha}{k}}\right|\left(|f'(b)|+\frac{n+t}{n+1}\eta\Big(|f'(a)|,|f'(b)|\Big)\right)dt \\
&\quad + \int_0^1 \left|t^{\frac{\alpha}{k}}-2(1-t)^{\frac{\alpha}{k}}\right|\left(|f'(a)|+\frac{n+t}{n+1}\eta\Big(|f'(b)|,|f'(a)|\Big)\right)dt\Bigg\} \\
&= \frac{b-a}{6(n+1)}\Bigg\{|f'(b)|\int_0^1 \left|2(1-t)^{\frac{\alpha}{k}}-t^{\frac{\alpha}{k}}\right|dt+\frac{1}{n+1}\eta\Big(|f'(a)|,|f'(b)|\Big) \\
&\quad \times \int_0^1 (n+t)\left|2(1-t)^{\frac{\alpha}{k}}-t^{\frac{\alpha}{k}}\right|dt + |f'(a)|\int_0^1 \left|t^{\frac{\alpha}{k}}-2(1-t)^{\frac{\alpha}{k}}\right|dt \\
&\quad + \frac{1}{n+1}\eta\Big(|f'(b)|,|f'(a)|\Big)\int_0^1 (n+t)\left|t^{\frac{\alpha}{k}}-2(1-t)^{\frac{\alpha}{k}}\right|dt\Bigg\} \\
&= \frac{b-a}{6(n+1)}\Bigg\{\mathcal{A}(\alpha,k)\Big(|f'(a)|+|f'(b)|\Big) \\
&\quad + \frac{\mathcal{B}(\alpha,n,k)}{n+1}\Big(\eta\Big(|f'(a)|,|f'(b)|\Big)+\eta\Big(|f'(b)|,|f'(a)|\Big)\Big)\Bigg\},
\end{aligned}$$

where

$$\begin{aligned}
\int_0^1 \left|2(1-t)^{\frac{\alpha}{k}}-t^{\frac{\alpha}{k}}\right|dt &= \int_0^{\mathcal{Q}(\alpha,k)} \left(2(1-t)^{\frac{\alpha}{k}}-t^{\frac{\alpha}{k}}\right)dt \\
&\quad + \int_{\mathcal{Q}(\alpha,k)}^1 \left(t^{\frac{\alpha}{k}}-2(1-t)^{\frac{\alpha}{k}}\right)dt = \mathcal{A}(\alpha,k)
\end{aligned} \tag{2.1}$$

and

$$\begin{aligned}
\int_0^1 (n+t)\left|2(1-t)^{\frac{\alpha}{k}}-t^{\frac{\alpha}{k}}\right|dt &= \int_0^{\mathcal{Q}(\alpha,k)} (n+t)\left(2(1-t)^{\frac{\alpha}{k}}-t^{\frac{\alpha}{k}}\right)dt \\
&\quad + \int_{\mathcal{Q}(\alpha,k)}^1 (n+t)\left(t^{\frac{\alpha}{k}}-2(1-t)^{\frac{\alpha}{k}}\right)dt = \mathcal{B}(\alpha,n,k).
\end{aligned} \tag{2.2}$$

□

Remark 2.4. If $k = 1$ and $\eta(x,y) = x - y$ in Theorem 2.3, then we recover [15, Theorem 2.2].

Theorem 2.5. *Under the conditions of Lemma 2.1, suppose that $|f'|^q$ is η-convex on $[a,b]$ for $q > 1$. Then we have the inequality*

$$
\begin{aligned}
|\mathcal{S}(\alpha,n,k)| \leq \frac{b-a}{6(n+1)} \left(\mathcal{A}(\alpha,k)\right)^{1-\frac{1}{q}} \Bigg\{ &\Big(\mathcal{A}(\alpha,k)|f'(b)|^q \\
&+\frac{\mathcal{B}(\alpha,n,k)}{n+1}\eta\Big(|f'(a)|^q,|f'(b)|^q\Big)\Big)^{\frac{1}{q}} + \Big(\mathcal{A}(\alpha,k)|f'(a)|^q \\
&+\frac{\mathcal{B}(\alpha,n,k)}{n+1}\eta\Big(|f'(b)|^q,|f'(a)|^q\Big)\Big)^{\frac{1}{q}} \Bigg\},
\end{aligned}
$$

where $\mathcal{A}(\alpha,k)$ and $\mathcal{B}(\alpha,n,k)$ are defined in Theorem 2.3.

