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Hermite–Hadamard type inequalities via k-fractional integrals concerning differentiable generalized η-convex mappings

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Abstract. The authors discover a new identity concerning differentiable mappings defined on $(\mathbf{m}, g; \theta)$-invex set via k-fractional integrals. By using the obtained identity as an auxiliary result, some new estimates with respect to Hermite–Hadamard type inequalities via k-fractional integrals for generalized-$\mathbf{m}$-$(((h_1 \circ g)^p, (h_2 \circ g)^q); (\eta_1, \eta_2))$-convex mappings are presented. It is pointed out that some new special cases can be deduced from the main results. Also, some applications to special means for different positive real numbers are provided.

1. Introduction

The following notations are used throughout this paper. By I we denote an interval $[a, b]$ with $a, b \in \mathbb{R} = (-\infty, +\infty)$ and $a < b$. For any subset $K \subseteq \mathbb{R}^n$, K° is the interior of K. The set of integrable functions on the interval $[a, b]$ is denoted by $L[a, b]$.

The following inequality, named Hermite–Hadamard inequality, is one of the most famous inequalities in the literature for convex functions.

Theorem 1.1. *Let $f \colon I \longrightarrow \mathbb{R}$ be a convex function on $I \subseteq \mathbb{R}$ and let $a, b \in I$ with $a < b$. Then*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

The inequality (1.1) is also known as trapezium inequality.

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For other recent results which generalize, improve, and extend the inequality (1.1) through various classes of convex functions, interested readers are referred to [1, 3, 6, 10].

Let us recall some special functions and evoke some basic definitions as follows.

Definition 1.2. For $k > 0$ and $x \in \mathbb{C}$, the *k-gamma function* (or k-Pochhammer's symbol) is defined by

$$\Gamma_k(x) = \lim_{n \longrightarrow \infty} \frac{n! k^n (nk)^{\frac{x}{k}-1}}{(x)_{n,k}},$$

where $(x)_{n,k} = x(x+k)\dots(x+(n-1)k)$. Its integral representation is given by

$$\Gamma_k(\alpha) = \int_0^{\infty} t^{\alpha-1} e^{-\frac{t^k}{k}} dt. \tag{1.2}$$

One can note that

$$\Gamma_k(\alpha + k) = \alpha \Gamma_k(\alpha).$$

For $k = 1$, (1.2) gives the integral representation of gamma function.

Definition 1.3 (see [9])**.** Let $f \in L[a, b]$. Then *k-fractional integrals* of order $\alpha, k > 0$ with $a \geq 0$ are defined as

$$I_{a^+}^{\alpha,k} f(x) = \frac{1}{k\Gamma_k(\alpha)} \int_a^x (x-t)^{\frac{\alpha}{k}-1} f(t) dt, \quad x > a,$$

and

$$I_{b^-}^{\alpha,k} f(x) = \frac{1}{k\Gamma_k(\alpha)} \int_x^b (t-x)^{\frac{\alpha}{k}-1} f(t) dt, \quad b > x.$$

For $k = 1$, k-fractional integrals give Riemann–Liouville integrals.

Definition 1.4 (see [14])**.** A set $S \subseteq \mathbb{R}^n$ is said to be *invex* with respect to the mapping $\eta : S \times S \longrightarrow \mathbb{R}^n$ if $x + t\eta(y, x) \in S$ for every $x, y \in S$ and $t \in [0, 1]$.

The invex set S is also termed as the η-connected set.

Definition 1.5 (see [8])**.** Let $h : [0, 1] \longrightarrow \mathbb{R}$ be a non-negative function such that $h \neq 0$. A positive function f on the invex set K is said to be *h-preinvex* with respect to η: $K \times K \longrightarrow \mathbb{R}$ if

$$f\big(x + t\eta(y, x)\big) \leq h(1-t) f(x) + h(t) f(y)$$

for each $x, y \in K$ and $t \in [0, 1]$.

If $\eta(y, x) = y - x$ in Definition 1.5, then the h-preinvex function f reduces to the h-convex mapping f (see [12]).

Definition 1.6 (see [13]). Let $S \subseteq \mathbb{R}^n$ be an invex set with respect to $\eta : S \times S \longrightarrow \mathbb{R}^n$. A function $f\colon S \longrightarrow \mathbb{R}_+ := [0, +\infty)$ is said to be *s-preinvex* (or s-Breckner-preinvex) with respect to η and $s \in (0, 1]$ if, for every $x, y \in S$ and $t \in [0, 1]$,

$$f\big(x + t\eta(y, x)\big) \le (1 - t)^s f(x) + t^s f(y).$$

Definition 1.7 (see [10]). A function $f\colon K \longrightarrow \mathbb{R}$ is said to be *s-Godunova–Levin–Dragomir-preinvex* of second kind, if

$$f\big(x + t\eta(y, x)\big) \le (1 - t)^{-s} f(x) + t^{-s} f(y),$$

for each $x, y \in K,\ t \in (0, 1)$ and $s \in (0, 1]$.

Definition 1.8 (see [11]). A non-negative function $f\colon K \longrightarrow \mathbb{R}$ is said to be *tgs-convex* on $K \subseteq \mathbb{R}$ if the inequality

$$f\big((1 - t)x + ty\big) \le t(1 - t)[f(x) + f(y)]$$

holds for all $x, y \in K$ and $t \in (0, 1)$.

Definition 1.9 (see [7]). A function $f\colon I \longrightarrow \mathbb{R}$ is said to be *MT-convex* if it is non-negative and, for all $x, y \in I$ and $t \in (0, 1)$,

$$f(tx + (1 - t)y) \le \frac{\sqrt{t}}{2\sqrt{1 - t}} f(x) + \frac{\sqrt{1 - t}}{2\sqrt{t}} f(y).$$

The concept of η-convex functions (which at the beginning are named as φ-convex functions) has been introduced in [5] as follows.

Definition 1.10. Consider a convex set $I \subseteq \mathbb{R}$ and a bifunction $\eta : f(I) \times f(I) \longrightarrow \mathbb{R}$. A function $f : I \longrightarrow \mathbb{R}$ is called *η-convex* if

$$f\big(\lambda x + (1 - \lambda)y\big) \le f(y) + \lambda\eta(f(x), f(y))$$

is valid for all $x, y \in I$ and $\lambda \in [0, 1]$.

Geometrically it says that if a function is η-convex on I, then for any $x, y \in I$, its graph is on or under the path starting from $(y, f(y))$ and ending at $(x, f(y) + \eta(f(x), f(y)))$. If $f(x)$ should be the end point of the path for every $x, y \in I$, then we have $\eta(x, y) = x - y$ and the function reduces to a convex one. For more results about η-convex functions, see [2]–[6].

Definition 1.11 (see [1]). Let $I \subseteq \mathbb{R}$ be an invex set with respect to $\eta_1 : I \times I \longrightarrow \mathbb{R}$. Consider $f : I \longrightarrow \mathbb{R}$ and $\eta_2 : f(I) \times f(I) \longrightarrow \mathbb{R}$. The function f is said to be *(η_1, η_2)-convex* if

$$f\big(x + \lambda\eta_1(y, x)\big) \le f(x) + \lambda\eta_2(f(y), f(x))$$

for all $x, y \in I$ and $\lambda \in [0, 1]$.

The main objective of this paper is to establish, in Section 2, some new estimates on Hermite–Hadamard type inequalities via k-fractional integrals associated with generalized-$\mathbf{m}$-$(((h_1 \circ g)^p, (h_2 \circ g)^q); (\eta_1, \eta_2))$-convex mappings. It is pointed out that some new special cases will be deduced from the main results. In Section 3, some applications to special means for different positive real numbers will be obtained.

2. Main results

The following definitions will be used in this section.

Definition 2.1. Let $\theta\colon I \longrightarrow \mathbb{R}$ and $g\colon [0,1] \longrightarrow [0,1]$ be continuous, and let $\mathbf{m}\colon [0,1] \longrightarrow (0,1]$. A set $K \subseteq \mathbb{R}$ is named $(\boldsymbol{m}, g; \theta)$*-invex* with respect to the mapping $\eta\colon \mathbb{R}\times\mathbb{R} \longrightarrow \mathbb{R}$ if $\mathbf{m}(t)\theta(x) + g(\xi)\eta(\theta(y), \mathbf{m}(t)\theta(x)) \in K$ holds for each $x, y \in I$ and any $t, \xi \in [0,1]$.

