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Necessary and sufficient Tauberian conditions for weighted mean methods of summability in two-normed spaces

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Abstract. We first define the concept of weighted mean method of summability and then present necessary and sufficient Tauberian conditions for the weighted mean summability of sequences in two-normed spaces. As corollaries, we establish two-normed analogues of two classical Tauberian theorems.

1. Introduction

The notion of two-normed spaces was first defined by Gähler [7], and has been developed extensively by several authors (see, for example, [3, 5, 6, 9, 18]).

Let X be a real vector space with the dimension $\dim X \geq 2$. A *two-norm* on X is a function $\|\cdot,\cdot\|\colon X \times X \to \mathbb{R}$ which satisfies the conditions

(i) $\|x, y\| = 0$ if and only if x and y are linearly dependent,
(ii) $\|x, y\| = \|y, x\|$ for each $x, y \in X$,
(iii) $\|\alpha x, y\| = |\alpha| \|x, y\|$ for each $x, y \in X$ and $\alpha \in \mathbb{R}$,
(iv) $\|x, y + z\| \leq \|x, y\| + \|x, z\|$ for each $x, y, z \in X$.

The pair $(X, \|\cdot,\cdot\|)$ is then called a *two-normed space*. Observe that in any two-normed space, $\|x, y\|$ is nonnegative and $\|x, y + \alpha x\| = \|x, y\|$ for each $x, y \in X$ and $\alpha \in \mathbb{R}$.

A standard example of a two-normed space is $\mathbb{R}^2$ being equipped with the two-norm

$$\|x, y\| = |x_1 y_2 - x_2 y_1|, \tag{1}$$

where $x = (x_1, x_2)$ and $y = (y_1, y_2)$.

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Let $(x_n : n = 0, 1, 2, \dots)$ be a sequence in a two-normed space $(X, \|\cdot,\cdot\|)$. The sequence (x_n) is said to be *convergent to* $l \in X$ if for every $y \in X$,

$$\lim_{n\to\infty} \|x_n - l, y\| = 0.$$

Furthermore, (x_n) is called *bounded* (notation $x_n = O(1)$) if for every $y \in X$ there exists $H > 0$ such that

$$\|x_n, y\| \leq H \quad (n = 0, 1, 2, \dots).$$

In 2017, Savas and Sezer [13] considered the $(C, 1)$ summability in 2-normed spaces. We introduce the weighted mean method of summability in 2-normed spaces.

Let (p_n) $(n \in \mathbb{N}_0 := \{0, 1, 2, \dots\})$ be a sequence of nonnegative numbers with $p_0 > 0$ such that

$$P_n := \sum_{k=0}^{n} p_k \to \infty, \quad \text{as } n \to \infty. \tag{2}$$

The *weighted means* of the sequence (x_n) are defined by

$$t_n := \frac{1}{P_n} \sum_{k=0}^{n} p_k x_k, \quad n = 0, 1, 2, \dots.$$

The sequence (x_n) is said to be *summable to* $l \in X$ *by the weighted mean method* determined by the sequence (p_n) (briefly, summable $(\overline{N}, p)$ to $l \in X$) if for every $y \in X$,

$$\lim_{n\to\infty} \|t_n - l, y\| = 0. \tag{3}$$

Notice that summability method $(\overline{N}, p)$ reduces to the Cesàro method $(C, 1)$ if $p_n = 1$ for all $n \in \mathbb{N}_0$, and to the logarithmic method $(\ell, 1)$ if $p_n = 1/(n + 1)$ for all $n \in \mathbb{N}_0$.

The following theorem indicates that the convergence of a sequence in a two-normed space always implies the convergence of its weighted means to the same limit.

