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A new characterization of the simple groups C_4(q), by its order and the largest order of elements

Behnam Ebrahimzadeh, Reza Mohammadyari, and Miryousef Sadeghi

Abstract. We prove that the simple group $C_4(q)$, where $q > 2$ and $(q^4+1)/2$ are prime numbers, can be uniquely determined by its order and the largest order of elements.

1. Introduction

All the groups that are considered in this article are finite. We assume that G is a finite group, the set of prime divisors of $|G|$ is denoted by $\pi(G)$, and the set of orders of elements of G is denoted by $\pi_e(G)$. Also, the largest order of elements of G is denoted by $k(G)$. Moreover, we denote by G_p a Sylow p-subgroup of G. The prime graph $\Gamma(G)$ of the group G is a graph whose vertex set is $\pi(G)$, and two vertices p and q are adjacent if and only if $pq \in \pi_e(G)$. Moreover, assume that $\Gamma(G)$ has $t(G)$ connected components π_i, for $i = 1, 2, \ldots, t(G)$. In the case where $|G|$ is of even order, we always assume that $2 \in \pi_1$.

The topic of this paper is group characterization by given properties, that is, showing that there exists only one group with given properties (up to isomorphism). There are different kinds of characterization, for example, the characterization by the set of orders of elements, the prime graph, the number of elements with same order, etc. One of the methods is group characterization by using the order of the group and the largest order of elements. In other words, we say that the group G is characterizable by the order of the group and the largest order of elements if, for any group H, $k(G) = k(H)$ and $|G| = |H|$ imply $G \cong H$. For example, the researchers in [7, 10, 8, 3, 4, 5] proved that the groups $PSL_2(q)$ with $q < 125$, sporadic

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Corresponding author: Behnam Ebrahimzadeh

simple groups $PSL_3(q)$ and $PSU_3(q)$, for some q, the Suzuki groups $Sz(q)$, where $q-1$ and $q\pm\sqrt{2q}+1$, the projective special unitary groups $PSU_3(3^n)$, where $3^{2n}-3^n+1$ is a prime number, and $PSP(8,q)$, where q is an odd prime number, are characterizable by using the order of group and the largest order of elements. In this paper, we prove that the simple groups $C_4(q)$, where $q>2$ and $(q^4+1)/2$ are prime numbers, can be uniquely determined by its order and the largest order of elements. In fact, we prove the following main theorem.

Main Theorem. *Let G be a group with $|G| = |C_4(q)|$ and $k(G) = k(C_4(q))$, where $q > 2$ and $\frac{q^4+1}{2}$ are prime numbers. Then $G \cong C_4(q)$.*

2. Notation and preliminaries

In this section, we give some useful lemmas which will be used in the proof of the main theorem.

Definition 2.1. A group G is called a *Frobenius group* if there is a non-trivial proper subgroup H such that $H \cap H^x = 1$, for each $x \in G - H$. The subgroup $N = G - \cup_{x\in G}(H-1)^x$ is the Frobenius kernel and H is the Frobenius complement.

Lemma 2.2 (see [6]). *Let G be a Frobenius group of even order with kernel K and complement H. Then*

(i) *$t(G) = 2$, $\pi(H)$ and $\pi(K)$ are vertex sets of the connected components of $\Gamma(G)$;*
(ii) *$|H|$ divides $|K|-1$;*
(iii) *K is nilpotent.*

Definition 2.3. A group G is called a 2-*Frobenius group* if there is a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that G/H and K are Frobenius groups with kernels K/H and H, respectively.

Lemma 2.4 (see [1]). *Let G be a 2-Frobenius group of even order. Then*

(i) *$t(G) = 2$, $\pi(H) \cup \pi(G/K) = \pi_1$ and $\pi(K/H) = \pi_2$;*
(ii) *G/K and K/H are cyclic groups such that $|G/K|$ divides $|Aut(K/H)|$.*

Lemma 2.5 (see [15]). *Let G be a finite group with $t(G) \geq 2$. Then one of the following statements holds:*

(i) *G is a Frobenius group;*
(ii) *G is a 2-Frobenius group;*
(iii) *G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that H and G/K are π_1-groups, K/H is a non-abelian simple group, H is a nilpotent group and $|G/K|$ divides $|Out(K/H)|$.*

Lemma 2.6 (see [16]). *Let q, k, l be natural numbers. Then*

(1) $(q^k - 1, q^l - 1) = q^{(k,l)} - 1$;

(2) $(q^k + 1, q^l + 1) = \begin{cases} q^{(k,l)} + 1 & \text{if both } \frac{k}{(k,l)} \text{ and } \frac{l}{(k,l)} \text{ are odd,} \\ (2, q+1) & \text{otherwise ;} \end{cases}$

(3) $(q^k - 1, q^l + 1) = \begin{cases} q^{(k,l)} + 1 & \text{if } \frac{k}{(k,l)} \text{ is even and } \frac{l}{(k,l)} \text{ is odd,} \\ (2, q+1) & \text{otherwise.} \end{cases}$

In particular, for every $q \geq 2$ and $k \geq 1$, the inequality $(q^k - 1, q^k + 1) \leq 2$ holds.