Remark 2.2. If $\mathbf{m}(t) = m$ for all $t \in [0,1]$, $g(\xi) = \xi$ for all $\xi \in [0,1]$, and $\theta(x) = x$ for all $x \in I$, then the $(\mathbf{m}, g; \theta)$-invex set degenerates to an m-invex set. In the special case $m = 1$ we get Definition 1.4.

We now introduce the concept of generalized-$\mathbf{m}$-$(((h_1 \circ g)^p, (h_2 \circ g)^q);$ $(\eta_1, \eta_2))$-convex mappings. We use the notation $\mathbb{R}_+ := [0, +\infty)$.

Definition 2.3. Let $K \subseteq \mathbb{R}$ be an $(\mathbf{m}, g; \theta)$-invex set with respect to the mapping $\eta_1\colon \mathbb{R}\times\mathbb{R} \longrightarrow \mathbb{R}$, where $\theta\colon I \longrightarrow \mathbb{R}$, $g\colon [0,1] \longrightarrow [0,1]$ are continuous, and $\mathbf{m}\colon [0,1] \longrightarrow (0,1]$. Consider the functions $f : K \longrightarrow (0, +\infty)$ and $\eta_2 : f(K) \times f(K) \longrightarrow \mathbb{R}_+$, and two continuous functions $h_1, h_2 : [0,1] \longrightarrow \mathbb{R}_+$. The mapping f is said to be generalized-$\mathbf{m}$-$(((h_1 \circ g)^p, (h_2 \circ g)^q); (\eta_1, \eta_2))$-convex if the inequality

$$f\big(\mathbf{m}(t)\theta(x) + g(\xi)\eta_1(\theta(y), \mathbf{m}(t)\theta(x))\big) \leq \Big[\mathbf{m}(\xi)(h_1 \circ g)^p(\xi) f^r(x) + (h_2 \circ g)^q(\xi)\eta_2(f^r(y), f^r(x))\Big]^{\frac{1}{r}},$$

holds for all $x, y \in I$, $r \neq 0$, $t, \xi \in [0,1]$, and for any fixed $p, q > -1$.

Remark 2.4. If we choose $p = q = 1$, $\mathbf{m}(t) = m$ $(t \in [0,1])$ and $g(\xi) = \xi$ $(\xi \in [0,1])$ in Definition 2.3, then we get Definition 2.3 in [6]. Setting $m = 1$, $h_1(t) = 1$, and $h_2(t) = t$, we get Definition 1.11. If, in addition, $\eta_1(x, y) = x - y$ and $\eta_2(f(x), f(y)) = \eta(f(x), f(y))$, then Definition 2.3 reduces to Definition 1.10. Under some suitable choices as we have done above, we can get also Definitions 1.6 and 1.7.

Remark 2.5. Let us discuss some special cases of Definition 2.3, where $g(\xi) = \xi$.

(I) Taking $h_1(t) = h(1-t)$ and $h_2(t) = h(t)$, we get generalized-$\mathbf{m}$-$((h^p(1-t), h^q(t)); (\eta_1, \eta_2))$-convex mappings.

(II) Taking $h_1(t) = (1-t)^s$ and $h_2(t) = t^s$ for $s \in (0, 1]$, we get generalized-$\mathbf{m}$-$(((1-t)^{sp}, t^{sq}); (\eta_1, \eta_2))$-Breckner-convex mappings.

(III) Taking $h_1(t) = (1-t)^{-s}$ and $h_2(t) = t^{-s}$ for $s \in (0, 1]$, we get generalized-$\mathbf{m}$-$(((1-t)^{-sp}, t^{-sq}); (\eta_1, \eta_2))$-Godunova–Levin–Dragomir-convex mappings.

(IV) Taking $h_1(t) = h_2(t) = t(1-t)$, we get generalized-$\mathbf{m}$-$((t(1-t))^{sp}, (t(1-t))^{sq}); (\eta_1, \eta_2))$-convex mappings.

(V) Taking $h_1(t) = \sqrt{1-t}/(2\sqrt{t})$ and $h_2(t) = \sqrt{t}/(2\sqrt{1-t})$, we get generalized-$\mathbf{m}$-$\Big(\Big(\Big(\frac{\sqrt{1-t}}{2\sqrt{t}}\Big)^p, \Big(\frac{\sqrt{t}}{2\sqrt{1-t}}\Big)^q\Big); (\eta_1, \eta_2)\Big)$-convex mappings.

It is worth to mention here that to the best of our knowledge all the special cases discussed above are new in the literature.

We give an example of a generalized-$\mathbf{m}$-$(((h_1 \circ g)^p, (h_2 \circ g)^q); (\eta_1, \eta_2))$-convex mapping which is not convex.

Example 2.6. Let θ be an identity function, $g(t) = t^r$ $(r > 0)$, and $h_1(t) = t^l$, $h_2(t) = t^s$ for all $l, s \in [0, 1]$. Consider the function $f\colon \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ defined by

$$f(x) = \begin{cases} x, & 0 \le x \le 2, \\ 4, & x > 2. \end{cases}$$

Define two bifunctions $\eta_1\colon \mathbb{R}_+ \times \mathbb{R}_+ \longrightarrow \mathbb{R}$ and $\eta_2\colon \mathbb{R}_+ \times \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ by

$$\eta_1(x, y) = \begin{cases} -y, & 0 \le y \le 2, \\ x + y, & y > 2, \end{cases}$$

and

$$\eta_2(x, y) = \begin{cases} x + y, & x \le y, \\ 4(x + y), & x > y. \end{cases}$$

Then, for $\mathbf{m} = \frac{1}{2}$ and $p, q \ge 1$, f is a generalized $\frac{1}{2}$-$((t^{rlp}, t^{rsq}); (\eta_1, \eta_2))$-convex mapping. But f is not preinvex with respect to η_1, and it is also not convex (consider $x = 0$, $y = 3$, and $t \in (0, 1]$).

For establishing our main results we need to prove the following lemma.

Lemma 2.7. *Let $\theta\colon I \longrightarrow \mathbb{R}$ be a continuous function, $g\colon [0, 1] \longrightarrow [0, 1]$ be a strictly increasing function on $(0, 1)$, and let $\boldsymbol{m}\colon [0, 1] \longrightarrow (0, 1]$. Suppose that $K = [\boldsymbol{m}(t)\theta(a), \boldsymbol{m}(t)\theta(a) + g(1)\Xi_t(a, b)] \subseteq \mathbb{R}$ is an $(\boldsymbol{m}, g; \theta)$-invex subset with respect to $\Psi\colon \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$, where $\Xi_t(a, b) = \Psi(\theta(b), \boldsymbol{m}(t)\theta(a)) > 0$ for all $t \in [0, 1]$. Assume that $f\colon K \longrightarrow \mathbb{R}$ is a differentiable mapping on K° such that $f' \in L(K)$. Then, for $\alpha, k > 0$ and $\lambda \in [0, 1]$, the equality*

$$T_{f,g}^{\alpha,k}(\Psi, \theta, \boldsymbol{m}; \lambda, a, b) := \frac{\Big[(1 - g(0))^{\frac{\alpha}{k}} - g^{\frac{\alpha}{k}}(0) - \frac{\alpha}{k}(1 - \lambda)\Big] f(\boldsymbol{\sigma}(g(0)))}{2}$$

$$+\frac{\left[g^{\frac{\alpha}{k}}(1)-(1-g(1))^{\frac{\alpha}{k}}+\frac{\alpha}{k}(1-\lambda)\right]f(\boldsymbol{\sigma}(g(1)))}{2}-\frac{\alpha}{2k\Xi_t^{\frac{\alpha}{k}}(a,b)}\int_{\boldsymbol{\sigma}(g(0))}^{\boldsymbol{\sigma}(g(1))}\Big[(w$$
$$-\boldsymbol{m}(t)\theta(a))^{\frac{\alpha}{k}-1}+(\boldsymbol{m}(t)\theta(a)+\Xi_t(a,b)-w)^{\frac{\alpha}{k}-1}\Big]f(w)dw$$

holds, where

$$\boldsymbol{\sigma}(g(\xi)):=\boldsymbol{m}(t)\theta(a)+g(\xi)\Xi_t(a,b)$$

and

$$T_{f,g}^{\alpha,k}(\Psi,\theta,\boldsymbol{m};\lambda,a,b):=\frac{\Xi_t(a,b)}{2}\int_0^1\left(g^{\frac{\alpha}{k}}(\xi)+\frac{\alpha}{k}(1-\lambda)-(1-g(\xi))^{\frac{\alpha}{k}}\right)$$
$$\times f'(\boldsymbol{\sigma}(g(\xi)))dg(\xi).$$

