On generalized Lagrange–Hermite–Bernoulli and related polynomials

Waseem A. Khan and M. A. Pathan

Abstract. We introduce a new class of generalized polynomials, ascribed to the family of Hermite, Lagrange, Bernoulli, Miller–Lee, and Laguerre polynomials and of their associated forms. These polynomials can be expressed in the form of generating functions, which allow a high degree of flexibility for the formulation of the relevant theory. We develop a point of view based on generating relations, exploited in the past, to study some aspects of the theory of special functions. We propose a fairly general analysis allowing a transparent link between different forms of special polynomials.

1. Introduction

The Lagrange polynomials in several variables, which are known as the Chan–Chyan–Srivastava polynomials [2], are defined by means of the generating function

$$\prod_{j=1}^{r}(1-x_j t)^{-\alpha_j} = \sum_{n=0}^{\infty} g_n^{(\alpha_1\ldots\alpha_r)}(x_1,\ldots,x_r)t^n \qquad (1.1)$$

where $\alpha_j \in \mathbb{C}$ $(j = 1,\ldots,r)$; $\mid t \mid < \min\{\mid x_1 \mid^{-1},\ldots,\mid x_r \mid^{-1}\}$. These polynomials are represented by

$$g_n^{(\alpha_1,\alpha_2\ldots\alpha_r)}(x_1,x_2,\ldots,x_r) = \sum_{k_{r-1}=0}^{n}\cdots\sum_{k_2=0}^{k_3}\sum_{k_1=0}^{k_2}(\alpha_1)_{k_1}(\alpha_2)_{k_2-k_1}\cdots$$

$$(\alpha_{r-1})_{k_{r-1}-k_{r-2}}(\alpha_r)_{n-k_{r-1}}\frac{x_1^{k_1}}{k_1!}\frac{x_2^{k_2-k_1}}{(k_2-k_1)!}\cdots\frac{x_{r-1}^{k_{r-1}-k_{r-2}}}{(k_{r-1}-k_{r-2})!}\frac{x_r^{n-k_r-1}}{(n-k_r-1)!} \qquad (1.2)$$

where $(\lambda)_0 := 1$ and $(\lambda)_n = \lambda(\lambda+1)\ldots(\lambda+n-1)$ $(n \in \mathbb{N} := \{1,2,\ldots\})$.

Altin and Erkus [1] presented a multivariable extension of the so called Lagrange–Hermite polynomials generated by (see [1, p. 239])

$$\prod_{j=1}^{r}(1 - x_j t^j)^{-\alpha_j} = \sum_{n=0}^{\infty} h_n^{(\alpha_1,\dots,\alpha_r)}(x_1,\dots x_r)t^n \qquad (1.3)$$

where $\alpha_j \in \mathbb{C}$ $(j = 1,\dots,r)$, $\mid t \mid < \min\{\mid x_1 \mid^{-1}, \mid x_2 \mid^{-\frac{1}{2}}, \dots, \mid x_r \mid^{-\frac{1}{r}}\}$, and

$$h_n^{(\alpha_1,\dots,\alpha_r)}(x_1,\dots,x_r) = \sum_{k_1+2k_2+\cdots+rk_r=n}(\alpha_1)_{k_1}\dots(\alpha_r)_{k_r}\frac{x_1^{k_1}}{k_1!}\dots\frac{x_r^{k_r}}{k_r!}.$$

The special case when r = 2 in (1.3) is essentially a case which corresponds to the familiar (two-variable) Lagrange–Hermite polynomials considered by Dattoli et al. [4]:

$$(1 - x_1 t)^{-\alpha_1}(1 - x_2 t^2)^{-\alpha_2} = \sum_{n=0}^{\infty} h_n^{(\alpha_1,\alpha_2)}(x_1,x_2)t^n. \qquad (1.4)$$

The multivariable (Erkus–Srivastava) polynomials $U_{n,l_1,\dots,l_r}^{(\alpha_1,\dots,\alpha_r)}(x_1,\dots,x_r)$, defined by the generating function [5, p. 268]

$$\prod_{j=1}^{r}(1 - x_j t^{l_j})^{-\alpha_j} = \sum_{n=0}^{\infty} U_{n,l_1,\dots,l_r}^{(\alpha_1\dots\alpha_r)}(x_1,\dots,x_r)t^n \qquad (1.5)$$

where $\alpha_j \in \mathbb{C}$ $(j = 1,\dots,r)$, $l_j \in \mathbb{N}$ $(j = 1,\dots,r)$, $\mid t \mid < \min\{\mid x_1 \mid^{-1}, \dots, \mid x_r \mid^{-1}\}$, are a unification (and generalization) of several known families of multivariable polynomials. It is evident that the Chan–Chyan–Srivastava polynomials $g_n^{(\alpha_1\dots\alpha_r)}(x_1,\dots,x_r)$ and the Lagrange–Hermite polynomials $h_n^{(\alpha_1\dots\alpha_r)}(x_1,\dots,x_r)$ are special cases of the Erkus–Srivastava polynomials $U_{n,l_1,\dots,l_r}^{(\alpha_1,\dots,\alpha_r)}(x_1,\dots,x_r)$ when $l_j = 1$ $(j = 1,\dots,r)$.

The generating function (1.5) yields the following explicit representation (see [5, p. 268]):

$$U_{n,l_1,\dots,l_r}^{(\alpha_1,\dots,\alpha_r)}(x_1,\dots,x_r) = \sum_{k_1,\dots,k_r\in\mathbb{N}_0,(l_1k_1+\cdots+l_rk_r=n)}(\alpha_1)_{k_1}\dots(\alpha_r)_{k_r}\frac{x_1^{k_1}}{k_1!}\dots\frac{x_r^{k_r}}{k_r!},$$

which, in the special case $l_j = 1$ $(j = 1,\dots,r)$, corresponds to (1.2).

