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Structure and classification of Hom-associative algebras

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Abstract. The purpose of this paper is to study the structure and the algebraic varieties of Hom-associative algebras. We characterize multiplicative simple Hom-associative algebras and give some examples deforming the 2×2-matrix algebra to simple Hom-associative algebras. We provide a classification of n-dimensional Hom-associative algebras for $n \leq 3$. Then we study irreducible components using deformation theory.

1. Introduction

The first motivation to study nonassociative Hom-algebras came from quasi-deformations of Lie algebras of vector fields, in particular, from q-deformations of Witt and Virasoro algebras. The deformed algebras arising when replacing usual derivations by σ-derivations are no longer Lie algebras. It was observed in the pioneering works, mainly by physicists, that in these examples a twisted Jacobi identity holds. Motivated by these examples and their generalizations on the one hand, and the desire to be able to treat within the same framework such well-known generalizations of Lie algebras as the color and Lie superalgebras on the other hand, quasi-Lie algebras and subclasses of quasi-Hom-Lie algebras and Hom-Lie algebras were introduced by Hartwig, Larsson and Silvestrov in [5, 6]. The Hom-associative algebras play the role of associative algebras in the Hom-Lie setting. They were introduced by the first author and Silvestrov in [7]. Usual functors between the categories of Lie algebras and associative algebras were extended to Hom-setting, see [10] for the construction of the enveloping algebra of a Hom-Lie algebra.

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A Hom-associative algebra (A, μ, α) is consisting of a vector space, a multiplication and a linear self-map. It may be viewed as a deformation of an associative algebra, in which the associativity condition is twisted by a linear map α and such that when $\alpha = id$, the Hom-associative algebra degenerates to exactly an associative algebra. We aim in this paper to study the structure of Hom-associative algebras. We give a characterization of multiplicative simple Hom-associative algebras and give some examples deforming the 2×2-matrix algebra to simple Hom-associative algebras. Moreover, we compute some invariants and discuss irreducible components of the corresponding algebraic varieties. Let A be an n-dimensional $\mathbb{K}$-linear space and $\{e_1, e_2, \dots, e_n\}$ be a basis of A. A Hom-algebra structure on A with product μ is determined by n^3 structure constants $\mathcal{C}_{ij}^k$, where $\mu(e_i, e_j) = \sum_{k=1}^n \mathcal{C}_{ij}^k e_k$, and by α which is identified by n^2 structure constants a_{ij}, where $\alpha(e_i) = \sum_{j=1}^n a_{ji} e_j$. Requiring the algebra structure to be Hom-associative and unital gives rise to a sub-variety $\mathcal{HA}ss_n$ (resp. $\mathcal{UHA}ss_n$) of $k^{n^3+n^2}$. Base changes in A result in the natural transport of structure action of $GL_n(\mathbb{K})$ on $\mathcal{HA}ss_n$. Thus isomorphism classes of n-dimensional Hom-algebras are in one-to-one correspondence with the orbits of the action of $GL_n(\mathbb{K})$ on $\mathcal{HA}ss_n$. The decomposition of $\mathcal{HA}ss_n$ into irreducible components with respect to Zariski topology is called the geometric classification of n-dimensional algebras.

The paper is organized as follows. In Section 2 we give the basics about Hom-associative algebras and provide some new properties. Moreover, we discuss unital Hom-associative algebras. Section 3 deals with simple multiplicative Hom-associative algebras. We present one of the main results of this paper, that is a characterization of simple multiplicative Hom-associative algebras. Indeed, we show that they are all obtained by twistings of simple associative algebras. Moreover, we give all simple Hom-associative algebras, which are related to 2×2 matrix algebra. Section 4 is dedicated to describing algebraic varieties of Hom-associative algebras and providing a classification, up to isomorphism, of 2-dimensional and 3-dimensional Hom-associative algebras. In the last section, we consider the geometric classification problem, using one-parameter formal deformations, and describe the irreducible components.

2. Structure of Hom-associative algebras

Let $\mathbb{K}$ be an algebraically closed field of characteristic 0, A be a linear space over $\mathbb{K}$. We refer to a Hom-algebra by a triple (A, μ, α), where $\mu : A\times A \to A$ is a bilinear map (multiplication) and α is a homomorphism of A (twist map).

2.1. Definitions.

Definition 2.1 (see [7]). A Hom-associative algebra is a triple (A, μ, α) consisting of a linear space A, a bilinear map $\mu : A \times A \to A$ and a linear space homomorphism $\alpha : A \to A$ satisfying

$$\mu(\alpha(x), \mu(y, z)) = \mu(\mu(x, y), \alpha(z)), \tag{2.1}$$
$$\alpha(\mu(x, y)) = \mu(\alpha(x), \alpha(y)).$$

Usually such Hom-associative algebras are called multiplicative. Since we are dealing only with multiplicative Hom-associative algebras, we shall call them Hom-associative algebras for simplicity. We denote the set of all Hom-associative algebras by $\mathcal{HA}ss$. Notice that in [3], the author introduced monoidal Hom-algebras which are Hom-associative algebras where α is an automorphism. In the language of Hopf algebras, the multiplication of a Hom-associative algebra over A consists of a linear map $\mu : A \otimes A \to A$ and condition (2.1) writes $\mu(\alpha(x) \otimes \mu(y \otimes z)) = \mu(\mu(x \otimes y) \otimes \alpha(z))$.

Definition 2.2. A unital Hom-associative algebra is given by a quadruple (A, μ, α, u), where $u \in A$, such that

- (A, μ, α) is a Hom-associative algebra,
- $\mu(x, u) = \mu(u, x) = \alpha(x), \quad x \in A,$
- $\alpha(u) = u$.

Remark 2.3. It was pointed out in [1, Proposition 1.2] that the multiplicativity condition is a direct consequence of the Hom-associativity together with the second and third conditions.

Definition 2.4. Let (A_1, μ_1, α_1) and (A_2, μ_2, α_2) be two Hom-associative algebras (respectively, unital Hom-associative algebras with u_1, u_2 the units). A linear map $\varphi : A_1 \to A_2$ is called a Hom-associative algebra morphism if

$$\varphi(\mu_1(x, y)) = \mu_2(\varphi(x), \varphi(y)) \text{ and } \alpha_2 \circ \varphi(x) = \varphi \circ \alpha_1(x), \; x, y \in A_1,$$

and $\varphi(u_1) = u_2$ for unital algebras.

In particular, Hom-associative algebras (A_1, μ_1, α_1) and (A_2, μ_2, α_2) are isomorphic if φ is also bijective.

Remark 2.5. Following [1], in the case of unital Hom-associative algebras, one may remove from the definition the condition $\alpha_2 \circ \varphi(x) = \varphi \circ \alpha_1(x)$. Indeed, for any $x \in A_1$, we have

$$\alpha_2 \circ \varphi(x) = \mu_2(u_2, \varphi(x)) = \mu_2(\varphi(u_1), \varphi(x)) = \varphi(\mu_1(u_1, x)) = \varphi \circ \alpha_1(x).$$

2.2. Structure of Hom-associative algebras. We state in this section some results on the structure of Hom-associative algebras which are not necessarily multiplicative.

Proposition 2.6 (see [11]). *Let (A, μ, α) be a Hom-associative algebra and $\beta : A \to A$ be a Hom-associative algebra morphism. Then $(A, \beta\mu, \beta\alpha)$ is a Hom-associative algebra. In particular, if (A, μ) is an associative algebra and β is an algebra morphism, then $(A, \beta\mu, \beta)$ is a Hom-associative algebra.*

Definition 2.7. Let (A, μ, α) be a Hom-associative algebra. If there is an associative algebra (A, μ') such that $\mu(x, y) = \alpha\mu'(x, y),\ x, y \in A$, then we say that (A, μ, α) is of associative type and (A, μ') is its compatible associative algebra or the untwist of (A, μ, α).