Proof. Integrating by parts and changing the variable $w=\boldsymbol{\sigma}(g(\xi))$, we get

$$T_{f,g}^{\alpha,k}(\Psi,\theta,\mathbf{m};\lambda,a,b)=\frac{\Xi_t(a,b)}{2}\left[\int_0^1 g^{\frac{\alpha}{k}}(\xi)f'(\boldsymbol{\sigma}(g(\xi)))dg(\xi)\right.$$
$$\left.+\frac{\alpha}{k}(1-\lambda)\int_0^1 f'(\boldsymbol{\sigma}(\xi))dg(\xi)-\int_0^1(1-g(\xi))^{\frac{\alpha}{k}}f'(\boldsymbol{\sigma}(g(\xi)))dg(\xi)\right]$$
$$=\frac{\Xi_t(a,b)}{2}\left[\frac{g^{\frac{\alpha}{k}}(\xi)f(\boldsymbol{\sigma}(g(\xi)))}{\Xi_t(a,b)}\Bigg|_0^1-\frac{\alpha}{k\Xi_t(a,b)}\int_0^1 g^{\frac{\alpha}{k}-1}(\xi)f(\boldsymbol{\sigma}(g(\xi)))dg(\xi)\right.$$
$$+\frac{\alpha(1-\lambda)}{k\Xi_t(a,b)}\left[f(\boldsymbol{\sigma}(g(1)))-f(\boldsymbol{\sigma}(g(0)))\right]-\frac{(1-g(\xi))^{\frac{\alpha}{k}}f(\boldsymbol{\sigma}(g(\xi)))}{\Xi_t(a,b)}\Bigg|_0^1$$
$$\left.-\frac{\alpha}{k\Xi_t(a,b)}\int_0^1(1-g(\xi))^{\frac{\alpha}{k}-1}f(\boldsymbol{\sigma}(g(\xi)))dg(\xi)\right]$$
$$=\frac{\left[(1-g(0))^{\frac{\alpha}{k}}-g^{\frac{\alpha}{k}}(0)-\frac{\alpha}{k}(1-\lambda)\right]f(\boldsymbol{\sigma}(g(0)))}{2}$$
$$+\frac{\left[g^{\frac{\alpha}{k}}(1)-(1-g(1))^{\frac{\alpha}{k}}+\frac{\alpha}{k}(1-\lambda)\right]f(\boldsymbol{\sigma}(g(1)))}{2}-\frac{\alpha}{2k\Xi_t^{\frac{\alpha}{k}}(a,b)}\int_{\boldsymbol{\sigma}(g(0))}^{\boldsymbol{\sigma}(g(1))}\Big[(w$$
$$-\mathbf{m}(t)\theta(a))^{\frac{\alpha}{k}-1}+(\mathbf{m}(t)\theta(a)+\Xi_t(a,b)-w)^{\frac{\alpha}{k}-1}\Big]f(w)dw.$$

This completes the proof of our lemma. □

Corollary 2.8. *Under the conditions of Lemma 2.7, for $g(\xi)=\xi$ the following identity for k-fractional integrals holds:*

$$T_f^{\alpha,k}(\Psi,\theta,\boldsymbol{m};\lambda,a,b):=\frac{\Xi_t(a,b)}{2}\int_0^1\left(\xi^{\frac{\alpha}{k}}+\frac{\alpha}{k}(1-\lambda)-(1-\xi)^{\frac{\alpha}{k}}\right)f'(\boldsymbol{\sigma}(\xi))d\xi$$

$$= \frac{\left(1 - \frac{\alpha}{k}(1-\lambda)\right) f(\boldsymbol{\sigma}(0)) + \left(1 + \frac{\alpha}{k}(1-\lambda)\right) f(\boldsymbol{\sigma}(1))}{2}$$

$$- \frac{\Gamma_k(\alpha+k)}{2\Xi_t^{\frac{\alpha}{k}}(a,b)} \left[I^{\alpha,k}_{\boldsymbol{\sigma}(0)^+} f(\boldsymbol{\sigma}(1)) + I^{\alpha,k}_{\boldsymbol{\sigma}(1)^-} f(\boldsymbol{\sigma}(0)) \right],$$

where

$$\boldsymbol{\sigma}(\xi) := \boldsymbol{m}(t)\theta(a) + \xi \Xi_t(a,b).$$

Remark 2.9. Using Corollary 2.8, for $\Xi_t(a,b) = \theta(b) - \mathbf{m}(t)\theta(a)$, where $\mathbf{m}(t) \equiv 1$ and $\lambda = 1$, we get the following Hermite–Hadamard k-fractional integral identity:

$$\frac{f(\theta(a)) + f(\theta(b))}{2} - \frac{\Gamma_k(\alpha+k)}{2(\theta(b)-\theta(a))^{\frac{\alpha}{k}}} \times \left[I^{\alpha,k}_{\theta(a)^+} f(\theta(b)) + I^{\alpha,k}_{\theta(b)^-} f(\theta(a)) \right]$$

$$= \frac{(\theta(b)-\theta(a))}{2} \times \int_0^1 \left(\xi^{\frac{\alpha}{k}} - (1-\xi)^{\frac{\alpha}{k}} \right) f'(\theta(a) + \xi(\theta(b)-\theta(a))) d\xi.$$

Using Lemma 2.7, we now state the following theorem for the corresponding version for a power of the first derivative.

Theorem 2.10. *With the assumptions of Lemma 2.7 let* h_1, $h_2\colon [0,1] \longrightarrow \mathbb{R}_+$ *be continuous functions and let* $\Psi_1\colon f(K) \times f(K) \longrightarrow \mathbb{R}_+$. *If* $(f'(x))^q$ *is a positive generalized-*$\boldsymbol{m}$*-*$(((h_1 \circ g)^{p_1}, (h_2 \circ g)^{p_2}); (\Psi, \Psi_1))$*-convex mapping with* $p_1, p_2 > -1$ *and* $q > 1$*, then the inequality*

$$\left| T^{\alpha,k}_{f,g}(\Psi_1, \theta, \boldsymbol{m}; \lambda, a, b) \right| \leq \frac{\Xi_t(a,b)}{2} \left[\frac{\alpha}{k}(1-\lambda) \sqrt[p]{g(1)-g(0)} \right.$$

$$\left. + \sqrt[p]{\frac{g^{\frac{p\alpha}{k}+1}(1) - g^{\frac{p\alpha}{k}+1}(0)}{\frac{p\alpha}{k}+1}} + \sqrt[p]{\frac{(1-g(0))^{\frac{p\alpha}{k}+1} - (1-g(1))^{\frac{p\alpha}{k}+1}}{\frac{p\alpha}{k}+1}} \right] \tag{2.1}$$

$$\times \sqrt[rq]{(f'(a))^{rq} \mathcal{I}_1^r(h_1 \circ g) + \Psi_1\left((f'(b))^{rq}, (f'(a))^{rq}\right) \mathcal{I}_2^r(h_2 \circ g)},$$

holds for α, $k > 0$ *and* $\lambda \in [0,1]$*, where* $p^{-1} + q^{-1} = 1$,

$$\mathcal{I}_1(h) := \int_0^1 \boldsymbol{m}^{\frac{1}{r}}(\xi) h^{\frac{p_1}{r}}(\xi) dg(\xi),$$