3. Proof of the Main Theorem

We denote the group $C_4(q)$ and the number $(q^4 + 1)/2$ by C and p, respectively. We recall that G is a group with

$$|G| = |C| = \frac{q^{16}(q^8 - 1)(q^6 - 1)(q^4 - 1)(q^2 - 1)}{2}$$

and $k(G) = k(C) = (q^3 - 1)(q + 1)/2$. To prove the main theorem, we prove the following lemmas.

Lemma 3.1. *The number p is an isolated vertex of $\Gamma(G)$.*

Proof. We prove that p is an isolated vertex, in contrary. If we assume that p is not an isolated vertex, then there is a prime number $t \in \pi(G) - p$, so that $tp \in \pi_e(G)$. As a result we have

$$tp \geq 2p = 2(\frac{q^4 + 1}{2}) > \frac{(q^3 - 1)(q + 1)}{2}.$$

Thus $k(G) > (q^3 - 1)(q + 1)/2$, which is a contradiction. Hence $t(G) \geq 2$. □

Lemma 3.2. *The group G is not a Frobenius group.*

Proof. Let G be a Frobenius group with kernel K and complement H. Then by Lemma 2.2, $t(G) = 2$, and $\pi(H)$ and $\pi(K)$ are vertex sets of the connected components of $\Gamma(G)$. Since p is an isolated vertex of $\Gamma(G)$, we have (i) $|H| = |G|/p$ and $|K| = p$, or (ii) $|H| = p$ and $|K| = |G|/p$. Assume that $|H| = |G|/p$ and $|K| = p$. Then Lemma 2.2 implies that $|G|/p$ divides $p - 1$, which is impossible. So the case $|H| = p$ and $|K| = |G|/p$ must occur. Lemma 2.2 implies that p divides $|G|/p - 1$, in other words

$$\frac{q^4 + 1}{2} \mid \frac{q^{16}(q^8 - 1)(q^6 - 1)(q^4 - 1)(q^2 - 1) - 1}{q^4 + 1} - 1.$$

Thus

$$q^4 + 1 \mid 2q^{32} - 2q^{30} - 4q^{28} + 2q^{26} + 4q^{24} + 2q^{22} - 4q^{20} - 2q^{18} + 2q^{16} - 2.$$

As a result,

$$q^4+1 \mid (q^4+1)(2q^{28}-2q^{26}-6q^{24}+4q^{22}+10q^{20}-2q^{18}-14q^{16} + 16q^{12}-16q^8+16q^4-16)+14.$$

So q^4+1 must divide 14, which is a contradiction. Therefore, we deduce that G is not a Frobenius group. □

Lemma 3.3. *The group G is not a 2-Frobenius group.*

Proof. Let G be a 2-Frobenius group. Then by Lemma 2.4, there is a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that G/H and K are Frobenius groups with kernels K/H and H, respectively. Also, we have $t(G)=2$, $\pi(G/K)\cup\pi(H)=\pi_1$, $\pi(K/H)=\pi_2$, and $|G/K|$ divides $|Aut(K/H)|$. Since p is an isolated vertex of $\Gamma(G)$, we deduce that $\pi_2=\{p\}$ and $|K/H|=p$. Hence $|Aut(K/H)|=p-1$. As $|G/K|$ divides $|Aut(K/H)|$, we deduce that $|G/K|$ divides $(q^4-1)/2$. By Lemma 2.6, we have $(p-1,p)=1$ as $(\frac{q^4-1}{2},\frac{q^4+1}{2})=1$. Since $|G/K|$ divides $(q^4-1)/2$, there is a prime divisor t of $\frac{q^4+1}{2}$ such that $t \mid |H|$. Now $K/H \rtimes H_t$ is a Frobenius group with kernel H_t, so $(q^4+1)/2 \mid |H_t|-1$. As a result $(q^4+1)/2 \mid (q^4-1)/2$, which is a contradiction. So G is not a 2-Frobenius group. □

Lemma 3.4. *The group G is isomorphic to the group C.*

Proof. Lemmas 3.2 and 3.3 imply that G satisfies only the statement (c) of Lemma 2.5. Let G have a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that H and G/K are π_1-groups, K/H is a non-abelian simple group, H is a nilpotent group, and $|G/K|$ divides $|Out(K/H)|$. Since p is an isolated vertex of $\Gamma(G)$, according to the classification of the finite simple groups we know that the possibilities for K/H are an alternating group A_m, $m\geq 5$, 26 sporadic groups, and a simple group of Lie types. First, we suppose that K/H is isomorphic to an alternating group.