Corollary 2.8. *Let (A, μ, α) be a multiplicative Hom-associative algebra where α is invertible. Then $(A, \mu' = \alpha^{-1} \circ \mu)$ is an associative algebra and α is an automorphism with respect to μ'. Hence, (A, μ, α) is of associative type and $(A, \mu' = \alpha^{-1} \circ \mu)$ is its compatible associative algebra.*

Proof. We prove that $(A, \alpha^{-1} \circ \mu)$ is an associative algebra. Indeed,

$$\begin{aligned}\mu'(\mu'(x, y), z) &= \alpha^{-1} \circ \mu(\alpha^{-1}\mu(x, y), z) = \alpha^{-1} \circ \mu(\alpha^{-1}\mu(x, y), \alpha^{-1} \circ \alpha(z)) \\ &= \alpha^{-2} \circ \mu(\mu(x, y), \alpha(z)) = \alpha^{-2} \circ \mu(\alpha(x), \mu(y, z)) \\ &= \alpha^{-1} \circ \mu(x, \alpha^{-1} \circ \mu(x, y)) = \mu'(x, \mu'(y, z)).\end{aligned}$$

Moreover, α is an automorphism with respect to μ'. Indeed,

$$\mu'(\alpha(x), \alpha(y)) = \alpha^{-1} \circ \mu(\alpha(x), \alpha(y)) = \alpha \circ \alpha^{-1} \circ \mu(x, y) = \alpha \circ \mu'(x, y).$$

□

Remark 2.9. Notice that if α is not invertible, assuming $\mu = \alpha\tilde{\mu}$ leads to

$$\begin{aligned}\mu(\alpha(x), \mu(y, z)) &= \mu(\mu(x, y), \alpha(z)), \\ \alpha\tilde{\mu}(\alpha(x), \alpha\tilde{\mu}(y, z)) &= \alpha\tilde{\mu}(\alpha\tilde{\mu}(x, y), \alpha(z)), \\ \alpha^2(\tilde{\mu}(x, \tilde{\mu}(y, z))) &= \alpha^2(\tilde{\mu}(\tilde{\mu}(x, y), z)),\end{aligned}$$

which means that $\tilde{\mu}$ is associative up to α^2.

Proposition 2.10. *Let (A_1, μ_1, α_1) and (A_2, μ_2, α_2) be two Hom-associative algebras and let $\phi : A_1 \to A_2$ be an invertible Hom-associative algebra morphism. If (A_1, μ_1, α_1) is of associative type and (A_1, μ_1') is its compatible associative algebra, then (A_2, μ_2, α_2) is of associative type with compatible associative algebra $(A_2, \mu_2' = \phi \circ \mu_1 \circ (\phi^{-1} \otimes \phi^{-1}))$ such that $\phi : (A_1, \mu_1') \to (A_2, \mu_2')$ is an algebra morphism.*

Proof. Because ϕ is a homomorphism from (A_1, μ_1, α_1) to (A_2, μ_2, α_2), $\alpha_2\phi = \phi\alpha_1$, and ϕ defines μ_2 by $\mu_2(\phi(x), \phi(y)) = \phi\mu_1(x, y),\ x, y \in A_1$. It is easy to check that (A_2, μ_2) is an associative algebra. Furthermore,

$$\begin{aligned}\mu_2(\phi(x), \phi(y)) &= \phi \circ \mu_1(x, y) = \phi \circ \alpha_1 \circ \mu_1'(x, y)) \\ &= \alpha_2 \circ \phi\mu_1'(x, y) = \alpha_2\mu_2'(\phi(x), \phi(y)).\end{aligned}$$

We show that μ_2 is an associative algebra such that

$$\mu_2(u, v) = \phi \circ \mu_1(\phi^{-1}(u), \phi^{-1}(v))$$

with $x = \phi^{-1}(u), y = \phi^{-1}(v)$ and $z = \phi^{-1}(w)$ for all $x, y, z \in A_1$. One has

$$\mu_2(\mu_2(u, v), w) = \phi \circ \mu_1(\phi^{-1} \otimes \phi^{-1})(\phi \circ \mu_1(\phi^{-1} \otimes \phi^{-1})(u, v), w)$$

$$
\begin{aligned}
&= \phi \circ \mu_1(\phi^{-1} \otimes \phi^{-1})(\phi \circ \mu_1(\phi^{-1}(u), \phi^{-1}(v)), w) \\
&= \phi \circ \mu_1(\mu_1(\phi^{-1}(u), \phi^{-1}(v)), \phi^{-1}(w)) \\
&= \phi \circ \mu_1(\phi^{-1}(u), \mu_1(\phi^{-1}(v), \phi^{-1}(w))) \\
&= \phi \circ \mu_1(\phi^{-1} \otimes \phi^{-1})(\phi \otimes \phi)(\phi^{-1}(u), \mu_1(\phi^{-1}(v), \phi^{-1}(w)) \\
&= \phi \circ \mu_1(\phi^{-1} \otimes \phi^{-1})(u, \phi\mu_1(\phi^{-1}(v), \phi^{-1}(w))) = \mu_2(u, \mu_2(v, w)).
\end{aligned}
$$

Hence, (A_2, μ_2) is an associative algebra. □

Proposition 2.11. *Let (A, μ, α) be an n-dimensional Hom-associative algebra and let $\phi : A \to A$ be an invertible linear map. Then there is an isomorphism with an n-dimensional Hom-associative algebra $(A, \mu', \phi\alpha\phi^{-1})$, where $\mu' = \phi \circ \mu \circ (\phi^{-1} \otimes \phi^{-1})$. Furthermore, if $\left\{C_{ij}^k\right\}$ are the structure constants of μ with respect to the basis $\{e_1, \ldots, e_n\}$, then μ' has the same structure constants with respect to the basis $\{\phi(e_1), \ldots, \phi(e_n)\}$ when $\phi(e_p) = \sum_{k=1}^{n} a_{kp} e_k$.*

Proof. We prove, for any invertible linear map ϕ: $A \to A$, that $(A, \mu', \phi\alpha\phi^{-1})$ is a Hom-associative algebra. We have

$$
\begin{aligned}
&\mu'(\mu'(x, y), \phi\alpha\phi^{-1}(z)) = \phi\mu(\phi^{-1} \otimes \phi^{-1})(\phi\mu(\phi^{-1} \otimes \phi^{-1})(x, y), \phi\alpha\phi^{-1}(z)) \\
&= \phi\mu(\mu(\phi^{-1}(x), \phi^{-1}(y)), \alpha\phi^{-1}(z)) = \phi\mu(\alpha\phi^{-1}(x), \mu(\phi^{-1}(y), \phi^{-1}(z))) \\
&= \phi\mu(\phi^{-1} \otimes \phi^{-1})(\phi \otimes \phi)(\alpha\phi^{-1}(x), \mu(\phi^{-1} \otimes \phi^{-1})(y, z))) \\
&= \phi\mu(\phi^{-1} \otimes \phi^{-1})(\phi\alpha\phi^{1}(x), \phi\mu(\phi^{-1} \otimes \phi^{-1})(y, z)) = \mu'(\phi\alpha\phi^{-1}(x), \mu'(y, z)).
\end{aligned}
$$

So $(A, \mu', \phi\alpha\phi^{-1})$ is a Hom-associative algebra.

It is also multiplicative. Indeed,

$$
\begin{aligned}
&\phi\alpha\phi^{-1}\mu'(x, y) = \phi\alpha\phi^{-1}\phi\mu(\phi^{-1} \otimes \phi^{-1})(x, y) = \phi\alpha\mu(\phi^{-1} \otimes \phi^{-1})(x, y) \\
&= \phi\mu(\alpha\phi^{-1}(x), \alpha\phi^{-1}(y)) = \phi\mu(\phi^{-1} \otimes \phi^{-1})(\phi \otimes \phi)(\alpha\phi^{-1}(x), \alpha\phi^{-1}(y)) \\
&= \mu'(\phi\alpha\phi^{-1}(x), \phi\alpha\phi^{-1}(y)).
\end{aligned}
$$

Therefore, $\phi : (A, \mu, \alpha) \to (A, \mu', \phi\alpha\phi^{-1})$ is a Hom-associative algebra morphism, since

$$\phi \circ \mu = \phi \circ \mu \circ (\phi^{-1} \otimes \phi^{-1}) \circ (\phi \otimes \phi) = \mu' \circ (\phi \otimes \phi)$$

and $(\phi\alpha\phi^{-1}) \circ \phi = \phi \circ \alpha$.

It is easy to see that $\{\phi(e_i), \cdots, \phi(e_n)\}$ is a basis of A. For $i, j = 1, \ldots, n$, we have

$$\mu_2(\phi(e_i), \phi(e_j)) = \phi\mu_1(\phi^{-1}(e_i), \phi^{-1}(e_j)) = \phi\mu(e_i, e_j) = \sum_{k=1}^{n} \mathcal{C}_{ij}^k \phi(e_k).$$

□

Remark 2.12. A Hom-associative algebra (A, μ, α) is isomorphic to an associative algebra if and only if $\alpha = id$. Indeed, $\phi \circ \alpha\phi^{-1} = id$ is equivalent to $\alpha = id$.