2. Definitions and basic properties of Lagrange–Hermite–Bernoulli and related polynomials

In this section, we link the Lagrange polynomials in several variables with Hermite and Bernoulli polynomials. The resulting formulae allow a considerable unification of various special results which appear in the literature.

Following Pathan [6[and Pathan and Khan [7], it seems quite appropriate to define Lagrange–Hermite–Bernoulli polynomials ${}_BH_n^{(\alpha_1,\ldots,\alpha_r)}(x|x_1,\ldots x_r)$ in several variables by means of the generating function

$$\frac{t}{e^t-1}e^{xt}\prod_{j=1}^r(1-x_jt^j)^{-\alpha_j} = \sum_{n=0}^\infty {}_BH_n^{(\alpha_1,\ldots,\alpha_r)}(x|x_1,\ldots x_r)t^n,$$

which, for ordinary case $r=2$, reduces to the Lagrange–Hermite–Bernoulli polynomials defined by

$$\frac{t}{e^t-1}e^{xt}(1-x_1t)^{-\alpha_1}(1-x_2t^2)^{-\alpha_2} = \sum_{n=0}^\infty {}_BH_n^{(\alpha_1,\alpha_2)}(x|x_1,x_2)t^n. \qquad (2.1)$$

In particular, when $x_1 = x_2 = 0$, (2.1) reduces to Bernoulli polynomials $B_n(x)$ defined by

$$\frac{t}{e^t-1}e^{xt} = \sum_{n=0}^\infty B_n(x)\frac{t^n}{n!}. \qquad (2.2)$$

The Lagrange–Hermite–Bernoulli polynomials of order α are defined by

$$\left(\frac{t}{e^t-1}\right)^\alpha e^{xt}(1-x_1t)^{-\alpha_1}(1-x_2t^2)^{-\alpha_2} = \sum_{n=0}^\infty {}_BH_n^{(\alpha|\alpha_1,\alpha_2)}(x|x_1,x_2)t^n, \quad (2.3)$$

so that

$$_BH_n^{(\alpha|\alpha_1,\alpha_2)}(x|x_1,x_2) = \sum_{m=0}^n h_m^{(\alpha_1,\alpha_2)}(x_1,x_2)\frac{B_{n-m}^{(\alpha)}(x)}{(n-m)!}.$$

Setting $\alpha = x = 0$ in (2.3), the result reduces to (1.4). For $\alpha = 1$, the equality (2.3) reduces to (2.1).

The Lagrange–Bernoulli polynomials of order α are defined by

$$\left(\frac{t}{e^t-1}\right)^\alpha e^{xt}(1-x_1t)^{-\alpha_1}(1-x_2t)^{-\alpha_2} = \sum_{n=0}^\infty {}_Bg_n^{(\alpha|\alpha_1,\alpha_2)}(x|x_1,x_2)t^n \qquad (2.4)$$

so that

$$_Bg_n^{(\alpha|\alpha_1,\alpha_2)}(x|x_1,x_2) = \sum_{m=0}^n g_m^{(\alpha_1,\alpha_2)}(x_1,x_2)\frac{B_{n-m}^{(\alpha)}(x)}{(n-m)!}.$$

Setting $x = 0$ in (2.4), the result reduces to

$$\left(\frac{t}{e^t-1}\right)^\alpha (1-x_1t)^{-\alpha_1}(1-x_2t)^{-\alpha_2} = \sum_{n=0}^\infty {}_Bg_n^{(\alpha|\alpha_1,\alpha_2)}(x_1,x_2)t^n$$

where

$$_Bg_n^{(\alpha|\alpha_1,\alpha_2)}(0|x_1,x_2) = {}_Bg_n^{(\alpha|\alpha_1,\alpha_2)}(x_1,x_2).$$

Theorem 2.1. *The following implicit summation formula for 2-variable Hermite polynomials* $h_n^{(\alpha_1,\alpha_2)}(x_1, x_2)$ *holds true:*

$$h_n^{(\alpha_1,\alpha_2)}(x_1, x_2) = \sum_{m=0}^{n} {}_B H_{n-m}^{(\alpha_1,\alpha_2)}(x|x_1, x_2)\frac{(-x)^m}{m!}. \qquad (2.5)$$

Proof. For $\alpha = 0$ in (2.3), we have

$$e^{-xt}\sum_{n=0}^{\infty} {}_B H_n^{(\alpha_1,\alpha_2)}(x|x_1, x_2)t^n = (1 - x_1 t)^{-\alpha_1}(1 - x_2 t^2)^{-\alpha_2},$$

which gives

$$\sum_{m=0}^{\infty} \frac{(-x)^m t^m}{m!} \sum_{n=0}^{\infty} {}_B H_n^{(\alpha_1,\alpha_2)}(x|x_1, x_2)t^n = \sum_{n=0}^{\infty} h_n^{(\alpha_1,\alpha_2)}(x_1, x_2)t^n.$$

Replacing n by $n - m$ in the left hand side, we have

$$\sum_{n=0}^{\infty}\sum_{m=0}^{n} {}_B H_{n-m}^{(\alpha_1,\alpha_2)}(x|x_1, x_2)\frac{(-x)^m}{m!}t^n = \sum_{n=0}^{\infty} h_n^{(\alpha_1,\alpha_2)}(x_1, x_2)t^n.$$