Step 1. Let $K/H \cong A_m$, where $m\geq 5$ and $m=r'$, $r'+1$, $r'+2$. Then, by [11], $k(A_m)=m$. So $m\geq (q^3-1)(q+1)/2$. On the other hand, $\frac{q^4+1}{2}\in \pi(K/H)$. Now, if $m=(q^3-1)(q+1)/2$, then we know that $|A_m| \mid |G|$, but $(q^3-1)(q+1)!/4 \nmid |G|$, which is a contradiction.

Step 2. If K/H is isomorphic to sporadic groups, then by [15], $\pi(S)=\{11,13,17,19,23,29,31,43,47,67,71\}$, where S is a sporadic group and these numbers are the largest orders elements of sporadic groups. Since the odd order components of a sporadic simple group are primes less than 71, it follows that $(q^4+1)/2<71$, and as a result $q=3$. But $|S| \nmid |C_4(3)|$, which is a contradiction.

Step 3. In this case, we consider K/H which is isomorphic to a group of Lie-type.

Case 3.1. If $K/H \cong {}^3D_4(q')$, then by [11], $k({}^3D_4(q')) = (q'^3-1)(q'+1)$. Also we know that $|{}^3D_4(q')| \mid |G|$. Now, we observe that $(q^3-1)(q+1)/2 = (q'^3-1)(q'+1)$, therefore $(q'^3-1)(q'+1) < (q^3-1)(q+1))$. Hence $q'^4 \leq q^4$. As a result, $q' \leq q$. But ${}^3D_4(q') \nmid |G|$, which is a contradiction.

Case 3.2 Let $K/H \cong E_6(q')$, $E_7(q')$, $E_8(q')$, $F_4(q')$. For example, if $K/H \cong E_8(q')$, then by [11],

$$k(E_8(q')) = (q'+1)(q'^2+q'+1)(q'^5-1).$$

On the other hand,

$$|E_8(q')| = q'^{120}(q'^{30}-1)(q'^{24}-1)(q'^{20}-1)(q'^{18}-1)(q'^{14}-1) \times (q'^{12}-1)(q'^8-1)(q'^2-1) \mid |G|.$$

Hence, we see that

$$\frac{(q^3-1)(q+1)}{2} = (q'+1)(q'^2+q'+1)(q'^5-1),$$

as a result $q'^8 \leq q^4$. Hence $q'^{120} \leq q^{60}$. On the other hand, we know that $q'^{120} \mid |G|$. But $q^{60} \nmid |G|$, which is a contradiction.

For $K/H \ncong E_6(q')$,$E_7(q')$, $F_4(q')$ we have a contradiction similarily.

Case 3.3. If $K/H \cong {}^2E_6(q')$, then by [11]

$$k({}^2E_6(q') = \frac{(q'+1)(q'^2+1)(q'^3-1)}{(3,q'+1)}$$

and also

$$|{}^2E_6(q')| = \frac{q'^{36}(q'^2-1)(q'^5+1)(q'^6-1)}{(3,q'+1)} \mid |G|.$$

Now, we observe that

$$\frac{(q^3-1)(q+1)}{2} = \frac{(q'+1)(q'^2+1)(q'^3-1)}{(3,q'+1)}.$$

If $(3, q'+1) = 1$, then

$$\frac{(q^3-1)(q+1)}{2} = (q'+1)(q'^2+1)(q'^3-1).$$

As a result we have $q'^6 \leq q^4$. Thus $q'^{36} \leq q^{24}$. On the other hand, we know that $q'^{36} \mid |G|$. But $q^{24} \nmid |G|$, which is a contradiction.

Case 3.4. If $K/H \cong {}^2G_2(3^{2m+1})$, where $m \geq 1$, then by [11],

$$k({}^2G_2(3^{2m+1}))) = 3^{2m+1} + 3^{m+1} + 1.$$

Since

$$\frac{(q^3-1)(q+1)}{2} = 3^{2m+1} + 3^{m+1} + 1,$$

so $(q^3-1)(q+1)/2 = 3^{m+1}(3^m+1)$. As a result $q^4+q^3-q = 2(3^{2m+1} + 2(3^{m+1}+3)$. Hence $q(q^3+q^2-1) = 3(2(3^{2m}+2(3^m)+1)$ and it follows that

$q = 3$, $q^3+q^2-1 = 2(3^{2m}+2(3^m)+1$. Therefore, $q^2(q+1) = 2(3^{2m}+3^m+1)$. So $q^2 = 2$, $q+1 = 3^{2m}+3^m+1$, which is a contradiction.