Remark 2.13. Proposition 2.11 is useful to make a classification of Hom-associative algebras. Indeed, we have to consider the class of morphisms which are conjugate. Representations of these classes are given by Jordan forms of the matrices corresponding to the morphisms. Any $n \times n$ matrix over $\mathbb{K}$ is equivalent, up to basis change, to a Jordan canonical form, then we choose ϕ such that the matrix of $\phi\alpha\phi^{-1} = \gamma$, where γ is a Jordan canonical form. Hence, to obtain the classification, we consider only Jordan forms for the structure map of Hom-associative algebras.

Proposition 2.14. *Let (A, μ, α) be a Hom-associative algebra. Let $(A, \mu', \phi\alpha\phi^{-1})$ be its isomorphic Hom-associative algebra described in Proposition 2.11. If ψ is an automorphism of (A, μ, α), then $\phi\psi\phi^{-1}$ is an automorphism of $(A, \mu, \phi\alpha\phi^{-1})$.*

Proof. Note that $\gamma = \phi\alpha\phi^{-1}$. We have

$$\phi\psi\phi^{-1}\gamma = \phi\psi\phi^{-1}\phi\alpha\phi^{-1} = \phi\psi\alpha\phi^{-1} = \phi\alpha\psi\phi^{-1} = \phi\alpha\phi^{-1}\phi\psi\phi^{-1} = \gamma\phi\psi\phi^{-1}.$$

For any $x, y \in A$, one has

$$\begin{aligned}\phi\psi\phi^{-1}\mu'(\phi(x), \phi(y)) &= \phi\psi\phi^{-1}\phi\mu(x, y) = \phi\psi\mu(x, y) = \phi\mu(\psi(x), \psi(y)) \\ &= \mu'(\phi\psi(x), \phi\psi(y)) = \mu'(\phi\psi\phi^{-1}(\phi(x)), \phi\psi\phi^{-1}(\phi(y))).\end{aligned}$$

By definition, $\phi\psi\phi^{-1}$ is an automorphism of $(A, \mu', \phi\alpha\phi^{-1})$. □

The following characterization was given for Hom-Lie algebras in [9].

Proposition 2.15. *Given two Hom-associative algebras (A, μ_A, α) and (B, μ_B, β), there is a Hom-associative algebra $(A \oplus B, \mu_{A\oplus B}, \alpha + \beta)$, where the bilinear map $\mu_{A\oplus B}(.,.)\colon (A \oplus B) \times (A \oplus B) \to (A \oplus B)$ is given by*

$$\mu_{A\oplus B}(a_1 + b_1, a_2 + b_2) = (\mu_A(a_1, a_2), \mu_B(b_1, b_2)), \quad a_1, a_2 \in A, \quad b_1, b_2 \in B,$$

and the linear map $(\alpha + \beta)\colon A \oplus B \to A \oplus B$ is given by

$$(\alpha + \beta)(a, b) = (\alpha(a), \beta(b)), \quad a \in A, \ b \in B.$$

Proof. For any $a_i \in A$, $b_i \in B$, by direct computation, we get

$$\begin{aligned}&\mu_{A\oplus B}((\alpha + \beta)(a_1, b_1), \mu_{A\oplus B}(a_2 + b_2, a_3 + b_3)) \\ &= \mu_{A\oplus B}((\alpha + \beta)(a_1, b_1), (\mu_A(a_2, a_3), \mu_B(b_2, b_3))) \\ &= \mu_{A\oplus B}((\alpha(a_1), \beta(b_1)), (\mu_A(a_2, a_3), \mu_B(b_2, b_3))) \\ &= (\mu_A(\alpha(a_1), \mu_A(a_2, a_3)), \mu_B(\beta(b_1), \mu_B(b_2, b_3))) \\ &= (\mu_A(\mu_A(a_1, a_2), \alpha(a_3)), \mu_B(\mu_B(b_1, b_2), \beta(b_3))) \\ &= \mu_{A\oplus B}(\mu_{A\oplus B}(a_1 + b_1, a_2 + b_2), (\alpha + \beta)(a_3, b_3))).\end{aligned}$$

This ends the proof. □

A Hom-associative algebra morphism $\phi\colon (A,\mu_A,\alpha)\to(B,\mu_B,\beta)$ is a linear map $\phi : A \to B$ such that $\phi\circ\mu_A(a,b) = \mu_B\circ(\phi(a),\phi(b)),\ \ a,b\in A,\ \ \phi\circ\alpha=\beta\circ\phi$. Denote by $\xi_\phi\subset A\oplus B$, the graph of a linear map $\phi : A\to B$.

Proposition 2.16. *A linear map $\phi : (A,\mu_A,\alpha)\to(B,\mu_B,\beta)$ is a Hom-associative algebra morphism if and only if the graph $\xi_\phi\subset A\oplus B$ is a Hom-associative subalgebra of $(A\oplus B,\mu_{A\oplus B},\alpha+\beta)$.*

Proof. Let $\phi\colon (A,\mu_A,\alpha)\to(B,\mu_B,\beta)$ be a Hom-associative algebra morphism. Then for any $a,b\in A$, we have

$$\mu_{A\oplus B}((a,\phi(a)),(b,\phi(b)) = (\mu_A(a,b),\mu_B(\phi(a),\phi(b))) = (\mu_A(a,b),\phi\mu_A(a,b)).$$

Thus the graph ξ_ϕ is closed under the product $\mu_{A\oplus B}$. Furthermore, since $\phi\circ\alpha=\beta\circ\phi$, we have

$$(\alpha+\beta)(a,\phi(a)) = (\alpha(a),\beta\circ\phi(a)) = (\alpha(a),\phi\circ\alpha(a)),$$

which implies that $(\alpha+\beta)\subset\xi_\phi$. Thus ξ_ϕ is a Hom-associative subalgebra of $(A\oplus B,\mu_{A\oplus B},\alpha+\beta)$.

Conversely, if the graph $\xi_\phi\subset A\oplus B$ is a Hom-associative subalgebra of $(A\oplus B,\mu_{A\oplus B},\alpha+\beta)$, then we have

$$\mu_{A\oplus B}((a,\phi(a)),(b,\phi(b))) = (\mu_A(a,b),\mu_B(\phi(a),\phi(b))\in\xi_\phi,$$

which implies that $\mu_B(\phi(a),\phi(b)) = \phi\circ\mu_A(a,b)$. Furthermore, $(\alpha+\beta)(\xi_\phi)\subset\xi_\phi$ yields that

$$(\alpha+\beta)(a,\phi(a)) = (\alpha(a),\beta\circ\phi(a))\in\xi_\phi,$$

which is equivalent to the condition $\beta\circ\phi(a)=\phi\circ\alpha(a)$. Therefore, ϕ is a Hom-associative algebra morphism. □

2.3. Unital Hom-associative algebras. In this section we discuss unital Hom-associative algebras. We denote by $\mathcal{UHA}ss_n$ the set of n-dimensional unital Hom-associative algebras.

Proposition 2.17. *Let (A,μ,α) be a Hom-associative algebra. We set $\tilde{A} = span(A,u)$ the vector space generated by elements of A and u. Assume that $\mu(x,u)=\mu(u,x)=\alpha(x),\ \ x\in A$, and $\alpha(u)=u$. Then $(\tilde{A},\mu,\alpha,u)$ is a unital Hom-associative algebra.*

Proof. It is straightforward to check the Hom-associativity. For example

$$\begin{aligned}\mu(\mu(x,y),\alpha(u)) &= \mu(\mu(x,y),u) = \alpha(\mu(x,y))\\ &= \mu(\alpha(x),\alpha(y)) = \mu(\alpha(x),\mu(y,u)).\end{aligned}$$

□

Remark 2.18. Some unital Hom-associative algebras cannot be obtained as an extension of a non-unital Hom-associative algebra.

Remark 2.19. Let (A, μ, α, u) be an n-dimensional unital Hom-associative algebra and let $\phi\colon A \to A$ be an invertible linear map such that $\phi(u) = u$. Then it is isomorphic to an n-dimensional Hom-associative algebra $(A, \mu', \phi\alpha\phi^{-1}, u)$ where $\mu' = \phi \circ \mu \circ (\phi^{-1} \otimes \phi^{-1})$. Moreover, if $\left\{C_{ij}^k\right\}$ are the structure constants of μ with respect to the basis $\{e_1, \dots, e_n\}$ with $e_1 = u$ being the unit, then μ' has the same structure constants with respect to the basis $\{\phi(e_1), \dots, \phi(e_n)\}$ with u the unit element.

