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Certain sufficient conditions for close-to-convexity and starlikeness of multivalent functions

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Abstract. By using Jack's lemma, we derive simple sufficient conditions for analytic functions to be multivalent close-to-convex and multivalent starlike.

1. Introduction

Denote by $\mathcal{A}(p)$, where $p \in \mathbb{N} := \{1, 2, \dots\}$, the class of multivalent analytic functions in the open unit disk $\mathbb{U} := \{z \in \mathbb{C} : |z| < 1\}$ of the form

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k,\ z \in \mathbb{U},$$

and let $\mathcal{A} := \mathcal{A}(1)$.

For $0 \leq \alpha < p$, we say that the function $f \in \mathcal{A}(p)$ belongs to the class of *p-valently starlike functions of order* α, denoted by $\mathbb{S}_p^*(\alpha)$, if it satisfies the inequality (see Owa [7] and Aouf [1, 2])

$$\operatorname{Re} \frac{zf'(z)}{f(z)} > \alpha,\ z \in \mathbb{U}. \tag{1.1}$$

Also, we say that the function $f \in \mathcal{A}(p)$ belongs to the class of of *p-valently convex functions of order* α, denoted by $\mathbb{K}_p(\alpha)$, if (see Owa [7])

$$\operatorname{Re} \left[1 + \frac{zf''(z)}{f'(z)}\right] > \alpha, z \in \mathbb{U}. \tag{1.2}$$

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For the special case $\alpha = 0$, we denote $\mathbb{S}_p^* := \mathbb{S}_p^*(0)$ and $\mathbb{K}_p := \mathbb{K}_p(0)$, and from the formulas (1.1) and (1.2) we have

$$f \in \mathbb{K}_p(\alpha) \Leftrightarrow \frac{zf'(z)}{p} \in \mathbb{S}_p^*(\alpha).$$

Furthermore, a function $f \in \mathcal{A}(p)$ is said to be in the class of *p-valently close-to-convex functions*, denoted by $\mathbb{C}(p)$, if there exists a function $g \in \mathbb{S}^*(p)$ such that (see Aouf [3] and Owa [8])

$$\operatorname{Re} \frac{zf'(z)}{g(z)} > 0,\ z \in \mathbb{U}.$$

Since $g(z) = z^p \in \mathbb{S}^*(p)$, it follows that a function $f \in \mathcal{A}(p)$ satisfying

$$\operatorname{Re} \frac{f'(z)}{z^{p-1}} > 0,\ z \in \mathbb{U},$$

or

$$\left| \frac{f'(z)}{z^{p-1}} - p \right| < p,\ z \in \mathbb{U}, \tag{1.3}$$

is a member of the class $\mathbb{C}(p)$.

In order to prove our results, we have to recall the following lemma of Jack [4] (generalized by Miller and Mocanu [5, 6]).

Lemma 1.1. *Let ω be a non-constant analytic function in $\mathbb{U}$ with $\omega(0) = 0$. If $|\omega|$ attains its maximum value on the circle $|z| = r$ at a point $z_0 \in \mathbb{U}$, then $z_0\omega'(z_0) = k\omega(z_0)$ where $k \geq 1$ is a real number.*

2. Main results

Theorem 2.1. *Let $f \in \mathcal{A}(p)$, and suppose that it satisfies, for $\gamma \geq 0$, the inequality*

$$\left| \frac{f'(z)}{z^{p-1}} - p \right|^{1-\gamma} \left| \frac{f''(z)}{pz^{p-2}} - (p-1) \right|^{\gamma} < p,\ z \in \mathbb{U}. \tag{2.1}$$

Then (1.3) *holds, i.e., f belongs to $\mathbb{C}(p)$ and is a bounded function in $\mathbb{U}$.*

Proof. For a function $f \in \mathcal{A}(p)$ satisfying the assumption (2.1), we define a function ω by

$$\omega(z) := \frac{1}{p} \left(\frac{f'(z)}{z^{p-1}} - p \right),\ z \in \mathbb{U}. \tag{2.2}$$

Then ω is analytic in $\mathbb{U}$ with $\omega(0) = 0$. To prove our conclusion (1.3) we will show that $|\omega(z)| < 1$, $z \in \mathbb{U}$.