Comparing the coefficients of t^n, we get (2.5). $\qquad\square$

Theorem 2.2. *For the Lagrange–Hermite–Bernoulli polynomials, the following implicit summation formula holds true:*

$$_B H_n^{(\alpha|\alpha_1,\alpha_2)}(x + y|x_1, x_2) = \sum_{m=0}^{n} {}_B H_{n-m}^{(\alpha|\alpha_1,\alpha_2)}(x|x_1, x_2)\frac{y^m}{m!}. \qquad (2.6)$$

Proof. From the definition (2.3), we have

$$\sum_{n=0}^{\infty} {}_B H_n^{(\alpha|\alpha_1,\alpha_2)}(x + y|x_1, x_2)t^n = \left(\frac{t}{e^t-1}\right)^{\alpha} e^{xt}e^{yt}(1 - x_1 t)^{-\alpha_1}(1 - x_2 t^2)^{-\alpha_2}$$

$$= \sum_{n=0}^{\infty} {}_B H_n^{(\alpha|\alpha_1,\alpha_2)}(x|x_1, x_2)t^n \sum_{m=0}^{\infty} y^m \frac{t^m}{m!}.$$

Replacing n by $n - m$ in the above equality and comparing the coefficients of t^n, we get the equality (2.6). $\qquad\square$

Theorem 2.3. *For the Lagrange–Hermite–Bernoulli polynomials, the following implicit summation formula holds true:*

$$_B H_n^{(\alpha+\beta|\alpha_1,\alpha_2)}(x + y|x_1, x_2) = \sum_{m=0}^{n} {}_B H_{n-m}^{(\alpha|\alpha_1,\alpha_2)}(x|x_1, x_2)B_m^{(\beta)}(y)\frac{1}{m!}. \qquad (2.7)$$

Proof. Using the definition (2.3), we have

$$\sum_{n=0}^{\infty} {}_B H_n^{(\alpha+\beta|\alpha_1,\alpha_2)}(x + y|x_1, x_2)t^n$$

$$= \left(\tfrac{t}{e^t-1}\right)^{\alpha+\beta} e^{(x+y)t}(1 - x_1 t)^{-\alpha_1}(1 - x_2 t^2)^{-\alpha_2}$$

$$= \sum_{n=0}^{\infty} {}_B H_n^{(\alpha|\alpha_1,\alpha_2)}(x|x_1,x_2)t^n \sum_{m=0}^{\infty} B_m^{(\beta)}(y)\tfrac{t^m}{m!}.$$

Replacing n by $n - m$ in the above equality, we have

$$\sum_{n=0}^{\infty} {}_B H_n^{(\alpha+\beta|\alpha_1,\alpha_2)}(x + y|x_1,x_2)t^n = \sum_{n=0}^{\infty}\sum_{m=0}^{n} {}_B H_{n-m}^{(\alpha|\alpha_1,\alpha_2)}(x|x_1,x_2)B_m^{(\beta)}(y)\tfrac{t^n}{m!}.$$

Comparing the coefficients of t^n, we get (2.7). $\square$

Theorem 2.4. *For the Lagrange–Hermite–Bernoulli polynomials, the following implicit summation formula holds true:*

$$_B H_n^{(\alpha|\alpha_1,\alpha_2)}(x|x_1,x_2) = \sum_{m=0}^{n} {}_B H_{n-m}^{(\alpha|\alpha_1,\alpha_2)}(x - z|x_1,x_2)\tfrac{z^m}{m!}. \qquad (2.8)$$

Proof. By exploiting the generating function (2.3), we have

$$\sum_{n=0}^{\infty} {}_B H_n^{(\alpha|\alpha_1,\alpha_2)}(x|x_1,x_2)t^n = \left(\tfrac{t}{e^t-1}\right)^{\alpha} e^{(x-z)t}e^{zt}(1 - x_1 t)^{-\alpha_1}(1 - x_2 t^2)^{-\alpha_2}$$

$$= \sum_{n=0}^{\infty} {}_B H_n^{(\alpha|\alpha_1,\alpha_2)}(x - z|x_1,x_2)t^n \sum_{m=0}^{\infty} z^m \tfrac{t^m}{m!}.$$

Replacing n by $n - m$ in the above equality, we have

$$\sum_{n=0}^{\infty} {}_B H_n^{(\alpha|\alpha_1,\alpha_2)}(x|x_1,x_2)t^n = \sum_{n=0}^{\infty}\sum_{m=0}^{n} {}_B H_{n-m}^{(\alpha|\alpha_1,\alpha_2)}(x - z|x_1,x_2)z^m \tfrac{t^n}{m!}.$$

Comparing the coefficients of t^n, we get the result (2.8). $\square$

Theorem 2.5. *For the Lagrange–Hermite–Bernoulli polynomials, the following implicit summation formula holds true:*

$$_B H_n^{(\alpha+\beta|\alpha_1,\alpha_2)}(x|x_1,x_2) = \sum_{m=0}^{n} {}_B H_{n-m}^{(\alpha|\alpha_1,\alpha_2)}(z|x_1,x_2)\tfrac{B_m^{(\beta)}(x-z)}{m!}. \qquad (2.9)$$