Indeed, we use Proposition 2.11 and Definition 2.2. The unit is preserved since $\mu'(x, e_1) = \phi \circ \mu(\phi^{-1}(x), \phi^{-1}(e_1)) = \phi \circ \alpha \circ \phi^{-1}(x)$.

Proposition 2.20. *Let $(A_1, \mu_1, \alpha_1, u_1)$ and $(A_2, \mu_2, \alpha_2, u_2)$ be two unital Hom-associative algebras. Suppose that there exists a Hom-associative algebra morphism $\phi\colon A_1 \to A_2$ with $\phi(u_1) = u_2$. If (A_1, μ_1', u_1') is an untwist of $(A_1, \mu_1, \alpha_1, u_1)$, then there exists an untwist of $(A_2, \mu_2, \alpha_2, u_2)$ such that $\phi\colon (A_1, \mu_1', u_1') \to (A_2, \mu_2', u_2')$ is an algebra morphism.*

Proof. Since ϕ is a homomorphism from $(A_1, \mu_1, \alpha_1, u_1)$ to $(A_2, \mu_2, \alpha_2, u_2)$, $\alpha_2\phi = \phi\alpha_1$, and for all $x \in A$ we have $\mu_2(\phi(x), \phi(u_1)) = \mu_2(\phi(x), u_2) = \alpha_2 \circ \phi(x)$ and $\phi \circ \mu_1(x, u_1) = \phi \circ \alpha_1(x)$. By Proposition 2.10, we can see that (A_2, μ_2, u_2) is also an associative algebra. Furthermore,

$$\begin{aligned} \mu_2'(\phi(x), \phi(u_1)) = \mu_2'(\phi(x), u_2) = \phi \circ \alpha_1' \circ \phi(x) = \phi \circ \alpha_1 \circ \mu_1(x, u_1) \\ = \alpha_2 \circ \phi \circ \mu_1(x, u_1) = \alpha_2 \circ \mu_2(\phi(x), u_2). \end{aligned}$$

□

3. Simple Hom-associative algebras

In this section, we study and characterize simple multiplicative Hom-associative algebras. Then we provide examples by considering 2×2 matrix algebra. This study is inspired by the study of simple Hom-Lie algebras in [4].

Definition 3.1. Let (A, μ, α) be a Hom-associative algebra. A subspace H of A is called a Hom-associative subalgebra of (A, μ, α) if $\alpha(H) \subseteq H$ and $\mu(H, H) \subseteq H$. In particular, a Hom-associative subalgebra H is said to be a two-sided ideal of (A, μ, α) if $\mu(H, A) \subseteq H$ and $\mu(A, H) \subseteq H$.

Definition 3.2. The set

$$C(A) = \{x \in A | \mu(x, y) = \mu(y, x),\ \mu(\alpha(x), y) = \mu(y, \alpha(x)),\ \ y \in A\}$$

is called the center of (A, μ, α).

Clearly, $C(A)$ is a two-sided ideal.

Lemma 3.3. *Let (A, μ, α) be a multiplicative Hom-associative algebra, then $(Ker(\alpha), \mu, \alpha)$ is a two-sided ideal.*

Proof. Obviously, $\alpha(x) = 0 \in Ker(\alpha)$ for any $x \in Ker(\alpha)$. Since $\alpha\mu(x, y) = \mu(\alpha(x), \alpha(y) = \mu(0, y) = 0$ for any $x \in Ker(\alpha)$ and $y \in A$, we get $\mu(x, y) \in Ker(\alpha)$.

On the other hand, we have $\alpha(y) = 0 \in Ker(\alpha)$ for any $y \in Ker(\alpha)$. Since $\alpha\mu(x, y) = \mu(\alpha(x), \alpha(y)) = \mu(x, 0) = 0$ for any $x \in Ker(\alpha)$ and $y \in A$, we get $\mu(x, y) \in Ker(\alpha)$. Therefore, $(Ker(\alpha), \mu, \alpha)$ is a two-sided ideal of (A, μ, α). □

Definition 3.4. Let $(A, \mu, \alpha)\,(\alpha \neq 0)$ be a non trivial Hom-associative algebra. It is said to be a simple Hom-associative algebra if it has no proper two-sided ideals.

Theorem 3.5. *Let (A, μ, α) be a finite dimensional simple Hom-associative algebra. Then α is an automorphism, the Hom-associative algebra is of associative type with a simple compatible associative algebra.*

Proof. According to Lemma 3.3, $Ker(\alpha)$ is a two-sided ideal. Since the Hom-associative algebra is simple, either $Ker(\alpha) = \{0\}$ or $Ker(\alpha) = A$. The Hom-associative algebra is nontrivial, therefore $Ker(\alpha) \neq A$.

Thus, A is of associative type. Let $(A, \mu' = \alpha^{-1}\mu)$ be the induced associative algebra of the multiplicative simple Hom-associative algebra (A, μ, α). Clearly, α is both an automorphism of (A, μ, α) and (A, μ'). Indeed,

$$\alpha\mu'(x, y) = \alpha\alpha^{-1}\mu(x, y) = \alpha^{-1}\mu(\alpha(x), \alpha(y)) = \mu'(\alpha(x), \alpha(y)).$$

Suppose that $A_1 \neq 0$ is the maximal two-sided ideal of (A, μ'). Because $\alpha(A_1)$ is also a two-sided ideal of (A, μ'), we have $\alpha(A_1) \subseteq A_1$. Moreover,

$$\mu(A_1, A) = \alpha\mu'(A_1, A) \subseteq \alpha(A_1) \subseteq A_1$$

and

$$\mu(A, A_1) = \alpha\mu'(A, A_1) \subseteq \alpha(A_1) \subseteq A_1.$$

So A_1 is a two-sided ideal of (A, μ, α). Then $A_1 = A$, and we have

$$\mu(A, A) = \mu(A_1, A) = \alpha\mu'(A_1, A) \subsetneqq \alpha(A_1) \subseteq A_1 = A$$

and

$$\mu(A, A) = \mu(A, A_1) = \alpha\mu'(A, A_1) \subsetneqq \alpha(A_1) \subseteq A_1 = A.$$

Furthermore, since (A, μ, α) is a multiplicative simple Hom-associative algebra, we clearly have $\mu(A, A) = A$. It is a contradiction. Hence $A_1 = 0$. □

By the above theorem, there exists an induced associative algebra for any multiplicative simple Hom-associative algebra (A, μ, α) and α is an automorphism of the induced associative algebra, in addition to this their products are mutually determined.

Theorem 3.6. *Two simple Hom-associative algebras* (A_1, μ_1, α) *and* (A_2, μ_2, β) *are isomorphic if and only if there exists an associative algebra isomorphism* $\varphi\colon A_1 \to A_2$ *(between their induced associative algebras) satisfying* $\varphi \circ \alpha = \beta \circ \varphi$. *In other words, the two associative algebra automorphisms* α *and* β *are conjugate.*

Proof. Let $(A_1, \tilde{\mu}_1)$ and $(A_2, \tilde{\mu}_2)$ be the induced associative algebras of (A_1, μ_1, α) and (A_2, μ_2, β), respectively. Suppose that $\varphi\colon (A_1, \mu_1, \alpha) \to (A_2, \mu_2, \beta)$ is an isomorphism of Hom-associative algebras. Then $\varphi\circ\alpha = \beta\circ\varphi$, thus $\varphi \circ \alpha^{-1} = \beta^{-1} \circ \varphi$. Moreover,

$$\begin{aligned}\varphi\tilde{\mu}_1(x, y) &= \varphi \circ \alpha^{-1} \circ \alpha\tilde{\mu}_1(x, y) = \varphi \circ \alpha^{-1}\mu_1(x, y) = \beta^{-1} \circ \varphi\mu_1(x, y)\\ &= \beta^{-1}(\mu_2(\varphi(x), \varphi(y))) = \tilde{\mu}_2(\varphi(x), \varphi(y)).\end{aligned}$$

So, φ is an isomorphism between the two induced associative algebras.