Theorem 1. *Let the condition* (2) *be satisfied. If a sequence* (x_n) *in* $(X, \|\cdot,\cdot\|)$ *converges to* $l \in X$*, then the sequence* (t_n) *of its weighted means also converges to* l*.*

Proof. If (x_n) converges to l, then, for every $\epsilon > 0$ and $y \in X$, there exists a positive integer n_0 such that $\|x_n - l, y\| \leq \epsilon/2$ if $n > n_0$, and there exists

$H > 0$ such that $\|x_n - l, y\| \le H$ if $n \le n_0$. Hence, for every $y \in X$, we get

$$\begin{aligned}
\|t_n - l, y\| &= \left\| \frac{1}{P_n} \sum_{k=0}^{n} p_k x_k - \frac{1}{P_n} \sum_{k=0}^{n} p_k l, y \right\| \\
&= \left\| \frac{1}{P_n} \sum_{k=0}^{n} p_k (x_k - l), y \right\| \le \frac{1}{P_n} \sum_{k=0}^{n} p_k \|x_k - l, y\| \\
&= \frac{1}{P_n} \sum_{k=0}^{n_0} p_k \|x_k - l, y\| + \frac{1}{P_n} \sum_{k=n_0+1}^{n} p_k \|x_k - l, y\| \\
&\le \frac{H P_{n_0}}{P_n} + \frac{\epsilon}{2}.
\end{aligned}$$

Since $P_{n_0}/P_n \to 0$ as $n \to \infty$, there exists a positive integer n_1 such that $|HP_{n_0}/P_n| \le \epsilon/2$ if $n > n_1$. Therefore, $\|t_n - l, y\| \le \epsilon$ if $n > \max\{n_0, n_1\}$. □

Theorem 1 expresses the so-called regularity property of the weighted mean method of summability in two-normed spaces.

Example 2. Let $X = \mathbb{R}^2$ be equipped with the two-norm (1). Define the sequence (x_n) in X by

$$x_n = \left(1 + (-1)^{n+1}, \tfrac{1}{2} + \tfrac{(-1)^n}{2}\right),$$

and let $y = (y_1, y_2)$. Then, putting $p_n = 1$ for all n, we find the $(C, 1)$ mean of (x_n) as

$$t_n = \begin{cases} \left(\frac{n}{n+1}, \frac{n+2}{2n+2}\right) & \text{if } n \text{ is even}, \\ \left(1, \frac{1}{2}\right) & \text{if } n \text{ is odd}. \end{cases}$$

In the case n is even, we obtain that, for all $y = (y_1, y_2) \in \mathbb{R}^2$,

$$\begin{aligned}
\lim_{n\to\infty} \|t_n - l, y\| &= \lim_{n\to\infty} \left\| \left(\tfrac{n}{n+1}, \tfrac{n+2}{2n+2}\right) - \left(1, \tfrac{1}{2}\right), (y_1, y_2) \right\| \\
&= \lim_{n\to\infty} \left\| \left(\tfrac{-1}{n+1}, \tfrac{1}{2n+2}\right), (y_1, y_2) \right\| \\
&= \lim_{n\to\infty} \left| \tfrac{-y_2}{n+1} - \tfrac{y_1}{2n+2} \right| = 0.
\end{aligned}$$

If n is odd, then $(t_n) = (1, 1/2)$. Therefore, it follows from (3) that (x_n) is $(C, 1)$ summable to $(1, 1/2)$.

Besides, if (x_n) is convergent, the limit should be $(1, 1/2)$. However, when $y = (0, 1)$, since

$$\begin{aligned}
\lim_{n\to\infty} \|x_n - L, y\| &= \lim_{n\to\infty} \left\| \left((-1)^{n+1}, \tfrac{(-1)^n}{2}\right), (y_1, y_2) \right\| \\
&= \lim_{n\to\infty} \left| (-1)^{n+1} y_2 - \tfrac{(-1)^n y_1}{2} \right| \\
&= \lim_{n\to\infty} \left| (-1)^{n+1} \right| = 1 \ne 0,
\end{aligned}$$

we have that the sequence (x_n) is not convergent.

Example 2 shows that the converse of Theorem 1 is not true in general. Thus we are led to the problem of finding additional conditions which, together with assumption (3), would assure the convergence of the sequence (x_n). Such conditions are called Tauberian conditions and the resulting theorem is called a Tauberian theorem. Tauberian type theorems have a long history: see, for example, the book [10] and the papers [1, 2, 4, 12, 16, 17].

In this work, our purpose is to obtain Tauberian conditions under which the convergence of (x_n) in a two-normed space follows from its $(\overline{N}, p)$ summability.