and

$$\mathcal{I}_2(h) := \int_0^1 h^{\frac{p_2}{r}}(\xi) dg(\xi).$$

Proof. From Lemma 2.7, using Hölder's and Minkowski's inequalities, we have

$$\left| T^{\alpha,k}_{f,g}(\Psi_1, \theta, \mathbf{m}; \lambda, a, b) \right| \leq \frac{|\Xi_t(a,b)|}{2} \int_0^1 \left| g^{\frac{\alpha}{k}}(\xi) + \frac{\alpha}{k}(1-\lambda) \right.$$

$$
\begin{aligned}
&- (1-g(\xi))^{\frac{\alpha}{k}}\Big|\big|f'(\boldsymbol{\sigma}(g(\xi)))\big|dg(\xi)\\
&\leq \frac{\Xi_t(a,b)}{2}\left[\int_0^1 \left(g^{\frac{\alpha}{k}}(\xi)+\frac{\alpha}{k}(1-\lambda)\right) f'(\boldsymbol{\sigma}(g(\xi)))dg(\xi)\right.\\
&\left.+\int_0^1 (1-g(\xi))^{\frac{\alpha}{k}} f'(\boldsymbol{\sigma}(g(\xi)))dg(\xi)\right]\\
&\leq \frac{\Xi_t(a,b)}{2}\left[\left(\int_0^1 \left(g^{\frac{\alpha}{k}}(\xi)+\frac{\alpha}{k}(1-\lambda)\right)^p dg(\xi)\right)^{\frac{1}{p}} \left(\int_0^1 \left(f'(\boldsymbol{\sigma}(g(\xi)))\right)^q dg(\xi)\right)^{\frac{1}{q}}\right.\\
&\left.+\left(\int_0^1 (1-g(\xi))^{\frac{p\alpha}{k}} dg(\xi)\right)^{\frac{1}{p}} \left(\int_0^1 \left(f'(\boldsymbol{\sigma}(g(\xi)))\right)^q dg(\xi)\right)^{\frac{1}{q}}\right]\\
&\leq \frac{\Xi_t(a,b)}{2}\left[\left(\int_0^1 g^{\frac{p\alpha}{k}}(\xi)dg(\xi)\right)^{\frac{1}{p}}+\left(\int_0^1 \left(\frac{\alpha}{k}(1-\lambda)\right)^p dg(\xi)\right)^{\frac{1}{p}}\right.\\
&\left.+\left(\int_0^1 (1-g(\xi))^{\frac{p\alpha}{k}} dg(\xi)\right)^{\frac{1}{p}}\right]\left\{\int_0^1 \Big[\mathbf{m}(\xi)(h_1\circ g)^{p_1}(\xi)(f'(a))^{rq}\right.\\
&\left.+(h_2\circ g)^{p_2}(\xi)\Psi_2\left((f'(b))^{rq},(f'(a))^{rq}\right)\Big]^{\frac{1}{r}} dg(\xi)\right\}^{\frac{1}{q}}\\
&\leq \frac{\Xi_t(a,b)}{2}\left[\frac{\alpha}{k}(1-\lambda)\sqrt[p]{g(1)-g(0)}+\sqrt[p]{\frac{g^{\frac{p\alpha}{k}+1}(1)-g^{\frac{p\alpha}{k}+1}(0)}{\frac{p\alpha}{k}+1}}\right.\\
&\left.+\sqrt[p]{\frac{(1-g(0))^{\frac{p\alpha}{k}+1}-(1-g(1))^{\frac{p\alpha}{k}+1}}{\frac{p\alpha}{k}+1}}\right]\\
&\times\left[\left(\int_0^1 \mathbf{m}^{\frac{1}{r}}(\xi)(f'(a))^q(h_1\circ g)^{\frac{p_1}{r}}(\xi)dg(\xi)\right)^r\right.\\
&\left.+\left(\int_0^1 \Psi_1^{\frac{1}{r}}\left((f'(b))^{rq},(f'(a))^{rq}\right)(h_2\circ g)^{\frac{p_2}{r}}(\xi)dg(\xi)\right)^r\right]^{\frac{1}{rq}},
\end{aligned}
$$

which gives (2.1) after simple calculations. □

We point out some special cases of Theorem 2.10.

Corollary 2.11. *In Theorem 2.10, for $p=q=2$ we get*

$$
\begin{aligned}
&\left|T_{f,g}^{\alpha,k}(\Psi_1,\theta,\boldsymbol{m};\lambda,a,b)\right| \leq \frac{\Xi_{t,1}(a,b)}{2}\left[\frac{\alpha}{k}(1-\lambda)\sqrt{g(1)-g(0)}\right.\\
&\left.+\sqrt{\frac{g^{\frac{2\alpha}{k}+1}(1)-g^{\frac{2\alpha}{k}+1}(0)}{\frac{2\alpha}{k}+1}}+\sqrt{\frac{(1-g(0))^{\frac{2\alpha}{k}+1}-(1-g(1))^{\frac{2\alpha}{k}+1}}{\frac{2\alpha}{k}+1}}\right]
\end{aligned}
$$

$$\times \sqrt[2r]{(f'(a))^{2r}\mathcal{I}_1^r(h_1\circ g)+\Psi_1\left((f'(b))^{2r},(f'(a))^{2r}\right)\mathcal{I}_2^r(h_2\circ g)}.$$

Corollary 2.12. *In Theorem 2.10, for $g(\xi)=\xi$ we get the following inequality for k-fractional integrals:*

$$\left|T_f^{\alpha,k}(\Psi_1,\theta,\boldsymbol{m};\lambda,a,b)\right|\le\frac{\Xi_{t,1}(a,b)}{2}\left[\frac{\alpha}{k}(1-\lambda)+2\sqrt[p]{\frac{k}{p\alpha+k}}\right]$$
$$\times\sqrt[rq]{(f'(a))^{rq}\mathcal{I}_1^r(h_1)+\Psi_1\left((f'(b))^{rq},(f'(a))^{rq}\right)\mathcal{I}_2^r(h_2)}.$$

Corollary 2.13. *Under the conditions of Remark 2.9, using Corollary 2.12, we get the following Hermite–Hadamard type inequality:*

$$\left|\frac{f(\theta(a))+f(\theta(b))}{2}-\frac{\Gamma_k(\alpha+k)}{2(\theta(b)-\theta(a))^{\frac{\alpha}{k}}}\left[I^{\alpha,k}_{(\theta(a))^+}f(\theta(b))+I^{\alpha,k}_{(\theta(b))^-}f(\theta(a))\right]\right|$$
$$\le(\theta(b)-\theta(a))\sqrt[p]{\frac{k}{p\alpha+k}}\sqrt[rq]{(f'(a))^{rq}\mathcal{I}_1^r(h_1)+\Psi_1\left((f'(b))^{rq},(f'(a))^{rq}\right)\mathcal{I}_2^r(h_2)}.$$

Corollary 2.14. *In Corollary 2.12, for $h_1(t)=h(1-t)=:\overline{h}(t)$, $h_2(t)=h(t)$, and $\boldsymbol{m}(t)=m\in(0,1]$ for all $t\in[0,1]$, we get the following k-fractional inequality for generalized-m-$((h^{p_1}(1-t),h^{p_2}(t));\ (\Psi,\Psi_1))$-convex mappings:*

$$\left|T_f^{\alpha,k}(\Psi,\theta,m;\lambda,a,b)\right|\le\frac{\Xi_1(a,b)}{2}\left[\frac{\alpha}{k}(1-\lambda)+2\sqrt[p]{\frac{k}{p\alpha+k}}\right]$$
$$\times\sqrt[rq]{m(f'(a))^{rq}\mathcal{I}_1^r(\overline{h})+\Psi_1\left((f'(b))^{rq},(f'(a))^{rq}\right)\mathcal{I}_2^r(h)}.$$