Proof. Going back to the generating function (2.3), the use of (2.2) gives

$$\sum_{n=0}^{\infty} {}_B H_n^{(\alpha+\beta|\alpha_1,\alpha_2)}(x|x_1,x_2)t^n$$

$$= \left(\tfrac{t}{e^t-1}\right)^{\alpha} e^{zt}(1 - x_1 t)^{-\alpha_1}(1 - x_2 t^2)^{-\alpha_2} \left(\tfrac{t}{e^t-1}\right)^{\beta} e^{(x-z)t}$$

$$= \sum_{n=0}^{\infty} {}_B H_n^{(\alpha|\alpha_1,\alpha_2)}(z|x_1,x_2)t^n \sum_{m=0}^{\infty} \tfrac{B_m^{(\beta)}(x-z)t^m}{m!}.$$

Replacing n by $n-m$ in the above equation and comparing the coefficients of t^n, we get (2.9). $\qquad\square$

Theorem 2.6. *For the Lagrange–Hermite–Bernoulli polynomials, the following implicit summation formula holds true:*

$$\sum_{m=0}^{n} {}_BH_{n-m}^{(\alpha|\alpha_1,\alpha_2)}(x|x_1,x_2)h_m^{(\beta_1,\beta_2)}(x_1,x_2) = {}_BH_n^{(\alpha|\alpha_1+\beta_1,\alpha_2+\beta_2)}(x|x_1,x_2).$$

$$(2.10)$$

Proof. Using (2.4) and (1.4), we have

$$\sum_{n=0}^{\infty} {}_BH_n^{(\alpha|\alpha_1+\beta_1,\alpha_2+\beta_2)}(x|x_1,x_2)t^n$$

$$= \left(\frac{t}{e^t-1}\right)^{\alpha} e^{xt}(1-x_1t)^{-\alpha_1-\beta_1}(1-x_2t^2)^{-\alpha_2-\beta_2}$$

$$= \sum_{n=0}^{\infty} {}_BH_n^{(\alpha|\alpha_1,\alpha_2)}(x|x_1,x_2)t^n \sum_{m=0}^{\infty} h_m^{(\beta_1,\beta_2)}(x_1,x_2)t^m.$$

Replacing n by $n-m$ in the above equality, we have

$$\sum_{n=0}^{\infty}\sum_{m=0}^{n} {}_BH_{n-m}^{(\alpha|\alpha_1,\alpha_2)}(x|x_1,x_2)h_m^{(\beta_1,\beta_2)}(x_1,x_2)t^n$$

$$= \sum_{n=0}^{\infty} {}_BH_n^{(\alpha|\alpha_1+\beta_1,\alpha_2+\beta_2)}(x|x_1,x_2)t^n.$$

Comparing the coefficients of t^n, we get the equality (2.10). $\qquad\square$

Theorem 2.7. *The following implicit summation formula, involving the Lagrange–Hermite–Bernoulli polynomials and Lagrange–Bernoulli polynomials, holds true:*

$$\sum_{m=0}^{n} {}_BH_{n-m}^{(\alpha|\alpha_1,\alpha_2)}(x|x_1,x_2)\frac{(\beta)_m y^m}{m!} = \sum_{m=0}^{n} {}_Bg_{n-m}^{(\alpha|\alpha_1,\beta)}(x|x_1,y)\frac{(x_2)^m(\alpha_2)_m}{m!}.$$

Proof. We start with the generating function (2.3). Multiplying both sides by $(1-yt)^{-\beta}$ and interpreting the result, using (2.4) and series expansion of $(1-yt)^{-\beta}$, we get the required result. $\qquad\square$

3. Generalized Bernoulli–Miller–Lee polynomials

The definitions (2.3) and (2.4) can be exploited in a number of ways. As a first example, we set $\alpha = \alpha_2 = 0$, $\alpha_1 = m+1$, $x_1 = 1$ in (2.3) to get

$$e^{xt}(1-t)^{-m-1} = \sum_{n=0}^{\infty} G_n^{(m)}(x)t^n, \quad |t| < 1.$$

Here $G_n^{(m)}(x)$ are called the Miller–Lee polynomials [3, p. 21].

Another example is the definition of Bernoulli–Hermite–Miller–Lee polynomials ${}_BHG_n^{(m,\alpha)}(x,y)$ given by the generating function

$$\left(\frac{t}{e^t-1}\right)^\alpha \frac{e^{xt+yt^2}}{(1-t)^{m+1}} = \sum_{n=0}^\infty {}_BHG_n^{(m,\alpha)}(x,y)\frac{t^n}{n!}, \tag{3.1}$$

which, for $\alpha = 0$, reduces to

$$\frac{e^{xt+yt^2}}{(1-t)^{m+1}} = \sum_{n=0}^\infty {}_HG_n^{(m)}(x,y)\frac{t^n}{n!}$$

where ${}_HG_n^{(m)}(x,y)$ are called the Hermite–Miller–Lee polynomials.

Putting $y = 0$ into (3.1) gives

$$\left(\frac{t}{e^t-1}\right)^\alpha \frac{e^{xt}}{(1-t)^{m+1}} = \sum_{n=0}^\infty {}_BG_n^{(m,\alpha)}(x)\frac{t^n}{n!} \tag{3.2}$$

where ${}_BG_n^{(m,\alpha)}(x)$ are called the Bernoulli–Miller–Lee polynomials.