2. Main results

Theorem 3. *Let (p_n) be a sequence of nonnegative numbers such that $p_0 > 0$ and*

$$\liminf_{n\to\infty} \frac{P_{[\lambda n]}}{P_n} > 1 \quad \text{for every} \quad \lambda > 1, \tag{4}$$

where $[\lambda n]$ denotes the integer part of the product λn, and let (x_n) be a sequence in $(X, ||\cdot,\cdot||)$ which is $(\overline{N}, p)$ summable to $l \in X$. Then (x_n) converges to l if and only if one of the following two conditions is satisfied:

$$\inf_{\lambda>1} \limsup_{n\to\infty} \left\| \frac{1}{P_{[\lambda n]} - P_n} \sum_{k=n+1}^{[\lambda n]} p_k(x_k - x_n), y \right\| = 0, \tag{5}$$

$$\inf_{0<\lambda<1} \limsup_{n\to\infty} \left\| \frac{1}{P_n - P_{[\lambda n]}} \sum_{k=[\lambda n]+1}^{n} p_k(x_n - x_k), y \right\| = 0 \tag{6}$$

for all $y \in X$.

Remark 4. It is clear that the condition (4) implies (2).

Remark 5. Consider the special cases of (5) and (6) for $p_n = 1$ for all $n \in \mathbb{N}_0$. In this case, conditions (5) and (6) reduce to

$$\inf_{\lambda>1} \limsup_{n\to\infty} \left\| \frac{1}{[\lambda n] - n} \sum_{k=n+1}^{[\lambda n]} (x_k - x_n), y \right\| = 0 \tag{7}$$

and

$$\inf_{0<\lambda<1} \limsup_{n\to\infty} \left\| \frac{1}{n - [\lambda n]} \sum_{k=[\lambda n]+1}^{n} (x_n - x_k), y \right\| = 0, \tag{8}$$

respectively.

Savaş and Sezer [13] used Tauberian conditions (7) and (8) to retrieve convergence of (x_n) from the $(C, 1)$ summability.

Remark 6. Following Schmidt [14] (or Stanojević [15]), a sequence (x_n) is said to be *slowly oscillating in two-norm* if for every $y \in X$,

$$\inf_{\lambda>1} \limsup_{n\to\infty} \max_{n<k\leq[\lambda n]} \|x_k - x_n, y\| = 0 \tag{9}$$

or equivalently,

$$\inf_{0<\lambda<1} \limsup_{n\to\infty} \max_{[\lambda n]<k\leq n} \|x_n - x_k, y\| = 0.$$

Notice that, if (x_n) is slowly oscillating in two-norm, then (7) and (8) hold.

Remark 7. If the two-sided condition $\frac{P_n}{p_n}\Delta x_n = O(1)$ of Hardy [8] type, where $\Delta x_n = x_n - x_{n-1}$, is satisfied under some appropriate condition imposed on (p_n), then (x_n) is slowly oscillating in two-norm.

Taking into account the Remarks 6 and 7, we obtain the following two-normed analogues of some classical Tauberian theorems.

Corollary 8. *Let* (4) *be satisfied and let* (x_n) *be* $(\overline{N}, p)$ *summable to* $l \in X$. *If* (x_n) *is slowly oscillating in two-norm, then* (x_n) *converges to* l.

Corollary 9. *Let*

$$\lim_{n\to\infty} \frac{P_{[\lambda n]}}{P_n} = \lambda^\delta, \quad \lambda > 1, \tag{10}$$

be satisfied for some $\delta > 0$ *and let* (x_n) *be* $(\overline{N}, p)$ *summable to* $l \in X$. *If*

$$\frac{P_n}{p_n}\Delta x_n = O(1), \tag{11}$$

then (x_n) *converges to* l.

Remark 10. It is clear that the condition (10) implies (4).

3. Auxiliary results

Lemma 11 (see [11])**.** *If* (P_n) *is a nondecreasing sequence of positive numbers, then* (4) *is equivalent to the condition*

$$\liminf_{n\to\infty} \frac{P_n}{P_{[\lambda n]}} > 1 \quad \textit{for every} \quad 0 < \lambda < 1. \tag{12}$$

In the proof of the main theorem we need the following lemma on the so-called moving weighted averages.