Differentiating (2.2), we have

$$\frac{f''(z)}{z^{p-2}} - p(p-1) = p(p-1)\omega(z) + pz\omega'(z),\ z \in \mathbb{U}. \tag{2.3}$$

From (2.2) and (2.3) we obtain that

$$\begin{aligned}\left|\frac{f'(z)}{z^{p-1}}-p\right|^{1-\gamma}&\left|\frac{f''(z)}{z^{p-2}}-p(p-1)\right|^{\gamma}\\ &=|p\omega(z)|^{1-\gamma}\left|p(p-1)\omega(z)+pz\omega'(z)\right|^{\gamma}\\ &=p\,|\omega(z)|\left|p-1+\frac{z\omega'(z)}{\omega(z)}\right|^{\gamma},\ z\in\mathbb{U}.\end{aligned}\tag{2.4}$$

Supposing that there exists a point $z_0\in\mathbb{U}$ such that $\max\limits_{|z|\le|z_0|}|\omega(z)|=|\omega(z_0)|=1$, from Lemma 1.1 we obtain that $z_0\omega'(z_0)=k\omega(z_0)$ where $k\ge 1$. Hence, from (2.4) we have

$$\left|\frac{f'(z_0)}{z_0^{p-1}}-p\right|^{1-\gamma}\left|\frac{f''(z_0)}{z_0^{p-2}}-p(p-1)\right|^{\gamma}=p\,|p-1+k|^{\gamma}\ge p^{\gamma+1},$$

which contradicts (2.1). Therefore, $|\omega(z)|<1$ for all $z\in\mathbb{U}$, and the conclusion (1.3) has been proved.

Finally, from (1.3) it follows that $|f'(z)|\le 2p\,|z|^{p-1}<2p$, $z\in\mathbb{U}$, hence

$$|f(z)|=\left|\int_0^z f'(\zeta)d\zeta\right|\le\int_0^r\left|f'(\rho e^{i\theta})\right|d\rho\le 2pr<2p,$$
$$z=re^{i\theta}\in\mathbb{U},\ \theta\in[0,2\pi).$$

Consequently, f is bounded in $\mathbb{U}$. □

For the special case $\gamma=1$, Theorem 2.1 reduces to the next result.

Corollary 2.1. *If $f\in\mathcal{A}(p)$ satisfies*

$$\left|\frac{f''(z)}{pz^{p-2}}-(p-1)\right|<p,\ z\in\mathbb{U},$$

then the inequality (1.3) *holds, i.e., $f\in\mathbb{C}(p)$ and it is a bounded function in $\mathbb{U}$.*

Theorem 2.2. *Let $f\in\mathcal{A}(p)$, and suppose that f satisfies, for $\gamma\ge 0$, the inequality*

$$\left|\frac{f'(z)}{pz^{p-1}}-1\right|^{1-\gamma}\left|1+\frac{zf''(z)}{f'(z)}-p\right|^{\gamma}<\left(\frac{1}{2}\right)^{\gamma},\ z\in\mathbb{U}.\tag{2.5}$$

Then the inequality (1.3) *holds, i.e., $f\in\mathbb{C}(p)$ and it is a bounded function in $\mathbb{U}$.*

Proof. For a function $f\in\mathcal{A}(p)$ satisfying the assumption (2.5), we define a function ω by (2.2). Then, ω is analytic in $\mathbb{U}$ with $\omega(0)=0$, and differentiating (2.2), we get

$$1+\frac{zf''(z)}{f'(z)}-p=\frac{z\omega'(z)}{1+\omega(z)},\ z\in\mathbb{U}. \tag{2.6}$$

From the assumption (2.5), it follows that the left-hand side of (2.6) is an analytic function in $\mathbb{U}$, hence $\omega(z)\neq-1$ for all $z\in\mathbb{U}$. From (2.2) and (2.6) we have

$$\left|\frac{f'(z)}{pz^{p-1}}-1\right|^{1-\gamma}\left|1+\frac{zf''(z)}{f'(z)}-p\right|^{\gamma}=|\omega(z)|^{1-\gamma}\left|\frac{z\omega'(z)}{1+\omega(z)}\right|^{\gamma},\ z\in\mathbb{U}. \tag{2.7}$$