Corollary 2.15. *In Corollary 2.14, for $h_1(t)=(1-t)^s$ and $h_2(t)=t^s$ we get the following k-fractional integral inequality for generalized-m-$(((1-t)^{sp_1},t^{sp_2});(\Psi,\Psi_1))$-Breckner-convex mappings:*

$$\left|T_f^{\alpha,k}(\Psi,\theta,m;\lambda,a,b)\right|\le\frac{\Xi_1(a,b)}{2}\left[\frac{\alpha}{k}(1-\lambda)+2\sqrt[p]{\frac{k}{p\alpha+k}}\right]$$
$$\times\sqrt[rq]{m(f'(a))^{rq}\left(\frac{r}{r+sp_1}\right)^r+\Psi_1\left((f'(b))^{rq},(f'(a))^{rq}\right)\left(\frac{r}{r+sp_2}\right)^r}.$$

Corollary 2.16. *In Corollary 2.14, for $h_1(t)=(1-t)^{-s}$, $h_2(t)=t^{-s}$ and $r>s\cdot\max\{p_1,p_2\}$, we get the following k-fractional integral inequality for generalized-m-$(((1-t)^{-sp_1},t^{-sp_2});(\Psi,\Psi_1))$-Godunova–Levin–Dragomir-convex mappings:*

$$\left|T_f^{\alpha,k}(\Psi,\theta,m;\lambda,a,b)\right|\le\frac{\Xi_1(a,b)}{2}\left[\frac{\alpha}{k}(1-\lambda)+2\sqrt[p]{\frac{k}{p\alpha+k}}\right]$$

$$\times \sqrt[rq]{m(f'(a))^{rq}\left(\frac{r}{r-sp_1}\right)^r + \Psi_1\left((f'(b))^{rq},(f'(a))^{rq}\right)\left(\frac{r}{r-sp_2}\right)^r}.$$

Corollary 2.17. *In Corollary 2.14, for $h_1(t) = h_2(t) = t(1-t)$ and $\boldsymbol{m}(t) = m \in (0,1]$ for all $t \in [0,1]$, we get the following k-fractional integral inequality for generalized-m-$((t(1-t))^{sp_1},(t(1-t))^{sp_2});(\Psi,\Psi_1))$-convex mappings:*

$$\begin{aligned}\left|T_f^{\alpha,k}(\Psi,\theta,m;\lambda,a,b)\right| &\leq \frac{\Xi_1(a,b)}{2}\left[\frac{\alpha}{k}(1-\lambda)+2\sqrt[p]{\frac{k}{p\alpha+k}}\right]\\ &\times\left[m(f'(a))^{rq}\beta^r\left(1+\frac{p_1}{r},1+\frac{p_1}{r}\right)\right.\\ &\left.+\Psi_1\left((f'(b))^{rq},(f'(a))^{rq}\right)\beta^r\left(1+\frac{p_2}{r},1+\frac{p_2}{r}\right)\right]^{\frac{1}{rq}}.\end{aligned}$$

Corollary 2.18. *In Corollary 2.14, for $h_1(t) = \frac{\sqrt{1-t}}{2\sqrt{t}}$, $h_2(t) = \frac{\sqrt{t}}{2\sqrt{1-t}}$ and $r > \frac{1}{2}\max\{p_1,p_2\}$, we get the following k-fractional integral inequality for generalized-m-$\left(\left(\left(\frac{\sqrt{1-t}}{2\sqrt{t}}\right)^{p_1},\left(\frac{\sqrt{t}}{2\sqrt{1-t}}\right)^{p_2}\right);(\Psi,\Psi_1)\right)$-convex mappings:*

$$\begin{aligned}\left|T_f^{\alpha,k}(\Psi,\theta,m;\lambda,a,b)\right| &\leq \frac{\Xi_1(a,b)}{2}\left[\frac{\alpha}{k}(1-\lambda)+2\sqrt[p]{\frac{k}{p\alpha+k}}\right]\\ &\times\left[m(f'(a))^{rq}\left(\frac{1}{2}\right)^{\frac{p_1}{r}}\beta^r\left(1-\frac{p_1}{2r},1+\frac{p_1}{2r}\right)\right.\\ &\left.+\Psi_1\left((f'(b))^{rq},(f'(a))^{rq}\right)\left(\frac{1}{2}\right)^{\frac{p_2}{r}}\beta^r\left(1-\frac{p_2}{2r},1+\frac{p_2}{2r}\right)\right]^{\frac{1}{rq}}.\end{aligned}$$

Theorem 2.19. *With the assumptions of Lemma 2.7 let $h_1, h_2\colon [0,1] \longrightarrow \mathbb{R}_+$ be continuous functions and let $\Psi_1\colon f(K)\times f(K) \longrightarrow \mathbb{R}_+$. If $(f'(x))^q$ is a positive generalized-$\boldsymbol{m}$-$(((h_1\circ g)^{p_1},(h_2\circ g)^{p_2});(\Psi,\Psi_1))$-convex mapping with $p_1,p_2 > -1$ and $q \geq 1$, then the inequality*

$$\begin{aligned}&\left|T_{f,g}^{\alpha,k}(\Psi,\theta,\boldsymbol{m};\lambda,a,b)\right|\\ &\leq \frac{\Xi_t(a,b)}{2}\left\{\left(\frac{\alpha}{k}(1-\lambda)\left(g(1)-g(0)\right)+\frac{g^{\frac{\alpha}{k}+1}(1)-g^{\frac{\alpha}{k}+1}(0)}{\frac{\alpha}{k}+1}\right)^{1-\frac{1}{q}}\right.\\ &\times\sqrt[rq]{(f'(a))^{rq}\mathcal{F}_1^r(h_1\circ g)+\Psi_1\left((f'(b))^{rq},(f'(a))^{rq}\right)\mathcal{F}_2^r(h_2\circ g)} \qquad (2.2)\\ &+\left(\frac{(1-g(0))^{\frac{\alpha}{k}+1}-(1-g(1))^{\frac{\alpha}{k}+1}}{\frac{\alpha}{k}+1}\right)^{1-\frac{1}{q}}\end{aligned}$$

$$\times \sqrt[rq]{(f'(a))^{rq}\mathcal{G}_1^r(h_1\circ g)+\Psi_1\left((f'(b))^{rq},(f'(a))^{rq}\right)\mathcal{G}_2^r(h_2\circ g)}\Bigg\}, \qquad (2.3)$$

holds for α, $k>0$ *and* $\lambda\in[0,1]$, *where*

$$\mathcal{F}_1(h):=\int_0^1 \boldsymbol{m}^{\frac{1}{r}}(\xi)\left(g^{\frac{\alpha}{k}}(\xi)+\frac{\alpha}{k}(1-\lambda)\right)h^{\frac{p_1}{r}}(\xi)dg(\xi),$$
$$\mathcal{F}_2(h):=\int_0^1 \left(g^{\frac{\alpha}{k}}(\xi)+\frac{\alpha}{k}(1-\lambda)\right)h^{\frac{p_2}{r}}(\xi)dg(\xi),$$

and

$$\mathcal{G}_1(h):=\int_0^1 \boldsymbol{m}^{\frac{1}{r}}(\xi)(1-g(\xi))^{\frac{\alpha}{k}}h^{\frac{p_1}{r}}(\xi)dg(\xi),$$
$$\mathcal{G}_2(h):=\int_0^1 (1-g(\xi))^{\frac{\alpha}{k}}h^{\frac{p_2}{r}}(\xi)dg(\xi).$$