Proof. Using Lemma 2.1, Hölder's inequality, and the η-convexity of $|f'|^q$, we have

$$
\begin{aligned}
&|\mathcal{S}(\alpha,n,k)| \\
&\leq \frac{b-a}{6(n+1)}\Bigg\{ \int_0^1 \left|2(1-t)^{\frac{\alpha}{k}} - t^{\frac{\alpha}{k}}\right| \left|f'\left(\frac{n+t}{n+1}a + \frac{1-t}{n+1}b\right)\right| dt \\
&\quad + \int_0^1 \left|t^{\frac{\alpha}{k}} - 2(1-t)^{\frac{\alpha}{k}}\right| \left|f'\left(\frac{1-t}{n+1}a + \frac{n+t}{n+1}b\right)\right| dt\Bigg\} \\
&\leq \frac{b-a}{6(n+1)}\Bigg\{ \left(\int_0^1 \left|2(1-t)^{\frac{\alpha}{k}} - t^{\frac{\alpha}{k}}\right| dt\right)^{1-\frac{1}{q}} \\
&\quad \times \left(\int_0^1 \left|2(1-t)^{\frac{\alpha}{k}} - t^{\frac{\alpha}{k}}\right| \left|f'\left(\frac{n+t}{n+1}a + \frac{1-t}{n+1}b\right)\right|^q dt\right)^{\frac{1}{q}} \\
&\quad + \left(\int_0^1 \left|t^{\frac{\alpha}{k}} - 2(1-t)^{\frac{\alpha}{k}}\right| dt\right)^{1-\frac{1}{q}} \\
&\quad \times \left(\int_0^1 \left|t^{\frac{\alpha}{k}} - 2(1-t)^{\frac{\alpha}{k}}\right| \left|f'\left(\frac{1-t}{n+1}a + \frac{n+t}{n+1}b\right)\right|^q dt\right)^{\frac{1}{q}} \Bigg\} \\
&\leq \frac{b-a}{6(n+1)} \left(\int_0^1 \left|2(1-t)^{\frac{\alpha}{k}} - t^{\frac{\alpha}{k}}\right| dt\right)^{1-\frac{1}{q}} \\
&\quad \times \Bigg\{ \left(\int_0^1 \left|2(1-t)^{\frac{\alpha}{k}} - t^{\frac{\alpha}{k}}\right| \Big(|f'(b)|^q + \frac{n+t}{n+1}\eta\Big(|f'(a)|^q,|f'(b)|^q\Big)\Big) dt\right)^{\frac{1}{q}} \\
&\quad + \left(\int_0^1 \left|t^{\frac{\alpha}{k}} - 2(1-t)^{\frac{\alpha}{k}}\right| \Big(|f'(a)|^q + \frac{n+t}{n+1}\eta\Big(|f'(b)|^q,|f'(a)|^q\Big)\Big) dt\right)^{\frac{1}{q}} \Bigg\},
\end{aligned}
$$

which proves, in view of (2.1) and (2.2), the desired inequality. □

Remark 2.6. If $k = 1$, and $\eta(x, y) = x - y$ in Theorem 2.5, then we recover [15, Theorem 2.4].

Theorem 2.7. *Under the conditions of Lemma 2.1, suppose that $|f'|^q$ is η-convex on $[a, b]$ for $q > 1$. Then*

$$\begin{aligned} |\mathcal{S}(\alpha, n, k)| \leq & \frac{b-a}{6(n+1)} \left(\mathcal{C}(\alpha, k, p)\right)^{\frac{1}{p}} \\ & \times \left\{ \left(|f'(b)|^q + \frac{2n+1}{2(n+1)} \eta\Big(|f'(a)|^q, |f'(b)|^q\Big) \right)^{\frac{1}{q}} \right. \\ & \left. + \left(|f'(a)|^q + \frac{2n+1}{2(n+1)} \eta\Big(|f'(b)|^q, |f'(a)|^q\Big) \right)^{\frac{1}{q}} \right\}, \end{aligned} \tag{2.3}$$

where $1/p + 1/q = 1$ and

$$\mathcal{C}(\alpha, k, p) := \int_0^1 \left| 2(1-t)^{\frac{\alpha}{k}} - t^{\frac{\alpha}{k}} \right|^p dt.$$

In addition, if $\alpha/k \in [0, 1]$, then

$$\begin{aligned} |\mathcal{S}(\alpha, n, k)| \leq & \frac{b-a}{6(n+1)} \left(\frac{2^{\frac{k}{\alpha}+p} + 1}{\left(\frac{\alpha}{k}p + 1\right)\left(2^{\frac{k}{\alpha}} + 1\right)} \right)^{\frac{1}{p}} \\ & \times \left\{ \left(|f'(b)|^q + \frac{2n+1}{2(n+1)} \eta\Big(|f'(a)|^q, |f'(b)|^q\Big) \right)^{\frac{1}{q}} \right. \\ & \left. + \left(|f'(a)|^q + \frac{2n+1}{2(n+1)} \eta\Big(|f'(b)|^q, |f'(a)|^q\Big) \right)^{\frac{1}{q}} \right\}. \end{aligned} \tag{2.4}$$