If we suppose that there exists a point $z_0\in\mathbb{U}$ such that $\max\limits_{|z|\leq|z_0|}|\omega(z)|=|\omega(z_0)|=1$, from Lemma 1.1 we obtain that $z_0\omega'(z_0)=k\omega(z_0)$ with $k\geq1$. Hence, from (2.7) we obtain

$$\begin{aligned}\left|\frac{f'(z_0)}{pz_0^{p-1}}-1\right|^{1-\gamma}\left|1+\frac{z_0f_0''(z)}{f_0'(z)}-p\right|^{\gamma}&=|\omega(z_0)|^{1-\gamma}\left|\frac{z\omega'(z_0)}{1+\omega(z_0)}\right|^{\gamma}\\&=|\omega(z_0)|\left|\frac{k}{1+\omega(z_0)}\right|^{\gamma}\geq\left(\frac{1}{2}\right)^{\gamma},\end{aligned}$$

which contradicts (2.5). Therefore, $|\omega(z)|<1$ for all $z\in\mathbb{U}$, and our conclusion has been proved.

Since under the assumption (2.5) the inequality (1.3) holds, as in the proof of the previous theorem it follows that f is bounded in $\mathbb{U}$. □

Putting $\gamma=1$ in Theorem 2.2, we obtain the next special case.

Corollary 2.2. *If $f\in\mathcal{A}(p)$ satisfies*

$$\left|1+\frac{zf''(z)}{f'(z)}-p\right|<\frac{1}{2},\ z\in\mathbb{U},$$

then the inequality (1.3) *holds, i.e., $f\in\mathbb{C}(p)$ and it is a bounded function in $\mathbb{U}$.*

Remark 2.1. For the special case $p=1$, the above corollary gives us the following criteria for close-to-convexity. If $f\in\mathcal{A}$, then

$$\left|\frac{zf''(z)}{f'(z)}\right|<\frac{1}{2},\ z\in\mathbb{U}\Rightarrow\left|f'(z)-1\right|<1,\ z\in\mathbb{U},$$

i.e., f lies in $\mathbb{C}(1)$ and is a bounded function in $\mathbb{U}$.

Theorem 2.3. *Let $f\in\mathcal{A}(p)$, and suppose that it satisfies, for $\gamma\geq0$, the inequality*

$$\left|\frac{zf'(z)}{f(z)}-p\right|^{1-\gamma}\left|1+\frac{zf''(z)}{f'(z)}-p\right|^{\gamma}<(p-\alpha)\left(1+\frac{1}{p+|p-2\alpha|}\right)^{\gamma},\ z\in\mathbb{U}. \tag{2.8}$$

Moreover, for $\gamma = 1$, assume that $f(z) \neq 0$ for all $z \in \dot{\mathbb{U}}$. Then $f \in \mathbb{S}_p^*(\alpha)$.

Proof. We have to prove that the assumption (2.8) implies the inequality (1.1). For a function $f \in \mathcal{A}(p)$ satisfying the assumption (2.8), we define a function ω by

$$\frac{zf'(z)}{f(z)} = \frac{p + (p-2\alpha)\omega(z)}{1-\omega(z)}, \; z \in \mathbb{U} \quad (0 \leq \alpha < p)\,. \tag{2.9}$$

We have that ω is analytic in $\mathbb{U}$ with $\omega(0) = 0$, and from the assumption (2.8) it follows that the left-hand side of (2.9) is an analytic function in $\mathbb{U}$, hence $\omega(z) \neq 1$ for all $z \in \mathbb{U}$.

Differentiating (2.9), we obtain

$$1 + \frac{zf''(z)}{f'(z)} - p = \frac{2(p-\alpha)\omega(z)}{1-\omega(z)} \left[1 + \frac{\frac{z\omega'(z)}{\omega(z)}}{p + (p-2\alpha)\omega(z)} \right], \; z \in \mathbb{U}. \tag{2.10}$$

Then from (2.9) and (2.10) we have

$$\begin{aligned} &\left| \frac{zf'(z)}{f(z)} - p \right|^{1-\gamma} \left| 1 + \frac{zf''(z)}{f'(z)} - p \right|^{\gamma} \\ &= 2(p-\alpha) \left| \frac{\omega(z)}{1-\omega(z)} \right| \left| 1 + \frac{\frac{z\omega'(z)}{\omega(z)}}{p + (p-2\alpha)\omega(z)} \right|^{\gamma}, \; z \in \mathbb{U}. \end{aligned} \tag{2.11}$$