Proof. From Lemma 2.7, generalized-$\mathbf{m}$-$(((h_1\circ g)^{p_1},(h_2\circ g)^{p_2});(\Psi,\Psi_1))$-convexity of $(f'(x))^q$, the well-known power mean inequality, and Minkowski's inequality we have

$$\begin{aligned}
&\left|T_{f,g}^{\alpha,k}(\Psi,\theta,\mathbf{m};\lambda,a,b)\right|\\
&\le \frac{|\Xi_t(a,b)|}{2}\int_0^1\left|g^{\frac{\alpha}{k}}(\xi)+\frac{\alpha}{k}(1-\lambda)-(1-g(\xi))^{\frac{\alpha}{k}}\right|\left|f'(\boldsymbol{\sigma}(g(\xi)))\right|dg(\xi)\\
&\le \frac{\Xi_t(a,b)}{2}\left[\int_0^1\left(g^{\frac{\alpha}{k}}(\xi)+\frac{\alpha}{k}(1-\lambda)\right)f'(\boldsymbol{\sigma}(g(\xi)))dg(\xi)\right.\\
&\quad\left.+\int_0^1(1-g(\xi))^{\frac{\alpha}{k}}f'(\boldsymbol{\sigma}(g(\xi)))dg(\xi)\right]\\
&\le \frac{\Xi_t(a,b)}{2}\left[\left(\int_0^1\left(g^{\frac{\alpha}{k}}(\xi)+\frac{\alpha}{k}(1-\lambda)\right)dg(\xi)\right)^{1-\frac{1}{q}}\right.\\
&\quad\times\left(\int_0^1\left(g^{\frac{\alpha}{k}}(\xi)+\frac{\alpha}{k}(1-\lambda)\right)\left(f'(\boldsymbol{\sigma}(g(\xi)))\right)^q dg(\xi)\right)^{\frac{1}{q}}\\
&\quad\left.+\left(\int_0^1(1-g(\xi))^{\frac{\alpha}{k}}dg(\xi)\right)^{1-\frac{1}{q}}\left(\int_0^1(1-g(\xi))^{\frac{\alpha}{k}}\left(f'(\boldsymbol{\sigma}(g(\xi)))\right)^q dg(\xi)\right)^{\frac{1}{q}}\right]\\
&\le \frac{\Xi_t(a,b)}{2}\left\{\left(\frac{\alpha}{k}(1-\lambda)\left(g(1)-g(0)\right)+\frac{g^{\frac{\alpha}{k}+1}(1)-g^{\frac{\alpha}{k}+1}(0)}{\frac{\alpha}{k}+1}\right)^{1-\frac{1}{q}}\right.\\
&\times\left[\int_0^1\left(g^{\frac{\alpha}{k}}(\xi)+\frac{\alpha}{k}(1-\lambda)\right)\left[\mathbf{m}(\xi)(h_1\circ g)^{p_1}(\xi)(f'(a))^{rq}\right.\right.
\end{aligned}$$

$$+(h_2\circ g)^{p_2}(\xi)\Psi_1\left((f'(b))^{rq},(f'(a))^{rq}\right)\Big]^{\frac{1}{r}}dg(\xi)\Big]^{\frac{1}{q}}$$

$$+\left(\frac{(1-g(0))^{\frac{\alpha}{k}+1}-(1-g(1))^{\frac{\alpha}{k}+1}}{\frac{\alpha}{k}+1}\right)^{1-\frac{1}{q}}$$

$$\times\Bigg[\int_0^1(1-g(\xi))^{\frac{\alpha}{k}}\Big[\mathbf{m}(\xi)(h_1\circ g)^{p_1}(\xi)(f'(a))^{rq}$$

$$+(h_2\circ g)^{p_2}(\xi)\Psi_1\left((f'(b))^{rq},(f'(a))^{rq}\right)\Big]^{\frac{1}{r}}dg(\xi)\Bigg]^{\frac{1}{q}}\Bigg\}$$

$$\leq\frac{\Xi_t(a,b)}{2}\Bigg\{\left(\frac{\alpha}{k}(1-\lambda)\left(g(1)-g(0)\right)+\frac{g^{\frac{\alpha}{k}+1}(1)-g^{\frac{\alpha}{k}+1}(0)}{\frac{\alpha}{k}+1}\right)^{1-\frac{1}{q}}$$

$$\times\Bigg[\left(\int_0^1\mathbf{m}^{\frac{1}{r}}(\xi)(f'(a))^q\left(g^{\frac{\alpha}{k}}(\xi)+\frac{\alpha}{k}(1-\lambda)\right)(h_1\circ g)^{\frac{p_1}{r}}(\xi)dg(\xi)\right)^r$$

$$+\left(\int_0^1\Psi_1^{\frac{1}{r}}\left((f'(b))^{rq},(f'(a))^{rq}\right)\left(g^{\frac{\alpha}{k}}(\xi)+\frac{\alpha}{k}(1-\lambda)\right)(h_2\circ g)^{\frac{p_2}{r}}(\xi)dg(\xi)\right)^r\Bigg]^{\frac{1}{rq}}$$

$$+\left(\frac{(1-g(0))^{\frac{\alpha}{k}+1}-(1-g(1))^{\frac{\alpha}{k}+1}}{\frac{\alpha}{k}+1}\right)^{1-\frac{1}{q}}$$

$$\times\Bigg[\left(\int_0^1\mathbf{m}^{\frac{1}{r}}(\xi)(f'(a))^q(1-g(\xi))^{\frac{\alpha}{k}}(h_1\circ g)^{\frac{p_1}{r}}(\xi)dg(\xi)\right)^r$$

$$+\left(\int_0^1\Psi_1^{\frac{1}{r}}\left((f'(b))^{rq},(f'(a))^{rq}\right)(1-g(\xi))^{\frac{\alpha}{k}}(h_2\circ g)^{\frac{p_2}{r}}(\xi)dg(\xi)\right)^r\Bigg]^{\frac{1}{rq}}\Bigg\}.$$

This gives the desired result after simple calculations. □

In what follows we point out some special cases of Theorem 2.19.

Corollary 2.20. *In Theorem 2.19, for $q=1$ we get the inequality*

$$\left|T_{f,g}^{\alpha,k}(\Psi,\theta,\boldsymbol{m};\lambda,a,b)\right|$$

$$\leq\frac{\Xi_t(a,b)}{2}\Bigg\{\sqrt[r]{(f'(a))^r\mathcal{F}_1^r(h_1\circ g)+\Psi_1\left((f'(b))^r,(f'(a))^r\right)\mathcal{F}_2^r(h_2\circ g)}$$

$$+\sqrt[r]{(f'(a))^r\mathcal{G}_1^r(h_1\circ g)+\Psi_1\left((f'(b))^r,(f'(a))^r\right)\mathcal{G}_2^r(h_2\circ g)}\Bigg\}.$$

Corollary 2.21. *In Theorem 2.19, for $g(\xi)=\xi$ we get the following inequality for k-fractional integrals:*

$$\left|T_{f}^{\alpha,k}(\Psi,\theta,\boldsymbol{m};\lambda,a,b)\right|$$

$$\leq \frac{\Xi_t(a,b)}{2}\left\{\left(\frac{k}{\alpha+k}+\frac{\alpha}{k}(1-\lambda)\right)^{1-\frac{1}{q}}\right.$$
$$\times \sqrt[rq]{(f'(a))^{rq}\mathcal{F}_1^r(h_1)+\Psi_1\left((f'(b))^{rq},(f'(a))^{rq}\right)\mathcal{F}_2^r(h_2)}$$
$$\left.+\left(\frac{k}{\alpha+k}\right)^{1-\frac{1}{q}}\sqrt[rq]{(f'(a))^{rq}\mathcal{G}_1^r(h_1)+\Psi_1\left((f'(b))^{rq},(f'(a))^{rq}\right)\mathcal{G}_2^r(h_2)}\right\}.$$

Corollary 2.22. *Under the conditions of Remark 2.9, using Corollary 2.21, we get the following Hermite–Hadamard type inequality:*

$$\left|\frac{f(\theta(a))+f(\theta(b))}{2}-\frac{\Gamma_k(\alpha+k)}{2(\theta(b)-\theta(a))^{\frac{\alpha}{k}}}\left[I^{\alpha,k}_{(\theta(a))^+}f(\theta(b))+I^{\alpha,k}_{(\theta(b))^-}f(\theta(a))\right]\right|$$
$$\leq \frac{(\theta(b)-\theta(a))}{2}\left(\frac{k}{\alpha+k}\right)^{1-\frac{1}{q}}$$
$$\times\left\{\sqrt[rq]{(f'(a))^{rq}\mathcal{F}_1^r(h_1)+\Psi_1\left((f'(b))^{rq},(f'(a))^{rq}\right)\mathcal{F}_2^r(h_2)}\right.$$
$$\left.+\sqrt[rq]{(f'(a))^{rq}\mathcal{G}_1^r(h_1)+\Psi_1\left((f'(b))^{rq},(f'(a))^{rq}\right)\mathcal{G}_2^r(h_2)}\right\}.$$