Theorem 3.1. *The following implicit summation formula, involving the Lagrange–Hermite–Bernoulli polynomials ${}_BH_n^{(\alpha|\alpha_1,\alpha_2)}(x|x_1,x_2)$, Bernoulli–Miller–Lee polynomials ${}_BG_n^{(m,\alpha)}(x)$, and Miller–Lee polynomials $G_n^{(m)}(x)$, holds true:*

$$\begin{aligned} {}_BG_n^{(m,\alpha)}(x) &= n!\sum_{r=0}^n B_{n-r}^{(\alpha)} G_r^{(m)}(x)\frac{1}{(n-r)!} \\ &= n!\sum_{r=0}^{[\frac{n}{2}]} \frac{(-\alpha_2)_r(x_2)^r}{r!}{}_BH_{n-2r}^{(\alpha|m+1,\alpha_2)}(x|1,x_2). \end{aligned} \tag{3.3}$$

Proof. Setting $x_1 = 1$ and $\alpha_1 = m+1$ in (2.3), and using (3.2), we have

$$\left(\frac{t}{e^t-1}\right)^\alpha e^{xt}(1-t)^{-m-1} = \sum_{n=0}^\infty {}_BG_n^{(m,\alpha)}(x)\frac{t^n}{n!}$$

$$= (1-x_2t^2)^{\alpha_2}\sum_{n=0}^\infty {}_BH_n^{(\alpha|m+1,\alpha_2)}(x|1,x_2)t^n,$$

which, using the binomial expansion, takes the form

$$\sum_{n=0}^\infty B_n^{(\alpha)}\frac{t^n}{n!}\sum_{r=0}^\infty G_r^{(m)}(x)t^r = \sum_{r=0}^\infty \frac{(-\alpha_2)_r(x_2)^r t^{2r}}{r!}\sum_{n=0}^\infty {}_BH_n^{(\alpha|m+1,\alpha_2)}(x|1,x_2)t^n.$$

Now

$$\sum_{n=0}^\infty {}_BG_n^{(m,\alpha)}(x)\frac{t^n}{n!} = \sum_{n=0}^\infty\sum_{r=0}^n B_{n-r}^{(\alpha)} G_r^{(m)}(x)\frac{t^n}{(n-r)!}$$

$$= \sum_{n=0}^{\infty} \sum_{r=0}^{[\frac{n}{2}]} \frac{(-\alpha_2)_r (x_2)^r}{r!} {}_B H_{n-2r}^{(\alpha|m+1,\alpha_2)}(x|1,x_2) t^n.$$

Finally, comparing the coefficients of t^n, we get (3.3). □

Remark 3.1. Equality (3.3) is obviously a series representation of the Bernoulli–Miller–Lee polynomials ${}_B G_n^{(m,\alpha)}(x)$ linking Lagrange–Hermite–Bernoulli and Miller–Lee polynomials.

Theorem 3.2. *The following implicit summation formula, involving the Lagrange–Hermite–Bernoulli polynomials* ${}_B H_n^{(\alpha|\alpha_1,\alpha_2)}(x|x_1,x_2)$ *and Miller–Lee polynomials* $G_n^{(m)}(x)$, *holds true:*

$$ {}_B H_n^{(\alpha|\alpha_1+m+1,\alpha_2)}(x+y|x_1,x_2) = \sum_{r=0}^{n} {}_B H_{n-r}^{(\alpha|\alpha_1,\alpha_2)}(y|x_1,x_2) G_r^{(m)}\left(\frac{x}{x_1}\right) x_1^r. $$

(3.4)

Proof. By replacing x by $x+y$ and α_1 by α_1+m+1 in (2.3), we have

$$\left(\frac{t}{e^t-1}\right)^\alpha e^{(x+y)t}(1-x_1t)^{-m-1}(1-x_1t)^{-\alpha_1}(1-x_2t^2)^{-\alpha_2}$$

$$= \sum_{n=0}^{\infty} {}_B H_n^{(\alpha|\alpha_1+m+1,\alpha_2)}(x+y|x_1,x_2) t^n,$$

which can be written as

$$\sum_{n=0}^{\infty} {}_B H_n^{(\alpha|\alpha_1,\alpha_2)}(y|x_1,x_2) t^n \sum_{r=0}^{\infty} G_r^{(m)}\left(\frac{x}{x_1}\right) x_1^r t^r$$

$$= \sum_{n=0}^{\infty} {}_B H_n^{(\alpha|\alpha_1+m+1,\alpha_2)}(x+y|x_1,x_2) t^n.$$

Now replacing n by $n-r$ in the left hand side of the above equality, we get

$$\sum_{n=0}^{\infty} \sum_{r=0}^{n} {}_B H_{n-r}^{(\alpha|\alpha_1,\alpha_2)}(y|x_1,x_2) G_r^{(m)}\left(\frac{x}{x_1}\right) x_1^r t^n$$

$$= \sum_{n=0}^{\infty} {}_B H_n^{(\alpha|\alpha_1+m+1,\alpha_2)}(x+y|x_1,x_2) t^n.$$

Finally, comparing the coefficients of t^n, we get the equality (3.4). □

Theorem 3.3. *The following implicit summation formula, involving the Lagrange-Hermite-Bernoulli polynomials* ${}_B H_n^{(\alpha|\alpha_1,\alpha_2)}(x|x_1,x_2)$ *and Miller-Lee polynomials* $G_n^{(m)}(x)$, *holds true:*

$$\sum_{r=0}^{n} B_{n-r}^{(\alpha)} {}_H G_r^{(m|\alpha_1,\alpha_2)}(x|x_1,x_2) \frac{1}{(n-r)!} = \sum_{r=0}^{n} (\alpha_1)_r x_1^r {}_B H_{n-r}^{(\alpha|m+1,\alpha_2)}(x|1,x_2) \frac{1}{r!}.$$