If we suppose that there exists a point $z_0 \in \mathbb{U}$ such that $\max\limits_{|z| \leq |z_0|} |\omega(z)| = |\omega(z_0)| = 1$, from Lemma 1.1 we obtain that $z_0\omega'(z_0) = k\omega(z_0)$ with $k \geq 1$. Therefore, from (2.11) we get

$$\begin{aligned} &\left| \frac{z_0 f'(z_0)}{f(z_0)} - p \right|^{1-\gamma} \left| 1 + \frac{z_0 f''(z_0)}{f'(z_0)} - p \right|^{\gamma} \\ &= 2(p-\alpha) \left| \frac{\omega(z_0)}{1-\omega(z_0)} \right| \left| 1 + \frac{\frac{z_0\omega'(z_0)}{\omega(z_0)}}{p + (p-2\alpha)\omega(z_0)} \right|^{\gamma} \\ &\geq (p-\alpha) \left| 1 + \frac{k}{p} \frac{1}{1 + \left(1 - \frac{2\alpha}{p}\right) w(z_0)} \right|^{\gamma}. \end{aligned} \tag{2.12}$$

Considering the function φ defined by

$$\varphi(z) := \frac{1}{1 + \left(1 - \frac{2\alpha}{p}\right) z}, \ z \in \mathbb{U},$$

it is easy to check that $|\varphi(z)| > \dfrac{p}{p + |p - 2\alpha|}$ for all $z \in \mathbb{U}$. Hence, using the fact that $\gamma \geq 0$, from (2.12) we obtain

$$\begin{aligned} \left|\frac{z_0 f'(z_0)}{f(z_0)} - p\right|^{1-\gamma} \left|1 + \frac{z_0 f''(z_0)}{f'(z_0)} - p\right|^{\gamma} &\geq (p - \alpha) \left(1 + \frac{k}{p} \frac{p}{p + |p - 2\alpha|}\right)^{\gamma} \\ &\geq (p - \alpha) \left(1 + \frac{1}{p + |p - 2\alpha|}\right)^{\gamma} \end{aligned}$$

which contradicts (2.8). This proves that $|\omega(z)| < 1$ for all $z \in \mathbb{U}$, and hence $f \in \mathbb{S}_p^*(\alpha)$. □

If we take $\alpha = 0$ in Theorem 2.3, then we obtain the next corollary.

Corollary 2.3. *Let $f \in \mathcal{A}(p)$, and suppose that f satisfies, for $\gamma \geq 0$, the inequality*

$$\left|\frac{z f'(z)}{f(z)} - p\right|^{1-\gamma} \left|1 + \frac{z f''(z)}{f'(z)} - p\right|^{\gamma} < p \left(\frac{2p + 1}{2p}\right)^{\gamma}, \ z \in \mathbb{U}.$$

Moreover, for $\gamma = 1$ assume that $f(z) \neq 0$ for all $z \in \dot{\mathbb{U}} := \mathbb{U} \setminus \{0\}$. Then $f \in \mathbb{S}_p^*$.

For $\gamma = 1$, Corollary 2.3 reduces to the next result.

Corollary 2.4. *If $f \in \mathcal{A}(p)$ satisfies the inequality*

$$\left|1 + \frac{z f''(z)}{f'(z)} - p\right| < p + \frac{1}{2}, \ z \in \mathbb{U},$$

and $f(z) \neq 0$ for all $z \in \dot{\mathbb{U}}$, then $f \in \mathbb{S}_p^*$.

Taking $p = 1$ in Corollaries 2.3 and 2.4, we get the results obtained by Singh and Singh [11, Theorem 3 and Corollary 3].

Putting $p = 1$ in Theorem 2.3, we have the following corollary.

Corollary 2.5. *If $f \in \mathcal{A}$ satisfies, for some $\gamma \geq 0$, the inequality*

$$\left|\frac{z f'(z)}{f(z)} - 1\right|^{1-\gamma} \left|\frac{z f''(z)}{f'(z)}\right|^{\gamma} < (1 - \alpha) \left(1 + \frac{1}{1 + |1 - 2\alpha|}\right)^{\gamma}, \ z \in \mathbb{U},$$

and, for $\gamma = 1$ we have $f(z) \neq 0$ for all $z \in \dot{\mathbb{U}}$, then $f \in \mathbb{S}^*(\alpha)$.

The above corollary is an improvement of the result obtained by Owa and Srivastava [10, Lemma 3].