Lemma 12. *Let* (p_n) *be a sequence of nonnegative real numbers such that* $p_0 > 0$ *and the condition* (4) *is satisfied. If* (x_n) *is* $(\overline{N}, p)$ *summable to* $l \in X$, *then, for each* $y \in X$,

$$\lim_{n\to\infty} \left\| \frac{1}{P_{[\lambda n]} - P_n} \sum_{k=n+1}^{[\lambda n]} p_k x_k - l, y \right\| = 0 \quad \textit{for every} \quad \lambda > 1 \tag{13}$$

and

$$\lim_{n\to\infty}\left\|\frac{1}{P_n-P_{[\lambda n]}}\sum_{k=[\lambda n]+1}^{n}p_kx_k-l,y\right\|=0 \quad \textit{for every} \quad 0<\lambda<1. \tag{14}$$

Proof. For brevity, we denote the moving weighted averages $(\tau_n^>)$ for $\lambda>1$ and $(\tau_n^<)$ for $0<\lambda<1$, respectively, by

$$\tau_n^> = \frac{1}{P_{[\lambda n]}-P_n}\sum_{k=n+1}^{[\lambda n]}p_kx_k \quad\text{and}\quad \tau_n^< = \frac{1}{P_n-P_{[\lambda n]}}\sum_{k=[\lambda n]+1}^{n}p_kx_k.$$

Consider the case $\lambda>1$. Then

$$\begin{aligned}\left\|\tau_n^>-l,y\right\| &= \left\|\tau_n^>+t_{[\lambda n]}-t_{[\lambda n]}-l,y\right\| = \Bigg\|\frac{1}{P_{[\lambda n]}-P_n}\sum_{k=0}^{[\lambda n]}p_kx_k \\ &\quad -\frac{1}{P_{[\lambda n]}-P_n}\sum_{k=0}^{n}p_kx_k+\frac{1}{P_{[\lambda n]}}\sum_{k=0}^{[\lambda n]}p_kx_k-\frac{1}{P_{[\lambda n]}}\sum_{k=0}^{[\lambda n]}p_kx_k-l,y\Bigg\| \\ &= \left\|\frac{P_n}{P_{[\lambda n]}-P_n}\frac{1}{P_{[\lambda n]}}\sum_{k=0}^{[\lambda n]}p_kx_k-\frac{P_n}{P_{[\lambda n]}-P_n}\frac{1}{P_n}\sum_{k=0}^{n}p_kx_k+\frac{1}{P_{[\lambda n]}}\sum_{k=0}^{[\lambda n]}p_kx_k-l,y\right\|.\end{aligned}$$

Thus we see that

$$\left\|\tau_n^>-l,y\right\| \le \frac{P_n}{P_{[\lambda n]}-P_n}\left\|t_{[\lambda n]}-t_n,y\right\|+\left\|t_{[\lambda n]}-l,y\right\|. \tag{15}$$

Moreover, from (4) we have

$$\limsup_{n\to\infty}\frac{P_n}{P_{[\lambda n]}-P_n}=\left(\liminf_{n\to\infty}\frac{P_{[\lambda n]}}{P_n}-1\right)^{-1}<\infty.$$

Now, (13) follows from (15) and the $(\overline{N},p)$ summability of (x_n) to l. The proof of (14) is similar to that of (13). □

4. Proofs of main results

Proof of Theorem 3. Necessity. Assume that (x_n) is convergent to l. Given any $\lambda>1$, by Lemma 12 we obtain

$$\begin{aligned}&\lim_{n\to\infty}\left\|\frac{1}{P_{[\lambda n]}-P_n}\sum_{k=n+1}^{[\lambda n]}p_k(x_k-x_n),y\right\| \\ &\le \lim_{n\to\infty}\left\|\frac{1}{P_{[\lambda n]}-P_n}\sum_{k=n+1}^{[\lambda n]}p_kx_k-l,y\right\|+\lim_{n\to\infty}\left\|x_n-l,y\right\|=0\end{aligned}$$

for every $y \in X$. Namely, we have an even stronger condition than (5). In a similar way, for any $0 < \lambda < 1$, we have

$$\lim_{n\to\infty} \left\| \frac{1}{P_n - P_{[\lambda n]}} \sum_{k=[\lambda n]+1}^{n} p_k(x_n - x_k), y \right\| = 0,$$

which is stronger than (6).