On the other hand, if there exists an isomorphism φ between the induced associative algebras $(A_1, \tilde{\mu}_1)$ and $(A_2, \tilde{\mu}_2)$ such that $\varphi \circ \alpha = \beta \circ \varphi$, then

$$\begin{aligned}\varphi\mu_1(x, y) &= \varphi \circ \alpha\tilde{\mu}_1(x, y) = \beta \circ \tilde{\mu}_2(\varphi(x), \varphi(y))\\ &= \beta(\mu_2(\varphi(x), \varphi(y)) = \mu_2(\varphi(x), \varphi(y)).\end{aligned}$$

□

3.1. Examples of simple Hom-associative algebras. We consider the simple associative algebra defined by 2×2 matrices, which we denote by $\mathcal{M}_2$. Let $\mathcal{B} = \{E_{ij}\}_{\substack{i=1,2\\ j=1,2}}$ be the canonical basis given by elementary matrices. We seek first for algebra morphisms φ of $\mathcal{M}_2$, that is linear maps such that

$$\varphi(E_{ij}).\varphi(E_{kl}) = \varphi(E_{ij}.E_{kl}) = \delta_{jk}\varphi(E_{il}),$$

where δ_{ij} is the Kronecker symbol. Then we apply the previous theorem to construct families of 4-dimensional simple Hom-associative algebras. By straightforward calculation we obtain the following algebra morphisms:

Morphism 1

$$\left\{\begin{array}{ll}\varphi(E_{11}) = E_{11} - i\frac{\sqrt{\beta_2}}{\sqrt{\beta_1}}E_{21} & \varphi(E_{12}) = i\sqrt{\beta_1}\sqrt{\beta_2}E_{11} + \beta_1 E_{12} + \beta_2 E_{21}\\ & \qquad -i\sqrt{\beta_1}\sqrt{\beta_2}E_{22}\\ \varphi(E_{21}) = \frac{E_{21}}{\beta_1} & \varphi(E_{22}) = i\frac{\sqrt{\beta_2}}{\sqrt{\beta_1}}E_{21} + E_{22};\end{array}\right.$$

Morphism 2

$$\left\{\begin{array}{ll}\varphi(E_{11}) = E_{11} + i\frac{\sqrt{\beta_2}}{\sqrt{\beta_1}}E_{21} & \varphi(E_{12}) = -i\sqrt{\beta_1}\sqrt{\beta_2}E_{11} + \beta_1 E_{12} + \beta_2 E_{21}\\ & \qquad +i\sqrt{\beta_1}\sqrt{\beta_2}E_{22}\\ \varphi(E_{21}) = \frac{E_{21}}{\beta_1} & \varphi(E_{22}) = -i\frac{\sqrt{\beta_2}}{\sqrt{\beta_1}}E_{21} + E_{22};\end{array}\right.$$

Morphism 3

$$\left\{\begin{array}{ll}\varphi(E_{11}) = E_{11} - \lambda_1 E_{21} & \varphi(E_{12}) = -\frac{\beta_2}{\lambda_1}E_{11} - \frac{\beta_2}{\gamma_2^2}E_{12} + \beta_2 E_{21} + \frac{\beta_2}{\lambda_1}E_{22}\\ \varphi(E_{21}) = -\frac{\lambda_1^2}{\beta_2}E_{21} & \varphi(E_{22}) = \lambda_1 E_{21} + E_{22};\end{array}\right.$$

Morphism 4

$$\begin{cases} \varphi(E_{11}) = E_{11} - \lambda_1 E_{21} & \varphi(E_{12}) = \beta_1\lambda_1 E_{11} + \beta_1 E_{12} - \beta_1\lambda_1^2 E_{21} - \beta_1 E_{22} \\ \varphi(E_{21}) = \frac{E_{21}}{\beta_1} & \varphi(E_{22}) = \lambda_1 E_{21} + E_{22}; \end{cases}$$

Morphism 5

$$\begin{cases} \varphi(E_{11}) = E_{11} + \beta_3\gamma_1 E_{21} & \varphi(E_{12}) = -\beta_3 E_{11} + \frac{E_{12}}{\gamma_1} - \beta_3\gamma_1 E_{21} + \beta_3 E_{22} \\ \varphi(E_{21}) = \gamma_1 E_{21} & \varphi(E_{22}) = -\beta_3\gamma_1 E_{21} + E_{22}; \end{cases}$$

Morphism 6

$$\begin{cases} \varphi(E_{11}) = i\sqrt{\beta_2}\sqrt{\gamma_1} E_{21} + E_{22} & \varphi(E_{12}) = \beta_2 E_{21} \\ \varphi(E_{21}) = i\frac{\sqrt{\gamma_1}}{\sqrt{\beta_2}} E_{11} + \frac{E_{12}}{\beta_2} + \gamma_1 E_{21} - i\frac{\sqrt{\gamma_1}}{\sqrt{\beta_2}} E_{22} & \varphi(E_{22}) = E_{11} - i\sqrt{\beta_2}\sqrt{\gamma_1} E_{21}; \end{cases}$$

Morphism 7

$$\begin{cases} \varphi(E_{11}) = \frac{\beta_4}{\beta_2} E_{12} + E_{22} & \varphi(E_{12}) = \beta_4 E_{11} - \frac{\beta_4^2}{\beta_2} E_{12} + \beta_2 E_{21} - \beta_4 E_{22} \\ \varphi(E_{21}) = \frac{E_{12}}{\beta_2} & \varphi(E_{22}) = E_{11} - \frac{\beta_4}{\beta_2} E_{12}; \end{cases}$$

Morphism 8

$$\begin{cases} \varphi(E_{11}) = E_{11} + \gamma_2 E_{12} & \varphi(E_{12}) = \frac{E_{12}}{\gamma_1} \\ \varphi(E_{21}) = -\gamma_2 E_{11} - \frac{\gamma_2^2}{\gamma_1} E_{12} + \gamma_1 E_{21} + \gamma_2 E_{22} & \varphi(E_{22}) = -\frac{\beta_2}{\beta_1} E_{12} + E_{22}; \end{cases}$$

Morphism 9

$$\begin{cases} \varphi(E_{11}) = -\gamma_2 E_{21} + E_{22} & \varphi(E_{12}) = \frac{E_{21}}{\gamma_4} \\ \varphi(E_{21}) = -\gamma_2 E_{11} + \gamma_4 E_{12} - \frac{\gamma_2^2}{\gamma_4} E_{21} + \gamma_2 E_{22} & \varphi(E_{22}) = E_{11} + \frac{\gamma_2}{\gamma_4} E_{21}, \end{cases}$$

where $\beta_1, \beta_2, \beta_3, \beta_4, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \gamma_1, \gamma_2, \gamma_3, \gamma_4 \in \mathbb{C}$ are parameters.

They lead to simple Hom-associative algebras $(\mathcal{M}_2, *, \varphi)$ where $E_{ij} * E_{pq} = \varphi(E_{ij}E_{pq})$. Therefore, the multiplication tables are given as follows:

Algebra 1

$$\begin{cases} E_{11} * E_{11} = E_{11} - i\frac{\sqrt{\beta_2}}{\sqrt{\beta_1}} E_{21} & E_{11} * E_{12} = i\sqrt{\beta_1}\sqrt{\beta_2} E_{11} + \beta_1 E_{12} + \beta_2 E_{21} \\ & \qquad -i\sqrt{\beta_1}\sqrt{\beta_2} E_{22} \\ E_{12} * E_{21} = E_{11} - i\frac{\sqrt{\beta_2}}{\sqrt{\beta_1}} E_{21} & E_{12} * E_{22} = i\sqrt{\beta_1}\sqrt{\beta_2} E_{11} + \beta_1 E_{12} + \beta_2 E_{21} \\ & \qquad -i\sqrt{\beta_1}\sqrt{\beta_2} E_{22} \\ E_{21} * E_{11} = \frac{E_{21}}{\beta_1} & E_{21} * E_{12} = -i\frac{\sqrt{\beta_2}}{\sqrt{\beta_1}} E_{21} + E_{22} \\ E_{22} * E_{21} = \frac{E_{11}}{\beta_1} & E_{22} * E_{22} = -i\frac{\sqrt{\beta_2}}{\sqrt{\beta_1}} E_{21} + E_{22}; \end{cases}$$