Theorem 2.4. *Let $f \in \mathcal{A}(p)$ be such that $f(z) \neq 0$ for all $z \in \dot{\mathbb{U}}$, and suppose that, for $0 \le \alpha < p$, the inequality*

$$\left|1 + \frac{zf''(z)}{f'(z)} - p\right| < \frac{(p-\alpha)(2p+1-\alpha)}{2p-\alpha}, \; z \in \mathbb{U}, \tag{2.13}$$

is satisfied. Then,

$$\left|\frac{zf'(z)}{f(z)} - p\right| < p - \alpha, \; z \in \mathbb{U}, \tag{2.14}$$

i.e., $f \in \mathbb{S}_p^*(\alpha)$.

Proof. Assume that $f \in \mathcal{A}(p)$, with $f(z) \neq 0$ for all $z \in \dot{\mathbb{U}}$, satisfies the inequality (2.13). Define a function ω by

$$\frac{zf'(z)}{f(z)} = p + (p-\alpha)\omega(z), \; z \in \mathbb{U}. \tag{2.15}$$

It follows that ω is analytic in $\mathbb{U}$ with $\omega(0) = 0$. Differentiating (2.15), we have

$$1 + \frac{zf''(z)}{f'(z)} - p = (p-\alpha)\left[\omega(z) + \frac{z\omega'(z)}{p + (p-\alpha)\omega(z)}\right], \; z \in \mathbb{U}. \tag{2.16}$$

Suppose that there exists a point $z_0 \in \mathbb{U}$ such that $\max\limits_{|z| \le |z_0|} |\omega(z)| = |\omega(z_0)| = 1$. From Lemma 1.1 we obtain that $z_0\omega'(z_0) = k\omega(z_0)$ with $k \ge 1$, and from (2.16) we obtain that

$$\begin{aligned}\left|1 + \frac{z_0 f''(z_0)}{f'(z_0)} - p\right| &= (p-\alpha)\,|\omega(z_0)|\left|1 + \frac{k}{p + (p-\alpha)\omega(z_0)}\right| \\ &= (p-\alpha)\left|1 + \frac{k}{p}\,\frac{1}{1 + \left(1 - \dfrac{\alpha}{p}\right)\omega(z_0)}\right|.\end{aligned} \tag{2.17}$$

If we define a function ψ by

$$\psi(z) := \frac{1}{1 + \left(1 - \dfrac{\alpha}{p}\right) z}, \; z \in \mathbb{U},$$

it is easy to check that $|\psi(z)| > \dfrac{p}{2p-\alpha}$ for all $z \in \mathbb{U}$. Hence, from (2.17) we obtain that

$$\left|1 + \frac{z_0 f''(z_0)}{f'(z_0)} - p\right| \ge (p-\alpha)\left|1 + \frac{k}{p}\,\frac{p}{2p-\alpha}\right| \ge \frac{(p-\alpha)(2p+1-\alpha)}{2p-\alpha},$$

which contradicts (2.13). Thus, we conclude that $|\omega(z)| < 1$ for all $z \in \mathbb{U}$, which proves that (2.14) holds. $\square$

Remarks 2.1. (i) For the special case $\gamma = 1$, Theorem 2.3 reduces to the next implication. Let $f \in \mathcal{A}(p)$ be such that $f(z) \neq 0$ for all $z \in \dot{\mathbb{U}}$. Then

$$\left|1+\frac{zf''(z)}{f'(z)}-p\right| < (p-\alpha)\left(1+\frac{1}{p+|p-2\alpha|}\right),\ z \in \mathbb{U}, \tag{2.18}$$

implies (1.1).

(ii) Comparing this result with Theorem 2.4, since $\alpha \in [0, p)$, and using the fact that

$$\frac{(p-\alpha)(2p+1-\alpha)}{2p-\alpha} \geq (p-\alpha)\left(1+\frac{1}{p+|p-2\alpha|}\right) \Leftrightarrow \alpha \in \left[\frac{2p}{3}, p\right),$$

we deduce that Theorem 2.4 gives, for the case $\alpha \in [2p/3, p)$, a better result than the implication (i).