Proof. For $\alpha_1 = m + 1$ and $x_1 = 1$ in (2.3), we have

$$\left(\tfrac{t}{e^t-1}\right)^\alpha e^{xt}(1-t)^{-m-1}(1-x_2t^2)^{-\alpha_2} = \sum_{n=0}^{\infty} {}_BH_n^{(\alpha|m+1,\alpha_2)}(x|1,x_2)t^n.$$

Multiplying both sides by $(1 - x_1t)^{-\alpha_1}$, we have

$$\sum_{n=0}^{\infty} B_n^{(\alpha)}\tfrac{t^n}{n!} \sum_{r=0}^{\infty} {}_HG_r^{(m|\alpha_1,\alpha_2)}(x|x_1,x_2)t^n$$

$$= (1 - x_1t)^{-\alpha_1} \sum_{n=0}^{\infty} {}_BH_n^{(\alpha|m+1,\alpha_2)}(x|1,x_2)t^n.$$

Now, replacing n by $n - r$ in the above equality, we get

$$\sum_{n=0}^{\infty}\sum_{r=0}^{n} B_{n-r}^{(\alpha)}\,{}_HG_r^{(m|\alpha_1,\alpha_2)}(x|x_1,x_2)\tfrac{t^n}{(n-r)!}$$

$$= \sum_{n=0}^{\infty}\sum_{r=0}^{n} (\alpha_1)_r(x_1)^r\,{}_BH_{n-r}^{(\alpha|m+1,\alpha_2)}(x|1,x_2)\tfrac{t^n}{r!}.$$

Comparing the coefficients of t^n, we get the desired result.   □

4. Generalized Laguerre–Bernoulli polynomials

We shall be interested in the relation between the Lagrange–Hermite–Bernoulli polynomials ${}_BH_n^{(\alpha|\alpha_1,\alpha_2)}(x|x_1,x_2)$ and Laguerre polynomials $L_n^{(m)}(x)$.

For $x_2 = 0$, $x_1 = -1$, $\alpha_1 = -m$, and $\alpha_2 = 0$ in equation (2.3), we have

$$\left(\tfrac{t}{e^t-1}\right)^\alpha e^{xt}(1+t)^m = \sum_{n=0}^{\infty} {}_BL_n^{(\alpha|m)}(x)\tfrac{t^n}{n!} \tag{4.1}$$

where ${}_BL_n^{(\alpha|m)}(x) = {}_BH_n^{(\alpha|-m,0)}(x|-1,0)$ are called the generalized Laguerre–Bernoulli polynomials.

When $\alpha = 0$ in (4.1), the polynomials ${}_BL_n^{(\alpha|m)}(x)$ reduce to ordinary Laguerre polynomials $L_n^{(m)}(x)$ (see [8]).

Theorem 4.1. *The following implicit summation formula, involving the Lagrange–Hermite–Bernoulli polynomials ${}_BH_n^{(\alpha|\alpha_1,\alpha_2)}(x|x_1,x_2)$ and Laguerre polynomials $L_n^{(m)}(x)$, holds true:*

$$\sum_{r=0}^{n} {}_BH_{n-r}^{(\alpha)}(x|x_1,x_2)L_r^{(m-r)}(y) = \sum_{r=0}^{n} (\alpha)_r x_1^r\,{}_BH_{n-r}^{(\alpha|-m,\alpha_2)}\left(x+y|-1,x_2\right)\tfrac{1}{r!}.$$

Proof. Replacing x by $x + y$ and setting $x_1 = -1$, $\alpha_1 = -m$ in (2.3), we have

$$\left(\tfrac{t}{e^t-1}\right)^\alpha e^{(x+y)t}(1+t)^m(1-x_2t^2)^{-\alpha_2} = \sum_{n=0}^{\infty} {}_BH_n^{(\alpha|-m,\alpha_2)}(x+y|-1,x_2)t^n.$$

Multiplying both sides by $(1-x_1t)^{-\alpha_1}$, we have

$$\left(\tfrac{t}{e^t-1}\right)^\alpha e^{(x+y)t}(1+t)^m(1-x_1t)^{-\alpha_1}(1-x_2t^2)^{-\alpha_2}$$

$$= (1-x_1t)^{-\alpha_1}\sum_{n=0}^{\infty} {}_BH_n^{(\alpha|-m,\alpha_2)}(x+y|-1,x_2)t^n,$$

which gives

$$\sum_{n=0}^{\infty} {}_BH_n^{(\alpha)}(x|x_1,x_2)t^n \sum_{r=0}^{\infty} L_r^{(m-r)}(y)t^r$$

$$= \sum_{r=0}^{\infty} \tfrac{(\alpha)_r(x_1)^rt^r}{r!} \sum_{n=0}^{\infty} {}_BH_n^{(\alpha|-m,\alpha_2)}(x+y|-1,x_2)t^n.$$

Replacing n by $n-r$ in the above equality, we have

$$\sum_{n=0}^{\infty}\sum_{r=0}^{n} {}_BH_{n-r}^{(\alpha)}(x|x_1,x_2)L_r^{(m-r)}(y)t^n$$

$$= \sum_{n=0}^{\infty}\sum_{r=0}^{n}(\alpha)_r x_1^r {}_BH_{n-r}^{(\alpha|-m,\alpha_2)}(x+y|-1,x_2)\tfrac{t^n}{r!}.$$

Comparing the coefficients of t^n, we get the desired result. $\square$

Theorem 4.2. *The following implicit summation formula, involving the Lagrange-Hermite-Bernoulli polynomials ${}_BH_n^{(\alpha|\alpha_1,\alpha_2)}(x|x_1,x_2)$ and Laguerre polynomials $L_n^{(m)}(x)$, holds true:*

$$\sum_{k=0}^{n} B_{n-k}^{(\alpha)}(x)L_k^{(m-k)}(y)\tfrac{1}{(n-k)!} = {}_BH_n^{(\alpha|-m,0)}(x+y|-1,x_2). \qquad (4.2)$$