Corollary 2.23. *In Corollary 2.21, for $h_1(t)=h(1-t)=:\overline{h}(t),\ h_2(t)=h(t)$, and $\boldsymbol{m}(t)=m\in(0,1]$ for all $t\in[0,1]$, we get the following k-fractional inequality for generalized-m-$((h^{p_1}(1-t),h^{p_2}(t));(\Psi,\Psi_1))$-convex mappings:*

$$\left|T_f^{\alpha,k}(\Psi,\theta,m;\lambda,a,b)\right|\leq \frac{\Xi_1(a,b)}{2}\left\{\left(\frac{k}{\alpha+k}+\frac{\alpha}{k}(1-\lambda)\right)^{1-\frac{1}{q}}\right.$$
$$\times\sqrt[rq]{m(f'(a))^{rq}\mathcal{F}_1^r\left(\overline{h}\right)+\Psi_1\left((f'(b))^{rq},(f'(a))^{rq}\right)\mathcal{F}_2^r(h)}$$
$$\left.+\left(\frac{k}{\alpha+k}\right)^{1-\frac{1}{q}}\sqrt[rq]{m(f'(a))^{rq}\mathcal{G}_1^r\left(\overline{h}\right)+\Psi_1\left((f'(b))^{rq},(f'(a))^{rq}\right)\mathcal{G}_2^r(h)}\right\}.$$

Corollary 2.24. *In Corollary 2.23, for $h_1(t)=(1-t)^s$ and $h_2(t)=t^s$ we get the following k-fractional integral inequality for generalized-m-$(((1-t)^{sp_1},t^{sp_2});(\Psi,\Psi_1))$–Breckner-convex mappings:*

$$\left|T_f^{\alpha,k}(\Psi,\theta,m;\lambda,a,b)\right|\leq \frac{\Xi_1(a,b)}{2}\left\{\left(\frac{k}{\alpha+k}+\frac{\alpha}{k}(1-\lambda)\right)^{1-\frac{1}{q}}\right.$$
$$\times\left[m(f'(a))^{rq}\left(\beta\left(\frac{sp_1}{r}+1,\frac{\alpha}{k}+1\right)+\frac{r\alpha}{k(r+sp_1)}(1-\lambda)\right)^r\right.$$
$$\left.+\Psi_1\left((f'(b))^{rq},(f'(a))^{rq}\right)\left(\frac{1}{\frac{sp_2}{r}+\frac{\alpha}{k}+1}+\frac{r\alpha}{k(r+sp_2)}(1-\lambda)\right)^r\right]^{\frac{1}{rq}}$$

$$+\left(\frac{k}{\alpha+k}\right)^{1-\frac{1}{q}}\times\left[m(f'(a))^{rq}\left(\frac{1}{\frac{sp_1}{r}+\frac{\alpha}{k}+1}\right)^{r}\right.$$
$$\left.+\Psi_1\left((f'(b))^{rq},(f'(a))^{rq}\right)\beta^{r}\left(\frac{sp_2}{r}+1,\frac{\alpha}{k}+1\right)\right]^{\frac{1}{rq}}\Bigg\}.$$

Corollary 2.25. *In Corollary 2.23, for $h_1(t)=(1-t)^{-s}$, $h_2(t)=t^{-s}$, and $r>s\max\{p_1,p_2\}$, we get the following k-fractional integral inequality for generalized-m-$(((1-t)^{-sp_1},t^{-sp_2});(\Psi,\Psi_1))$–Godunova–Levin–Dragomir-convex mappings:*

$$\left|T_f^{\alpha,k}(\Psi,\theta,m;\lambda,a,b)\right|\leq\frac{\Xi_1(a,b)}{2}\left\{\left(\frac{k}{\alpha+k}+\frac{\alpha}{k}(1-\lambda)\right)^{1-\frac{1}{q}}\right.$$
$$\times\left[m(f'(a))^{rq}\left(\beta\left(1-\frac{sp_1}{r},\frac{\alpha}{k}+1\right)+\frac{r\alpha}{k(r-sp_1)}(1-\lambda)\right)^{r}\right.$$
$$\left.+\Psi_1\left((f'(b))^{rq},(f'(a))^{rq}\right)\left(\frac{1}{\frac{\alpha}{k}-\frac{sp_2}{r}+1}+\frac{r\alpha}{k(r-sp_2)}(1-\lambda)\right)^{r}\right]^{\frac{1}{rq}}$$
$$+\left(\frac{k}{\alpha+k}\right)^{1-\frac{1}{q}}\times\left[m(f'(a))^{rq}\left(\frac{1}{\frac{\alpha}{k}-\frac{sp_1}{r}+1}\right)^{r}\right.$$
$$\left.\left.+\Psi_1\left((f'(b))^{rq},(f'(a))^{rq}\right)\beta^{r}\left(1-\frac{sp_2}{r},\frac{\alpha}{k}+1\right)\right]^{\frac{1}{rq}}\right\}.$$

Corollary 2.26. *In Corollary 2.23, for $h_1(t)=h_2(t)=t(1-t)$ and $\boldsymbol{m}(t)=m\in(0,1]$ for all $t\in[0,1]$, we get the following k-fractional integral inequality for generalized-m-$((t(1-t))^{sp_1},(t(1-t))^{sp_2});(\Psi,\Psi_1))$-convex mappings:*

$$\left|T_f^{\alpha,k}(\Psi,\theta,m;\lambda,a,b)\right|\leq\frac{\Xi_1(a,b)}{2}\left\{\left(\frac{k}{\alpha+k}+\frac{\alpha}{k}(1-\lambda)\right)^{1-\frac{1}{q}}\left[m(f'(a))^{rq}\Big(\beta\Big(\frac{p_1}{r}\right.\right.$$
$$+\frac{\alpha}{k}+1,\frac{p_1}{r}+1\Big)+\frac{\alpha}{k}(1-\lambda)\beta\left(\frac{p_1}{r}+1,\frac{p_1}{r}+1\right)\Big)^{r}+\Psi_1\big((f'(b))^{rq},(f'(a))^{rq}\big)$$
$$\left.\times\left(\beta\left(\frac{p_2}{r}+\frac{\alpha}{k}+1,\frac{p_2}{r}+1\right)+\frac{\alpha}{k}(1-\lambda)\beta\left(\frac{p_2}{r}+1,\frac{p_2}{r}+1\right)\right)^{r}\right]^{\frac{1}{rq}}$$
$$+\left(\frac{k}{\alpha+k}\right)^{1-\frac{1}{q}}\left[m(f'(a))^{rq}\beta^{r}\left(\frac{p_1}{r}+\frac{\alpha}{k}+1,\frac{p_1}{r}+1\right)\right.$$
$$\left.\left.+\Psi_1\left((f'(b))^{rq},(f'(a))^{rq}\right)\beta^{r}\left(\frac{p_2}{r}+\frac{\alpha}{k}+1,\frac{p_2}{r}+1\right)\right]^{\frac{1}{rq}}\right\}.$$