Theorem 2.5. *Suppose that $f \in \mathcal{A}(p)$ satisfies, for $0 \leq \alpha \leq 1$, the inequality*

$$\left|\alpha\left(\frac{zf'(z)}{f(z)}-p\right)+(1-\alpha)\left(\frac{z^2f''(z)}{f(z)}-p(p-1)\right)\right| < p[\alpha+(1-\alpha)p],\ z \in \mathbb{U}. \tag{2.19}$$

Then the inequality (1.3) *holds, i.e., $f \in \mathbb{S}_p^*$.*

Proof. Let $f \in \mathcal{A}(p)$ satisfy the inequality (2.19). Define a function ω by

$$\frac{zf'(z)}{f(z)} = p(1+\omega(z)),\ z \in \mathbb{U}. \tag{2.20}$$

The function ω is analytic in $\mathbb{U}$ with $\omega(0) = 0$. Differentiating (2.20), we have

$$\frac{zf''(z)}{f'(z)} = \frac{zf'(z)}{f(z)} - 1 + \frac{z\omega'(z)}{1+\omega(z)},\ z \in \mathbb{U}.$$

Therefore,

$$\frac{z^2f''(z)}{f(z)} - p(p-1) = 2p^2\omega(z) + p^2\omega^2(z) - p\omega(z) + pz\omega'(z),\ z \in \mathbb{U}. \tag{2.21}$$

From (2.20) and (2.21) we have

$$\begin{aligned}&\alpha\left(\frac{zf'(z)}{f(z)}-p\right)+(1-\alpha)\left(\frac{z^2f''(z)}{f(z)}-p(p-1)\right)\\&= p\omega(z)\left\{\alpha+(1-\alpha)\left[2p-1+\frac{z\omega'(z)}{\omega(z)}+p\omega(z)\right]\right\},\ z \in \mathbb{U},\end{aligned}$$

and hence

$$\begin{aligned}&\left|\alpha\left(\frac{zf'(z)}{f(z)}-p\right)+(1-\alpha)\left(\frac{z^2f''(z)}{f(z)}-p(p-1)\right)\right|\\&=p\,|\omega(z)|\left|\alpha+(1-\alpha)\left[2p-1+\frac{z\omega'(z)}{\omega(z)}\right]+(1-\alpha)p\omega(z)\right|,\ z\in\mathbb{U}.\end{aligned}\tag{2.22}$$

We will prove that that $|\omega(z)|<1$, $z\in\mathbb{U}$. If we suppose that there exists a point $z_0\in\mathbb{U}$ such that $\max\limits_{|z|\le|z_0|}|\omega(z)|=|\omega(z_0)|=1$, from Lemma 1.1 we get that $z_0\omega'(z_0)=k\omega(z_0)$ with $k\ge 1$. Therefore, from (2.22) we obtain

$$\begin{aligned}&\left|\alpha\left(\frac{zf_0'(z_0)}{f(z_0)}-p\right)+(1-\alpha)\left(\frac{z_0^2f''(z_0)}{f(z_0)}-p(p-1)\right)\right|\\&=p\,|\omega(z_0)|\left|\alpha+(1-\alpha)\left[2p-1+\frac{z_0\omega'(z_0)}{\omega(z_0)}\right]+(1-\alpha)p\omega(z_0)\right|\\&\ge p\,[\alpha+(1-\alpha)(2p-1+k)-(1-\alpha)p]\ge p[\alpha+(1-\alpha)p],\end{aligned}$$

which contradicts (2.19). Concluding, we have $|\omega(z)|<1$ for all $z\in\mathbb{U}$, and hence (1.3) holds. □

Putting $\alpha=0$ and $\alpha=1/2$ in Theorem 2.5, we obtain, respectively, the following results.

Corollary 2.6. *If* $f\in\mathcal{A}(p)$ *satisfies*

$$\left|\frac{z^2f''(z)}{f(z)}-p(p-1)\right|<p^2,\ z\in\mathbb{U},$$

then (1.3) *holds, i.e.,* $f\in\mathbb{S}_p^*$.

Corollary 2.7. *If* $f\in\mathcal{A}(p)$ *satisfies*

$$\left|\frac{zf'(z)}{f(z)}+\frac{z^2f''(z)}{f(z)}-p^2\right|<p(p+1),\ z\in\mathbb{U},$$

then (1.3) *holds, i.e.,* $f\in\mathbb{S}_p^*$.

Remarks 2.2. (i) Putting $p=1$ in Theorem 2.4, we obtain the result due to Owa [9, Theorem 1].

(ii) Putting $p=\gamma=1$ and $\alpha=0$ in Theorem 2.4, we obtain the result of Singh and Singh [11, Corollary 3].

(iii) Putting $p=1$ in Theorem 2.5 and Corollary 2.6, we get the results due to Singh and Singh [11, Theorem 4 and Corollary 4].

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