Proof. Replacing x by $x+y$ and setting $x_1 = -1$, $\alpha_1 = -m$, and $\alpha_2 = 0$ in (2.3), we have

$$\left(\tfrac{t}{e^t-1}\right)^\alpha e^{(x+y)t}(1+t)^m = \sum_{n=0}^{\infty} {}_BH_n^{(\alpha|-m,0)}(x+y|-1,x_2)t^n,$$

and thus

$$\sum_{n=0}^{\infty} B_n^{(\alpha)}(x)\tfrac{t^n}{n!} \sum_{k=0}^{\infty} L_k^{(m-k)}(y)t^k = \sum_{n=0}^{\infty} {}_BH_n^{(\alpha|-m,0)}(x+y|-1,x_2)t^n.$$

Replacing n by $n-k$ in the left hand side of the above equality, we have

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n} B_{n-k}^{(\alpha)}(x)L_k^{(m-k)}(y)\frac{t^n}{(n-k)!} = \sum_{n=0}^{\infty}{}_BH_n^{(\alpha|-m,0)}(x+y|-1,x_2)t^n.$$

Comparing the coefficients of t^n, we get (4.2). $\hfill\square$

Theorem 4.3. *The following implicit summation formula, involving the Lagrange–Hermite–Bernoulli polynomials* ${}_BH_n^{(\alpha|\alpha_1,\alpha_2)}(x|x_1,x_2)$ *and Laguerre polynomials* $L_n^{(m)}(x)$, *holds true:*

$$\sum_{k=0}^{n} {}_BH_{n-k}^{(\alpha|-m+\alpha_1,\alpha_2)}(x|x_1,x_2)(-x_1)^k L_k^{(m-k)}(y/x_1)$$

$$= {}_BH_n^{(\alpha|-m+\alpha_1,\alpha_2)}(x-y|x_1,x_2).$$

Proof. Replacing α_1 by α_1-m and x by $x-y$ in (2.3), we have

$$\sum_{n=0}^{\infty} {}_HB_n^{(\alpha|-m+\alpha_1,\alpha_2)}(x|x_1,x_2)t^n \sum_{k=0}^{\infty}(-x_1)^k t^k L_k^{(m-k)}(y/x_1)$$

$$= \sum_{n=0}^{\infty} {}_BH_n^{(\alpha|-m+\alpha_1,\alpha_2)}(x-y|x_1,x_2)t^n.$$

Replacing n by $n-k$ in the left hand side of the above equality, we have

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n} {}_HB_{n-k}^{(\alpha|-m+\alpha_1,\alpha_2)}(x|x_1,x_2)(-x_1)^k L_k^{(m-k)}(y/x_1)t^n$$

$$= \sum_{n=0}^{\infty} {}_BH_n^{(\alpha|-m+\alpha_1,\alpha_2)}(x-y|x_1,x_2)t^n.$$

Comparing the coefficients of t^n, we get the desired equality. $\hfill\square$

Theorem 4.4. *The following implicit summation formula, involving the Lagrange–Hermite–Bernoulli polynomials* ${}_BH_n^{(\alpha|\alpha_1,\alpha_2)}(x|x_1,x_2)$ *and Laguerre polynomials* $L_n^{(m)}(x)$, *holds true:*

$$\sum_{k=0}^{n} B_{n-k}^{(\alpha)}(x)L_k^{(m-k)}(y)\frac{1}{(n-k)!} = {}_BH_n^{(\alpha|-m,0)}(x+y|-1,x_2). \qquad (4.3)$$

Proof. Setting $x_1=-1$, $\alpha_1=-m$, $\alpha_2=0$, and replacing x by $x-y$ in (2.3), we have

$$\sum_{n=0}^{\infty} B_n^{(\alpha)}(x)\frac{t^n}{n!} \sum_{k=0}^{\infty} L_k^{(m-k)}(-y)t^k = \sum_{n=0}^{\infty} {}_BH_n^{(\alpha|-m,0)}(x-y|-1,x_2)t^n.$$

Replacing n by $n - k$ in the left hand side of the above equality, we have

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n} B_{n-k}^{(\alpha)}(x)L_k^{(m-k)}(-y)\frac{t^n}{(n-k)!} = \sum_{n=0}^{\infty} {}_BH_n^{(\alpha|-m,0)}(x-y|-1,x_2)t^n.$$

Finally, replacing y by $-y$ and comparing the coefficients of t^n, we get (4.3). $\square$

Theorem 4.5. *The following implicit summation formula, involving the Lagrange–Hermite–Bernoulli polynomials ${}_BH_n^{(\alpha|\alpha_1,\alpha_2)}(x|x_1,x_2)$ and Laguerre polynomials $L_n^{(m)}(x)$, holds true:*

$$\sum_{r=0}^{n} B_{n-r}^{(\alpha)}(y)(-x)^n L_r^{(m-r)}(z)\frac{1}{(n-r)!} = {}_BH_n^{(\alpha|-m,0)}(y-z|-1,x_2)(-x)^n.$$

Proof. For $x_1 = -1$, $\alpha_1 = -m$, $\alpha_2 = 0$, $x \longrightarrow y - x$ in (2.3), we have

$$\left(\frac{t}{e^t-1}\right)^{\alpha} e^{(y-x)t}(1+t)^m = \sum_{n=0}^{\infty} {}_BH_n^{(\alpha|-m,0)}(y-x|-1,x_2)t^n.$$