Corollary 2.27. *In Corollary 2.23, for* $h_1(t) = \dfrac{\sqrt{1-t}}{2\sqrt{t}}$, $h_2(t) = \dfrac{\sqrt{t}}{2\sqrt{1-t}}$, *and* $r > \frac{1}{2}\max\{p_1, p_2\}$, *we get the following* k*-fractional integral inequality for generalized-*m*-*$\left(\left(\left(\frac{\sqrt{1-t}}{2\sqrt{t}}\right)^{p_1}, \left(\frac{\sqrt{t}}{2\sqrt{1-t}}\right)^{p_2}\right); (\Psi, \Psi_1)\right)$*-convex mappings:*

$$\begin{aligned}
\left|T_f^{\alpha,k}(\Psi,\theta,m;\lambda,a,b)\right| &\le \frac{\Xi_1(a,b)}{2}\left\{\left(\frac{k}{\alpha+k}+\frac{\alpha}{k}(1-\lambda)\right)^{1-\frac{1}{q}}\right.\\
&\times\left[m(f'(a))^{rq}\left(\frac{1}{2}\right)^{\frac{p_1}{r}}\left(\beta\left(\frac{\alpha}{k}-\frac{p_1}{2r}+1,\frac{p_1}{2r}+1\right)\right.\right.\\
&\left.+\frac{\alpha}{k}(1-\lambda)\beta\left(1-\frac{p_1}{2r},\frac{p_1}{2r}+1\right)\right)^r+\Psi_1\left((f'(b))^{rq},(f'(a))^{rq}\right)\left(\frac{1}{2}\right)^{\frac{p_2}{r}}\\
&\left.\times\left(\beta\left(\frac{\alpha}{k}+\frac{p_2}{2r}+1,1-\frac{p_2}{2r}\right)+\frac{\alpha}{k}(1-\lambda)\beta\left(\frac{p_2}{2r}+1,1-\frac{p_2}{2r}\right)\right)^r\right]^{\frac{1}{rq}}\\
&+\left(\frac{k}{\alpha+k}\right)^{1-\frac{1}{q}}\times\left[m(f'(a))^{rq}\left(\frac{1}{2}\right)^{\frac{p_1}{r}}\beta^r\left(1-\frac{p_1}{2r},\frac{\alpha}{k}+\frac{p_1}{2r}+1\right)\right.\\
&\left.\left.+\Psi_1\left((f'(b))^{rq},(f'(a))^{rq}\right)\left(\frac{1}{2}\right)^{\frac{p_2}{r}}\beta^r\left(\frac{p_2}{2r}+1,\frac{\alpha}{k}-\frac{p_2}{2r}+1\right)\right]^{\frac{1}{rq}}\right\}.
\end{aligned}$$

Remark 2.28. By taking particular values of parameters α, k, λ, p_1, and p_2 in Theorems 2.10 and 2.19, several k-fractional integral inequalities associated with generalized-$\mathbf{m}$-$(((h_1 \circ g)^{p_1}, (h_2 \circ g)^{p_2}); (\Psi, \Psi_1))$-convex mappings can be obtained. In particular, for $k = 1$, by our Theorems 2.10 and 2.19 we can get some new special Hermite–Hadamard type inequalities via fractional integrals of order $\alpha > 0$. Also, for $\alpha = k = 1$, we can get some new special Hermite–Hadamard type inequalities via classical integrals. The new inequalities may have further applications in many domains of mathematics, statistics, physics and other sciences.

Remark 2.29. Also, applying our Theorems 2.10 and 2.19 for appropriate choices of functions g (for example, $g(x) = x^\alpha$, where $\alpha > 1$, $g(x) = \tan x$, etc.), several k-fractional integral inequalities can be obtained.

Remark 2.30. Finally, applying our Theorems 2.10 and 2.19 for $0 < f'(x) \le L$, $x \in I$, we can get some new k-fractional integral inequalities. The details are left to the interested reader.

3. Applications to special means

Definition 3.1. A function $M : \mathbb{R}_+^2 \longrightarrow \mathbb{R}_+$ is called a *mean function*, if it has the following properties:

(1) Homogeneity: $M(ax, ay) = aM(x, y)$, for all $a > 0$;
(2) Symmetry: $M(x, y) = M(y, x)$;
(3) Reflexivity: $M(x, x) = x$;
(4) Monotonicity: if $x \le x'$ and $y \le y'$, then $M(x, y) \le M(x', y')$;
(5) Internality: $\min\{x, y\} \le M(x, y) \le \max\{x, y\}$.

We consider some special means for arbitrary positive real numbers $a < b$ as follows: the arithmetic mean $A := A(a, b)$, the geometric mean $G := G(a, b)$, the harmonic mean $H := H(a, b)$, the power mean $P_r := P_r(a, b)$, the identric mean $I := I(a, b)$, the logarithmic mean $L := L(a, b)$, the generalized log-mean $L_p := L_p(a, b)$, and the weighted p-power mean M_p. Assume that h_1, h_2, θ, g, Ψ, and Ψ_1 are the same as in Theorems 2.10 and 2.19. Let

$$\begin{aligned}\overline{M} : &= M(\theta(a), \theta(b)) \colon [\theta(a), \theta(a) + g(1)\Psi(\theta(b), \theta(a))]\\ &\times [\theta(a), \theta(a) + g(1)\Psi(\theta(b), \theta(a))] \longrightarrow \mathbb{R}_+,\end{aligned}$$

which is one of the above mentioned means. Therefore, setting $\mathbf{m}(t) \equiv 1$ and $\Psi(\theta(y), \theta(x)) = M(\theta(x), \theta(y))$ for all $x, y \in I$ in (2.1) and (2.2), we obtain the following two inequalities involving means:

$$\begin{aligned}\left|T_{f,g}^{\alpha,k}(M(\cdot,\cdot), \theta, 1; \lambda, a, b)\right| &\le \frac{\overline{M}}{2}\left[\frac{\alpha}{k}(1-\lambda)\sqrt[p]{g(1) - g(0)}\right.\\ &+ \sqrt[p]{\frac{g^{\frac{p\alpha}{k}+1}(1) - g^{\frac{p\alpha}{k}+1}(0)}{\frac{p\alpha}{k}+1}} + \left.\sqrt[p]{\frac{(1-g(0))^{\frac{p\alpha}{k}+1} - (1-g(1))^{\frac{p\alpha}{k}+1}}{\frac{p\alpha}{k}+1}}\right]\\ &\times \sqrt[rq]{(f'(a))^{rq}\mathcal{I}_1^r(h_1 \circ g) + \Psi_1\left((f'(b))^{rq}, (f'(a))^{rq}\right)\mathcal{I}_2^r(h_2 \circ g)},\end{aligned}$$

$$\begin{aligned}&\left|T_{f,g}^{\alpha,k}(M(\cdot,\cdot), \theta, 1; \lambda, a, b)\right|\\ &\le \frac{\overline{M}}{2}\left\{\left(\frac{\alpha}{k}(1-\lambda)(g(1) - g(0)) + \frac{g^{\frac{\alpha}{k}+1}(1) - g^{\frac{\alpha}{k}+1}(0)}{\frac{\alpha}{k}+1}\right)^{1-\frac{1}{q}}\right.\\ &\times \sqrt[rq]{(f'(a))^{rq}\mathcal{F}_1^r(h_1 \circ g)\Psi_1\left((f'(b))^{rq}, (f'(a))^{rq}\right)\mathcal{F}_2^r(h_2 \circ g)}\\ &+ \left(\frac{(1-g(0))^{\frac{\alpha}{k}+1} - (1-g(1))^{\frac{\alpha}{k}+1}}{\frac{\alpha}{k}+1}\right)^{1-\frac{1}{q}}\\ &\times \left.\sqrt[rq]{(f'(a))^{rq}\mathcal{G}_1^r(h_1 \circ g) + \Psi_1\left((f'(b))^{rq}, (f'(a))^{rq}\right)\mathcal{G}_2^r(h_2 \circ g)}\right\}.\end{aligned}$$

Letting $\overline{M} := A, G, H, P_r, I, L, L_p, M_p$ in these inequalities, we get the inequalities involving means for particular choices of positive $(f'(x))^q$ that are generalized-1-$(((h_1 \circ g)^{p_1}, (h_2 \circ g)^{p_2}); (\Psi, \Psi_1))$-convex mappings.

Remark 3.2. Applying our Theorems 2.10 and 2.19 for appropriate choices of functions h_1 and h_2 (see Remark 2.5) such that $(f'(x))^q$ is a positive generalized-1-$(((h_1 \circ g)^{p_1}, (h_2 \circ g)^{p_2}); (\Psi, \Psi_1))$-convex mapping (as, for example, $f(x) = x^\alpha$ $(\alpha > 1,\ x > 0)$, $f(x) = e^x$ $(x \in \mathbb{R})$, $f(x) = \ln x$ $(x > 1)$, etc.), we can deduce some new k-fractional integral inequalities using special means given above. The details are left to the interested reader.

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