Algebra 2

$$\begin{cases} E_{11} * E_{11} = E_{11} + i\frac{\sqrt{\beta_2}}{\sqrt{\beta_1}} E_{21} & E_{11} * E_{12} = -i\sqrt{\beta_1}\sqrt{\beta_1} E_{11} + \beta_1 E_{12} + \beta_2 E_{21} \\ & \qquad +i\sqrt{\beta_1}\sqrt{\beta_1} E_{22} \\ E_{12} * E_{21} = E_{11} + i\frac{\sqrt{\beta_1}}{\sqrt{\beta_1}} E_{21} & E_{12} * E_{22} = -i\sqrt{\beta_1}\sqrt{\beta_2} E_{11} + \beta_2 E_{12} + \beta_2 E_{21} \\ & \qquad +i\sqrt{\beta_1}\sqrt{\beta_2} E_{22} \\ E_{21} * E_{11} = \frac{E_{21}}{\beta_1} & E_{21} * E_{12} = -i\frac{\sqrt{\beta_2}}{\sqrt{\beta_1}} E_{21} + E_{22} \\ E_{22} * E_{21} = \frac{E_{21}}{\beta_1} & E_{22} * E_{22} = -i\frac{\sqrt{\beta_2}}{\sqrt{\beta_1}} E_{21} + E_{22}; \end{cases}$$

Algebra 3

$$\left\{\begin{array}{ll} E_{11} * E_{11} = E_{11} - \lambda_1 E_{21} & E_{11} * E_{12} = -\frac{\beta_2}{\lambda_1} E_{11} - \frac{\beta_2}{{\gamma_2}^2} E_{12} + \beta_2 E_{21} + \frac{\beta_2}{\lambda_1} E_{22} \\ E_{12} * E_{21} = E_{11} - \lambda_1 E_{21} & E_{12} * E_{22} = -\frac{\beta_2}{\lambda_1} E_{11} - \frac{\beta_2}{{\gamma_2}^2} E_{12} + \beta_2 E_{21} + \frac{\beta_2}{\lambda_1} E_{22} \\ E_{21} * E_{11} = -\frac{\lambda_1^2}{\beta_2} E_{21} & E_{21} * E_{12} = \lambda_1 E_{21} + E_{22} \\ E_{22} * E_{21} = -\frac{\lambda_1^2}{\beta_2} E_{21} & E_{22} * E_{22} = \lambda_1 E_{21} + E_{22}; \end{array}\right.$$

Algebra 4

$$\left\{\begin{array}{ll} E_{11} * E_{11} = E_{11} - \lambda_1 E_{21} & E_{11} * E_{12} = \beta_1 \lambda_1 E_{11} + \beta_1 E_{12} - \beta_1 \lambda_1^2 E_{21} - \beta_1 E_{22} \\ E_{12} * E_{21} = E_{11} - \lambda_1 E_{21} & E_{12} * E_{22} = \beta_1 \lambda_1 E_{11} + \beta_1 E_{12} - \beta_1 \lambda_1^2 E_{21} - \beta_1 E_{22} \\ E_{21} * E_{11} = \frac{E_{21}}{\beta_1} & E_{21} * E_{12} = \lambda_1 E_{21} + E_{22} \\ E_{22} * E_{21} = \frac{E_{21}}{\beta_1} & E_{22} * E_{22} = \lambda_1 E_{21} + E_{22}; \end{array}\right.$$

Algebra 5

$$\left\{\begin{array}{ll} E_{11} * E_{11} = E_{11} + \beta_3 \gamma_1 E_{21} & E_{11} * E_{12} = -\beta_3 E_{11} + \frac{E_{12}}{\gamma_1} - \beta_3 \gamma_1 E_{21} + \beta_3 E_{22} \\ E_{12} * E_{21} = E_{11} + \beta_3 \gamma_1 E_{21} & E_{12} * E_{22} = -\beta_3 E_{11} + \frac{E_{12}}{\gamma_1} - \beta_3 \gamma_1 E_{21} + \beta_3 E_{22} \\ E_{21} * E_{11} = \gamma_1 E_{21} & E_{21} * E_{12} = -\beta_3 \gamma_1 E_{21} + E_{22} \\ E_{22} * E_{21} = \gamma_1 E_{21} & E_{22} * E_{22} = -\beta_3 \gamma_1 E_{21} + E_{22}; \end{array}\right.$$

Algebra 6

$$\left\{\begin{array}{ll} E_{11} * E_{11} = i\sqrt{\beta_2}\sqrt{\gamma_1} E_{21} + E_{22} & E_{11} * E_{12} = \beta_2 E_{21} \\ E_{12} * E_{21} = i\sqrt{\beta_2}\sqrt{\gamma_1} E_{21} + E_{22} & E_{12} * E_{22} = \beta_2 E_{21} \\ E_{21} * E_{11} = i\frac{\sqrt{\gamma_1}}{\sqrt{\beta_2}} E_{11} + \frac{E_{12}}{\beta_2} + \gamma_1 E_{21} - i\frac{\sqrt{\gamma_1}}{\sqrt{\beta_2}} E_{22} & E_{21} * E_{12} = E_{11} \\ & \qquad -i\sqrt{\beta_2}\sqrt{\gamma_1} E_{21} \\ E_{22} * E_{21} = i\frac{\sqrt{\gamma_1}}{\sqrt{\beta_2}} E_{11} + \frac{E_{12}}{\beta_2} + \gamma_1 E_{21} - i\frac{\sqrt{\gamma_1}}{\sqrt{\beta_2}} E_{22} & E_{22} * E_{22} = E_{11} \\ & \qquad -i\sqrt{\beta_2}\sqrt{\gamma_1} E_{21}; \end{array}\right.$$

Algebra 7

$$\left\{\begin{array}{ll} E_{11} * E_{11} = \frac{\beta_4}{\beta_2} E_{12} + E_{22} & E_{11} * E_{12} = \beta_4 E_{11} - \frac{\beta_4^2}{\beta_2} E_{12} + \beta_2 E_{21} - \beta_4 E_{22} \\ E_{12} * E_{21} = \frac{\beta_4}{\beta_2} E_{12} + E_{22} & E_{12} * E_{22} = \beta_4 E_{11} - \frac{\beta_4^2}{\beta_2} E_{12} + \beta_2 E_{21} - \beta_4 E_{22} \\ E_{21} * E_{11} = \frac{E_{12}}{\beta_2} & E_{21} * E_{12} = E_{11} - \frac{\beta_4}{\beta_2} E_{12} \\ E_{22} * E_{21} = \frac{E_{12}}{\beta_2} & E_{22} * E_{22} = E_{11} - \frac{\beta_4}{\beta_2} E_{12}; \end{array}\right.$$

Algebra 8

$$\left\{\begin{array}{ll} E_{11} * E_{11} = E_{11} + \gamma_2 E_{12} & E_{11} * E_{12} = \frac{E_{12}}{\gamma_1} \\ E_{12} * E_{21} = E_{11} + \gamma_2 E_{12} & E_{12} * E_{22} = \frac{E_{12}}{\gamma_1} \\ E_{21} * E_{11} = -\gamma_2 E_{11} - \frac{\gamma_2^2}{\gamma_1} E_{12} + \gamma_1 E_{21} + \gamma_2 E_{22} & E_{21} * E_{12} = -\frac{\beta_2}{\beta_1} E_{12} + E_{22} \\ E_{22} * E_{21} = -\gamma_2 E_{11} - \frac{\gamma_2^2}{\gamma_1} E_{12} + \gamma_1 E_{21} + \gamma_2 E_{22} & E_{22} * E_{22} = -\frac{\beta_2}{\beta_1} E_{12} + E_{22}; \end{array}\right.$$

Algebra 9

$$\left\{\begin{array}{ll} E_{11} * E_{11} = -\gamma_2 E_{21} + E_{22} & E_{11} * E_{12} = \frac{E_{21}}{\gamma_4} \\ E_{12} * E_{21} = -\gamma_2 E_{21} + E_{22} & E_{12} * E_{22} = \frac{E_{21}}{\gamma_4} \\ E_{21} * E_{11} = -\gamma_2 E_{11} + \gamma_4 E_{12} - \frac{\gamma_2^2}{\gamma_4} E_{21} + \gamma_2 E_{22} & E_{21} * E_{12} = E_{11} + \frac{\gamma_2}{\gamma_4} E_{21} \\ E_{22} * E_{21} = -\gamma_2 E_{11} + \gamma_4 E_{12} - \frac{\gamma_2^2}{\gamma_4} E_{21} + \gamma_2 E_{22} & E_{22} * E_{22} = E_{11} + \frac{\gamma_2}{\gamma_4} E_{21}. \end{array}\right.$$

4. Algebraic varieties of Hom-associative algebras and classification

In this section, we deal with algebraic varieties of Hom-associative algebras with a fixed dimension. A Hom-associative algebra is identified with its structure constants with respect to a fixed basis. Their set corresponds to an algebraic variety where the ideal is generated by polynomials corresponding to the Hom-associativity condition.