Writing $-zt$ instead of t in the above equality, and after that, replacing x by z and z by x, we have

$$\left(\frac{-xt}{e^{-xt}-1}\right)^{\alpha} e^{-xyt}(1-xt)^m e^{zxt} = \sum_{n=0}^{\infty} {}_BH_n^{(\alpha|-m,0)}(y-z|-1,x_2)(-xt)^n.$$

This gives

$$\sum_{n=0}^{\infty} B_n^{(\alpha)}(y)\frac{(-xt)^n}{n!} \sum_{r=0}^{\infty} (-xt)^r L_r^{(m-r)}(z)$$
$$= \sum_{n=0}^{\infty} {}_BH_n^{(\alpha|-m,0)}(y-z|-1,x_2)(-xt)^n.$$

Replacing n by $n - r$ in the left hand side of the above equality, we have

$$\sum_{n=0}^{\infty}\sum_{r=0}^{n} B_{n-r}^{(\alpha)}(y)(-x)^n L_r^{(m-r)}(z)\frac{t^n}{(n-r)!}$$
$$= \sum_{n=0}^{\infty} {}_BH_n^{(\alpha|-m,0)}(y-z|-1,x_2)(-xt)^n.$$

Finally, comparing the coefficients of t^n, we get the desired equality. $\square$

Theorem 4.6. *The following implicit summation formula, involving the Lagrange–Hermite–Bernoulli polynomials* ${}_BH_n^{(\alpha|\alpha_1,\alpha_2)}(x|x_1,x_2)$ *and generalized Laguerre–Bernoulli polynomials* ${}_BL_n^{(m)}(x)$, *holds true:*

$$\sum_{r=0}^n {}_BL_{n-r}^{(\alpha|m)}(x)\,{}_BL_r^{(\beta|k)}(y)\frac{1}{(n-r)!r!} = {}_BH_n^{(\alpha+\beta|-m-k,0)}(x+y|-1,x_2). \quad (4.4)$$

Proof. Replacing α by $\alpha+\beta$, x by $x+y$, and setting $x_1 = -1$, $\alpha_1 = -m-k$, $\alpha_2 = 0$ in (2.3), we have

$$\left(\tfrac{t}{e^t-1}\right)^\alpha e^{xt}(1+t)^m \left(\tfrac{t}{e^t-1}\right)^\beta e^{yt}(1+t)^k$$
$$= \sum_{n=0}^\infty {}_BH_n^{(\alpha+\beta|-m-k,0)}(x+y|-1,x_2)t^n,$$

which leads directly to

$$\sum_{n=0}^\infty {}_BL_n^{(\alpha|m)}(x)\frac{t^n}{n!} \sum_{r=0}^\infty {}_BL_r^{(\beta|k)}(y)\frac{t^r}{r!}$$
$$= \sum_{n=0}^\infty {}_BH_n^{(\alpha+\beta|-m-k,0)}(x+y|-1,x_2)t^n.$$

Replacing n by $n-r$ in the left hand side of the above equality, we have

$$\sum_{n=0}^\infty \sum_{r=0}^n {}_LB_{n-r}^{(\alpha|m)}(x)\,{}_LB_r^{(\beta|k)}(y)\frac{t^n}{(n-r)!r!}$$
$$= \sum_{n=0}^\infty {}_BH_n^{(\alpha+\beta|-m-k,0)}(x+y|-1,x_2)t^n.$$

Now, comparing the coefficients of t^n, we get (4.4). $\qquad\square$

Acknowledgements

References

[1] A. Altin and E. Erkus, *On a multivariable extension of the Lagrange-Hermite polynomials*, Integral Transforms Spec. Funct. **17** (2006), 239–244.

[2] W. Ch. C. Chan, Ch. J. Chyan, and H. M. Srivastava, *The Lagrange polynomials in several variables*, Integral Transforms Spec. Funct. **12**(2) (2001), 139–148.

[3] G. Dattoli, S. Lorenzutta, and D. Sacchetti, *Integral representations of new families of polynomials*, Ital. J. Pure Appl. Math. **15** (2004), 19–28.

[4] G. Dattoli, P. E. Ricci, and C. Cesarano, *The Lagrange polynomials the associated generalizations, and the umbral calculus*, Integral Transforms Spec. Funct. **14** (2003), 181–186.

[5] E. Erkus and H. M. Srivastava, *A unified presentation of some families of multivariable polynomials*, Integral Transforms Spec. Funct. **17** (2006), 267–273.

[6] M. A. Pathan, *A new class of generalized Hermite-Bernoulli polynomials*, Georgian Math. J. **19** (2012), 559–573.

[7] M. A. Pathan and W. A. Khan, *Some implicit summation formulas and symmetric identities for the generalized Hermite-Bernoulli polynomials*, Mediterr. J. Math. **12** (2015), 679–695.

[8] H. M. Srivastava and H. L. Manocha, *A Treatise on Generating Functions*, Ellis Horwood Ltd., Chichester; Halsted Press [John Wiley & Sons. Inc.], New York, 1984.

DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE, INTEGRAL UNIVERSITY, LUCKNOW-226026, INDIA
    E-mail address: `waseem08_khan@rediffmail.com`

CENTRE FOR MATHEMATICAL AND STATISTICAL SCIENCES (CMSS), KFRI, PEECHI P.O., THRISSUR, KERALA-680653, INDIA
    E-mail address: `mapathan@gmail.com`