Sufficiency. Suppose that (5) holds, and let $y \in X$ be arbitrarily fixed. Then, for any given $\epsilon > 0$, there exists $\lambda > 0$ such that

$$\limsup_{n\to\infty} \left\| \frac{1}{P_{[\lambda n]} - P_n} \sum_{k=n+1}^{[\lambda n]} p_k(x_k - x_n), y \right\| \le \epsilon.$$

So, using also $(\overline{N}, p)$ summability of (x_n) and Lemma 12, we obtain

$$\begin{aligned} \limsup_{n\to\infty} \|x_n - l, y\| \le \limsup_{n\to\infty} & \left\| \frac{1}{P_{[\lambda n]} - P_n} \sum_{k=n+1}^{[\lambda n]} p_k x_k - l, y \right\| \\ & + \limsup_{n\to\infty} \left\| \frac{1}{P_{[\lambda n]} - P_n} \sum_{k=n+1}^{[\lambda n]} p_k(x_k - x_n), y \right\| \le \epsilon \end{aligned}$$

for every $y \in X$. Since $\epsilon > 0$ is arbitrary, convergence of (x_n) to l follows.

A similar proof can be given if (6) is satisfied. □

Proof of Corollary 8. Suppose that (x_n) is slowly oscillating in two-norm. Then, for any given $y \in X$, we have

$$\begin{aligned} \left\| \frac{1}{P_{[\lambda n]} - P_n} \sum_{k=n+1}^{[\lambda n]} p_k(x_k - x_n), y \right\| & \le \frac{1}{P_{[\lambda n]} - P_n} \sum_{k=n+1}^{[\lambda n]} p_k \|x_k - x_n, y\| \\ & \le \max_{n<k\le[\lambda n]} \|x_k - x_n, y\| . \end{aligned}$$

Taking the lim sup of the last inequality as $n \to \infty$, we get

$$\limsup_{n\to\infty} \left\| \frac{1}{P_{[\lambda n]} - P_n} \sum_{k=n+1}^{[\lambda n]} p_k(x_k - x_n), y \right\| \le \limsup_{n\to\infty} \max_{n<k\le[\lambda n]} \|x_k - x_n, y\| .$$

Therefore, we conclude that

$$\inf_{\lambda>1} \limsup_{n\to\infty} \left\| \frac{1}{P_{[\lambda n]} - P_n} \sum_{k=n+1}^{[\lambda n]} p_k(x_k - x_n), y \right\| = 0$$

for every $y \in X$. The proof follows from Theorem 3. □

Proof of Corollary 9. Let (10) and (11) be satisfied. Then, for every $y \in X$, there exists $H > 0$ such that $\left\| \frac{P_n}{p_n} \Delta x_n, y \right\| \leq H \quad (n = 0, 1, \dots)$. Hence, slow oscillation of (x_n) follows. Indeed, for each $y \in X$,

$$\begin{aligned}\|x_k - x_n, y\| &= \left\| \sum_{j=n+1}^{k} (x_j - x_{j-1}), y \right\| = \left\| \sum_{j=n+1}^{k} \Delta x_j, y \right\| \\ &\leq \sum_{j=n+1}^{k} \frac{p_j}{P_j} \|\Delta x_j, y\| \leq H \sum_{j=n+1}^{k} \frac{p_j}{P_j} \\ &\leq H \frac{P_k - P_n}{P_n}.\end{aligned}$$

Taking the maximum of both sides of the inequality above, we have

$$\max_{n<k\leq[\lambda n]} \|x_k - x_n, y\| \leq H \frac{P_{[\lambda n]} - P_n}{P_n}.$$

Now, taking the lim sup of both sides of the last inequality, by (10) we have that

$$\limsup_{n\to\infty} \max_{n<k\leq[\lambda n]} \|x_k - x_n, y\| \leq H(\lambda^{\delta} - 1). \tag{16}$$

Finally, taking the infimum of both sides of (16) for $\lambda > 1$, we get

$$\inf_{\lambda>1} \limsup_{n\to\infty} \max_{n<k\leq[\lambda n]} \|x_k - x_n, y\| = 0.$$

Thus, the proof is completed by Corollary 8. □

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