Case 3.5. If $K/H \cong {}^2F_4(q')$, where $q' = 2^{2m+1} > 2$, then by [11],

$$k({}^2F_4(q') = 2^{4m+2} + 2^{3m+2} + 2^{2m+1} + 2^{m+1} + 1.$$

So we have

$$\frac{(q^3-1)(q+1)}{2} = 2^{4m+2)} + 2^{3m+2} + 2^{2m+1} + 2^{m+1} + 1.$$

As a result, $2^{4m+2} \leq q^4$. so $q'^2 \leq q^4$. Consequently, $q'^{12} \leq q^{24}$. On the other hand, we know that $q'^{12} \mid |G|$, but $q^{24} \nmid |G|$, which is a contradiction.

Case 3.6. If $K/H \cong {}^2B_2(2^{2m+1})$, where $m \geq 1$, then by [11],

$$k({}^2B_2(2^{2m+1}))) = 2^{2m+1} + 2^{m+1} + 1,$$

also $|{}^2B_2(2^{2m+1})| \mid |G|$. Now

$$\frac{(q^3-1)(q+1)}{2} = 2^{2m+1} + 2^{m+1} + 1,$$

As a result $q^4+q^3-q-1 = 2^{2m+2}+2^{m+2}+2)$, so

$$(q-1)(q^3+2q^2+2q+1) = 2(2^{2m+1} + (2^{m+1}+1).$$

It follows that $q-1 = 2$ and $q^3+2q^2+2q+1 = 2^{2m+1}+2^{m+1}+1$. Therefore, $q(q^2+2q+2) = 2^{m+1}(2^m+1)$. Hence $q = 2^{m+1}$, $q^2+2q+2 = 2^m+1$. On the other hand, from $q = 3$ we deduce that $3 = 2^{m+1}$, which is a contradiction.

Case 3.7. If $K/H \cong G_2(q')$, then by [11], $k(G_2(q') = q'^2+q'+1$ and also $|G_2(q')| = q'^6(q'^6-1)(q'^2-1) \mid |G|$. Since $\frac{(q^3-1)(q+1)}{2} = q'^2+q'+1$, we have that $q^4+q^3-q = (q'-1)(q'+2)$. So $q'-1 = q$ and $q'+2 = q^3+q^2-1$. Now $|G_2(q')| \nmid |G|$, which is a contradicion.

Case 3.8. If $K/H \cong {}^2A_n(q')$, where $n > 1$. then by [11],

$$k({}^2A_n(q')) = \frac{q'^{2n}-1}{(n+1, q'+1)}.$$

On the other hand, we know that $|{}^2A_n(q')| \mid |G|$. Hence

$$\frac{1}{(n+1, q'+1)} q'^{n(n+1)/2} \prod_{i=2}^{n+1} (q'^i - (-1)^i \mid |G|.$$

We note that

$$\frac{(q^3-1)(q+1)}{2} = \frac{q'^{2n}-1}{(n+1, q'+1)} < q'^{2n} - 1.$$

As a result, $q^4+q^3-q-1 < 2q'^{2n}$. It follows that $q^4 < 2q'^{2n}$. Since $q'^{2n} \mid |G|$, but $\frac{q^4}{2} \nmid |G|$, we have a contradiction.

Case 3.9. Let $K/H \cong D_n(q')$, where $n \geq 4$, then we have a contradiction similarily to the proof of case 3.8.

Case 3.10. Let $K/H \cong L_{n+1}(q')$, where $n \geq 1$. First we assume that $n = 1$, so $K/H \cong L_2(q')$. For this purpose, by [11], $k(L_2(q') = q' + 1, q'$, where q' is even and odd, respectively. Now we consider the equality

$$\frac{(q^3-1)(q+1)}{2} = q', q' + 1.$$

Consequently, $|L_2(q')| \nmid |G|$, which is a contradiction. For $n > 1$, $K/H \ncong L_{n+1}(q')$, similarily.

Case 3.11. Hence $K/H \cong C$. It follows that $|K/H| = |C|$. On the other hand, we know that $H \trianglelefteq K \trianglelefteq G$, where p is an isolated vertex of $\Gamma(G)$. Now, since $k(K/H) \mid k(G)$, we have

$$\frac{(q^3-1)(q+1)}{2} = \frac{(q'^3-1)(q'+1)}{2}.$$

This implies $q = q'$. Since $|K/H| = |C|$ and $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, we deduce that $H = 1$ and $G = K \cong C$. □

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Department of Mathematics, Buinzahra Branch , Islamic Azad University, Buinzahra, Iran

E-mail address: behnam.ebrahimzadeh@gmail.com

E-mail address: rezamohammadyari@gmail.com

Department of Mathematics, Payame Noor University, Tehran 19395-3697, Iran

E-mail address: my.sadeghi@pnu.ac.ir