4.1. Algebraic varieties $\mathcal{HA}ss_n$. Let A be a Hom-algebra defined by a multiplication μ and a twist map α. We set $\mu(e_i, e_j) = \sum_{k=1}^{n} \mathcal{C}_{ij}^k e_k$ and $\alpha(e_i) = \sum_{j=1}^{n} a_{ji} e_j$. Then the Hom-algebra structure is defined by the n^3 structure constants $\mathcal{C}_{ij}^k$ corresponding to μ and the n^2 structure constants a_{ji} corresponding to α with respect to the basis $\{e_1, \ldots, e_n\}$. If we require this algebra structure to be Hom-associative, then this limits the set of structure constants $(\mathcal{C}_{ij}^k, a_{ij})$ to a cubic sub-variety of the affine algebraic variety $\mathbb{K}^{n^3+n^2}$ defined by the polynomial equations system

$$\begin{cases} \displaystyle\sum_{l=1}^{n}\sum_{m=1}^{n} a_{il}\mathcal{C}_{jk}^m \mathcal{C}_{lm}^s - a_{mk}\mathcal{C}_{ij}^l \mathcal{C}_{lm}^s = 0, \quad i,j,k,s = 1,\ldots,n. \\ \displaystyle\sum_{p=1}^{n} a_{sp}\mathcal{C}_{ij}^p - \sum_{p=1}^{n}\sum_{q=1}^{n} a_{pi}a_{qj}\mathcal{C}_{pq}^s = 0, \quad i,j,s = 1,\ldots,n. \end{cases} \tag{4.1}$$

Moreover, if μ is commutative, then we have $\mathcal{C}_{ij}^k = \mathcal{C}_{ji}^k$ for $i,j,k = 1,\ldots,n$.

The first set of equations corresponds to the Hom-associative condition $\mu(\alpha(e_i), \mu(e_j, e_k)) = \mu(\mu(e_i, e_j), \alpha(e_k))$ and the second set to multiplicativity condition $\alpha \circ \mu(e_i, e_j) = \mu(\alpha(e_i), \alpha(e_j))$. We denote by $\mathcal{HA}ss_n$ the set of all n-dimensional multiplicative Hom-associative algebras.

Assume that $e_1 = u$, the unit, in the basis $\mathcal{B}$. It turns out that in addition to the system (4.1), we have the following condition with respect to unitality

$$u_1.e_i = e_i.u_1 = \alpha(e_i) \Rightarrow \sum_{k=1}^{n} \mathcal{C}_{1i}^k e_k = \sum_{k=1}^{n} \mathcal{C}_{i1}^k e_k = \sum_{k=1}^{n} a_{ki} e_k,$$

that is $\mathcal{C}_{i1}^k = \mathcal{C}_{1i}^k = a_{ki}$ for all i, k. We denote by $\mathcal{UHA}ss_n$ the algebraic varieties of n-dimensional unital Hom-associative algebras.

4.2. Action of linear group on the algebraic varieties $\mathcal{HA}ss_n$. The group $GL_n(\mathbb{K})$ acts on the algebraic varieties of Hom-structures by the so-called transport of structure action defined as follows. Let $A = (A, \mu, \alpha)$ be an n-dimensional Hom-associative algebra defined by multiplication μ and a linear map α. Given $f \in GL_n(\mathbb{K})$, the action $f \cdot A$ transports the structure

$$\begin{aligned}\Theta :GL_n(\mathbb{K}) \times \mathcal{HA}ss_n &\longrightarrow \mathcal{HA}ss_n,\\ (f,(A,\mu,\alpha)) &\longmapsto (A, f^{-1}\circ\mu\circ(f\otimes f), f\circ\alpha\circ f^{-1})\end{aligned}$$

defined for $x, y \in A$, by

$$f\cdot\mu(x,y) = f^{-1}\mu(f(x),f(y)), \quad f\cdot\alpha(x) = f^{-1}\alpha(f(x)).$$

The conjugate class is given by

$$\Theta(f,(A,\mu,\alpha)) = (A, f^{-1}\circ\mu\circ(f\otimes f), f\circ\alpha\circ f^{-1})) \text{ for } f\in GL_n(\mathbb{K}).$$

The orbit of a Hom-associative algebra A of $\mathcal{HA}ss_n$ is given by

$$\vartheta(A) = \left\{A' = f\cdot A,\ f\in GL_n(\mathbb{K})\right\}.$$

The orbits are in **1-1 correspondence** with the isomorphism classes of n-dimensional Hom-associative algebras.

The stabilizer is

$$Stab((A,\mu,\alpha)) = \left\{f\in GL_n(\mathbb{K}) | (f^{-1}\circ\mu\circ(f\otimes f) = \mu \text{ and } f\circ\alpha = \alpha\circ f\right\}.$$

We characterize in terms of structure constants the fact that two Hom-associative algebras are in the same orbit (or isomorphic). Let (A,μ_1,α_1) and (A,μ_2,α_2) be two n-dimensional Hom-associative algebras. They are isomorphic if there exists $\varphi\in GL_n(\mathbb{K})$ such that

$$\varphi\circ\mu_1 = \mu_2(\varphi\otimes\varphi) \quad\text{and}\quad \varphi\circ\alpha_1 = \alpha_2\circ\varphi. \tag{4.2}$$

Remark 4.1. Conditions (4.2) are equivalent to $\mu_1 = \varphi^{-1}\circ\mu_2\circ\varphi\otimes\varphi$ and $\alpha_1 = \varphi^{-1}\circ\alpha_2\circ\varphi$.

We set with respect to a basis $\{e_i\}_{i=1,\dots,n}$:

$$\varphi(e_i) = \sum_{p=1}^n a_{pi}e_p, \quad \alpha_1(e_i) = \sum_{j=1}^n \alpha_{ji}e_j, \quad \alpha_2(e_i) = \sum_{j=1}^n \beta_{ji}e_j,$$

$$\mu_1(e_i,e_j) = \sum_{k=1}^n \mathcal{C}_{ij}^k e_k, \quad \mu_2(e_i,e_j) = \sum_{k=1}^n \mathcal{D}_{ij}^k e_k, \quad i,j = 1,\dots,n.$$

Conditions (4.2) translate to the system of equations

$$\sum_{k=1}^n \mathcal{C}_{ij}^k a_{qk} - \sum_{k=1}^n\sum_{p=1}^n \mathcal{D}_{pk}^q a_{pi}a_{kj} = 0, \quad \sum_{k=1}^n \alpha_{ji}a_{qk} - \sum_{k=1}^n a_{ki}\beta_{qk} = 0, \quad i,j,q = 1,\dots,n.$$

4.3. Algebraic variety $\mathcal{HA}ss_2$. A Hom-associative algebra is identified with its structure constants $(C_{i,j}^k)$ and (a_{ij}) with respect to a given basis. They satisfy the first family of system (4.1), for which the solutions belong to the algebraic variety defined by the following Groebner basis:

$$\langle a_{21}c_{11}^1c_{12}^1 - a_{21}c_{11}^1c_{21}^1 - a_{11}c_{11}^2c_{12}^1 + a_{11}c_{11}^2c_{21}^1,$$
$$a_{21}c_{11}^1c_{12}^2 - a_{21}c_{11}^1c_{21}^2 - a_{11}c_{11}^2c_{12}^2 + a_{11}c_{11}^2c_{21}^2,$$