Proof. Using Lemma 2.1, Hölder's inequality, and the η-convexity of $|f'|^q$, we have

$$\begin{aligned} & |\mathcal{S}(\alpha, n, k)| \\ & \leq \frac{b-a}{6(n+1)} \left\{ \left(\int_0^1 \left| 2(1-t)^{\frac{\alpha}{k}} - t^{\frac{\alpha}{k}} \right|^p dt \right)^{\frac{1}{p}} \left(\int_0^1 \left| f'\left(\frac{n+t}{n+1} a + \frac{1-t}{n+1} b \right) \right|^q dt \right)^{\frac{1}{q}} \right. \\ & \quad \left. + \left(\int_0^1 \left| t^{\frac{\alpha}{k}} - 2(1-t)^{\frac{\alpha}{k}} \right|^p dt \right)^{\frac{1}{p}} \left(\int_0^1 \left| f'\left(\frac{1-t}{n+1} a + \frac{n+t}{n+1} b \right) \right|^q dt \right)^{\frac{1}{q}} \right\} \\ & \leq \frac{b-a}{6(n+1)} \left(\int_0^1 \left| 2(1-t)^{\frac{\alpha}{k}} - t^{\frac{\alpha}{k}} \right|^p dt \right)^{\frac{1}{p}} \\ & \quad \times \left\{ \left(\int_0^1 \left(|f'(b)|^q + \frac{n+t}{n+1} \eta\Big(|f'(a)|^q, |f'(b)|^q\Big) \right) dt \right)^{\frac{1}{q}} \right. \end{aligned}$$

$$+\left(\int_0^1\left(|f'(a)|^q+\frac{n+t}{n+1}\eta\Big(|f'(b)|^q,|f'(a)|^q\Big)\right)dt\right)^{\frac{1}{q}}\Bigg\}$$

$$=\frac{b-a}{6(n+1)}\left(\int_0^1\left|2(1-t)^{\frac{\alpha}{k}}-t^{\frac{\alpha}{k}}\right|^p dt\right)^{\frac{1}{p}}$$

$$\times\Bigg\{\left(|f'(b)|^q\int_0^1 1\ dt+\frac{1}{n+1}\eta\Big(|f'(a)|^q,|f'(b)|^q\Big)\int_0^1(n+t)dt\right)^{\frac{1}{q}}$$

$$+\left(|f'(a)|^q\int_0^1 1\ dt+\frac{1}{n+1}\eta\Big(|f'(b)|^q,|f'(a)|^q\Big)\int_0^1(n+t)dt\right)^{\frac{1}{q}}\Bigg\}.$$

This gives the inequality (2.3) after simple calculations.

Now, from the assumption $\alpha/k\in[0,1]$, we have that

$$\left|x^{\frac{\alpha}{k}}-y^{\frac{\alpha}{k}}\right|\leq\left|x-y\right|^{\frac{\alpha}{k}}\text{ for all }x,y\in[0,1].$$

So, it follows that

$$\left|2(1-t)^{\frac{\alpha}{k}}-t^{\frac{\alpha}{k}}\right|=2\left((1-t)^{\frac{\alpha}{k}}-\left(\frac{t}{2^{\frac{k}{\alpha}}}\right)^{\frac{\alpha}{k}}\right)\leq 2\left|1-\left(1+\frac{1}{2^{\frac{k}{\alpha}}}\right)t\right|^{\frac{\alpha}{k}}.$$

Thus,

$$\mathcal{C}(\alpha,k,p)\leq\int_0^1\left(2\left|1-\left(1+\frac{1}{2^{\frac{k}{\alpha}}}\right)t\right|^{\frac{\alpha}{k}}\right)^p dt$$

$$=2^p\left(\int_0^{\mathcal{Q}(\alpha,k)}\left(1-\left(\frac{2^{\frac{k}{\alpha}}+1}{2^{\frac{k}{\alpha}}}\right)t\right)^{\frac{\alpha}{k}p}dt+\int_{\mathcal{Q}(\alpha,k)}^1\left(\left(\frac{2^{\frac{k}{\alpha}}+1}{2^{\frac{k}{\alpha}}}\right)t-1\right)^{\frac{\alpha}{k}p}dt\right)$$

$$=2^p\left(\frac{2^{\frac{k}{\alpha}}}{\left(\frac{\alpha}{k}p+1\right)\left(2^{\frac{k}{\alpha}}+1\right)}+\frac{2^{\frac{k}{\alpha}}}{\left(\frac{\alpha}{k}p+1\right)\left(2^{\frac{k}{\alpha}}+1\right)}\left(\frac{1}{2^{\frac{k}{\alpha}}}\right)^{\frac{\alpha}{k}p+1}\right)$$

$$=\frac{2^{\frac{k}{\alpha}+p}+1}{\left(\frac{\alpha}{k}p+1\right)\left(2^{\frac{k}{\alpha}}+1\right)}.$$

This proves, in view of (2.3), the inequality (2.4). □

Remark 2.8. If $k=1$, and $\eta(x,y)=x-y$ in the inequality (2.3) of Theorem 2.7, then we recover [15, Theorem 2.3].

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Department of Mathematics and Computer Science, Alabama State University, Montgomery, AL 36101, USA

E-mail address: skermausour@alasu.edu