$$a_{12}(c_{11}^1)^2 + a_{11}c_{11}^1c_{12}^1 - a_{22}c_{11}^1c_{12}^1 + a_{11}c_{12}^1c_{12}^2 - a_{12}c_{11}^2c_{21}^1 + a_{21}c_{12}^1c_{21}^1$$
$$- a_{22}c_{11}^2c_{22}^1 + a_{21}c_{12}^2c_{22}^1,$$
$$a_{12}(c_{11}^2)^2 + a_{12}c_{11}^2c_{12}^1 - a_{11}c_{11}^1c_{21}^1 + a_{22}c_{11}^1c_{21}^1 - a_{21}c_{12}^1c_{21}^1 - a_{11}c_{21}^1c_{21}^2$$
$$+ a_{22}c_{11}^2c_{22}^1 - a_{21}c_{21}^2c_{22}^1,$$
$$- a_{11}c_{11}^1c_{12}^1 - a_{21}(c_{12}^1)^2 + a_{11}c_{11}^1c_{21}^1 - a_{11}c_{12}^2c_{21}^1 + a_{21}(c_{21}^1)^2 + a_{11}c_{12}^1c_{21}^2$$
$$- a_{21}c_{12}^2c_{22}^1 + a_{21}c_{21}^2c_{22}^1,$$
$$a_{12}c_{11}^1c_{12}^1 + a_{12}c_{12}^1c_{12}^2 - a_{12}c_{11}^1c_{21}^1 - a_{12}c_{21}^1c_{21}^2 + a_{22}c_{12}^2c_{22}^1 - a_{22}c_{21}^2c_{22}^1,$$
$$- a_{22}c_{12}^1c_{21}^1 + a_{12}c_{12}^1c_{22}^2 + a_{22}c_{21}^1c_{22}^1 - a_{12}c_{21}^1c_{22}^2,$$
$$a_{22}c_{12}^2c_{22}^1 - a_{12}c_{12}^2c_{22}^2 - a_{22}c_{12}^2c_{22}^2 + a_{22}c_{21}^2c_{22}^2,$$
$$a_{12}c_{11}^1c_{11}^2 + a_{11}c_{11}^2c_{12}^1 - a_{22}c_{11}^1c_{12}^2 + a_{11}(c_{12}^2)^2 - a_{12}c_{11}^2c_{21}^2 + a_{21}c_{12}^1c_{21}^2$$
$$- a_{22}c_{11}^2c_{22}^2 + a_{21}c_{12}^2c_{22}^2,$$
$$a_{12}c_{11}^1c_{11}^2 + a_{12}c_{11}^2c_{12}^2 - a_{11}c_{11}^2c_{21}^1 - a_{21}c_{12}^2c_{21}^1 + a_{22}c_{11}^1c_{21}^2$$
$$- a_{11}(c_{21}^2)^2 + a_{22}c_{11}^2c_{22}^2 - a_{21}c_{21}^2c_{22}^2,$$
$$- a_{11}c_{11}^2c_{12}^1 - a_{21}c_{12}^1c_{12}^2 + a_{11}c_{11}^2c_{21}^1 + a_{21}c_{21}^1c_{21}^2 - a_{21}c_{12}^2c_{22}^2 + a_{21}c_{21}^2c_{22}^2,$$
$$a_{12}c_{11}^2c_{12}^1 + a_{12}(c_{12}^2)^2 - a_{12}c_{11}^2c_{21}^1 - a_{22}c_{12}^2c_{21}^1 + a_{22}c_{12}^1c_{21}^2 - a_{12}(c_{21}^2)^2$$
$$+ a_{22}c_{12}^2c_{22}^2 - a_{22}c_{21}^2c_{22}^2,$$
$$a_{12}c_{11}^1c_{21}^1 + a_{22}(c_{21}^1)^2 + a_{12}c_{12}^1c_{21}^2 - a_{11}c_{11}^1c_{22}^1 - a_{21}c_{12}^1c_{22}^1 + a_{22}c_{21}^2c_{22}^1$$
$$- a_{11}c_{21}^1c_{22}^2 - a_{21}c_{22}^1c_{22}^2,$$
$$- a_{12}c_{11}^1c_{12}^1 - a_{22}(c_{12}^1)^2 - a_{12}c_{12}^2c_{21}^1 + a_{11}c_{11}^1c_{22}^1 - a_{22}c_{12}^2c_{22}^1 + a_{21}c_{21}^1c_{22}^1$$
$$+ a_{11}c_{12}^1c_{22}^2 + a_{21}c_{22}^1c_{22}^2,$$
$$a_{12}c_{11}^2c_{21}^1 + a_{12}c_{12}^2c_{21}^2 + a_{22}c_{21}^1c_{21}^2 - a_{11}c_{11}^2c_{22}^1 - a_{21}c_{12}^2c_{22}^1 - a_{11}c_{21}^2c_{22}^2$$
$$+ a_{22}c_{21}^2c_{22}^2 - a_{21}(c_{22}^2)^2,$$
$$- a_{12}c_{11}^2c_{12}^1 - a_{22}c_{12}^1c_{12}^2 - a_{12}c_{12}^2c_{21}^2 + a_{11}c_{11}^2c_{22}^1 + a_{21}c_{21}^2c_{22}^1 + a_{11}c_{12}^2c_{22}^2$$
$$- a_{22}c_{12}^2c_{22}^2 + a_{21}(c_{22}^2)^2\rangle.$$

If the Hom-associative algebra is multiplicative, it should satisfy further the second family of (4.1), that is, it belongs to the intersection with the algebraic variety defined by the Groebner basis:

$$\langle a_{11}c_{11}^1 - a_{11}^2c_{11}^1 + a_{12}c_{11}^2 - a_{11}a_{21}c_{12}^1 - a_{11}a_{21}c_{21}^1 - a_{21}^2c_{22}^1,$$
$$a_{11}a_{12}c_{11}^1 + a_{11}c_{12}^1 - a_{11}a_{2,2}c_{12}^1 + a_{12}c_{12}^2 - a_{12}a_{21}c_{21}^1 - a_{21}a_{22}c_{22}^1$$
$$a_{11}a_{12}c_{11}^1 - a_{12}a_{21}c_{12}^1 + a^{11}c_{21}^1 - a_{11}a_{22}c_{21}^1 + a_{12}c_{21}^2 - a_{21}a_{22}c_{22}^1,$$
$$a_{12}^2c_{11}^1 - a_{12}a_{22}c_{12}^1 - a_{12}a_{22}c_{21}^1 + a_{11}c_{22}^1 - a_{22}^2c_{22}^1 + a_{12}c_{22}^2,$$

$$a_{21}c_{11}^1 - a_{11}^2c_{11}^2 + a_{22}c_{11}^2 - a_{11}a_{21}c_{12}^2 - a_{11}a_{21}c_{21}^2 - a_{21}^2c_{22}^2,$$
$$a_{11}a_{12}c_{11}^2 + a_{21}c_{12}^1 + a_{22}c_{12}^2 - a_{11}a_{22}c_{12}^2 - a_{12}a_{21}c_{21}^2 - a_{21}a_{22}c_{22}^2,$$
$$a_{11}a_{12}c_{11}^2 - a_{12}a_{21}c_{12}^2 + a_{21}c_{21}^1 + a_{22}c_{21}^2 - a_{11}a_{22}c_{21}^2 - a_{21}a_{22}c_{22}^2,$$
$$a_{12}^2c_{11}^2 - a_{12}a_{22}c_{12}^2 - a_{12}a_{22}c_{21}^2 + a_{21}c_{22}^1 + a_{22}c_{22}^2 - a_{22}^2c_{22}^2\rangle.$$

Describing the algebraic varieties by solving such systems lead to the classification of 2-dimensional and 3-dimensional Hom-associative algebras.

4.4. Classification of 2-dimensional Hom-associative algebras. We have to consider two classes of morphisms which are given by Jordan forms, namely they are represented by the matrices $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ and $\begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}$. We check whether the previous are isomorphic. We provide all 2-dimensional Hom-associative algebras, corresponding to solutions of the system (4.1). To this end, we use a computer algebra system.

Lemma 4.2. *Let α be a diagonal morphism such that $\alpha(e_1) = pe_1$, $\alpha(e_2) = qe_2$, $p \neq q$, with respect to basis $\{e_1, e_2\}$. Then any $\varphi : A \to A$ such that $\varphi \circ \alpha = \alpha \circ \varphi$ is of the form $\varphi(e_1) = \lambda e_1$ and $\varphi(e_2) = \rho e_2$ with respect to the same basis.*

Proof. Let $\varphi(e_1) = \lambda_1 e_1 + \lambda_2 e_2$ and $\varphi(e_2) = \rho_1 e_1 + \rho_2 e_2$. On the one hand, $\varphi \circ \alpha(e_1) = \lambda_1 pe_1 + \lambda_2 pe_2$ and $\alpha' \circ \varphi(e_1) = \lambda_1 p' e_1 + \lambda_2 q' e_2$. So we have $\lambda_1 p = \lambda_1 p'$ and $\lambda_2 p = \lambda_2 q'$. On the other hand, $\varphi \circ \alpha(e_2) = q\rho_1 e_1 + q\rho_2 e_2$ and $\alpha' \circ \varphi(e_2) = \rho_1 p' e_1 + \rho_2 q' e_2$. We have $\rho_1 q = \rho_1 p'$ and $\rho_2 q = \rho_2 q'$. Then we have $\lambda_1(p - p') = 0, \quad \lambda_2(p - q') = 0, \quad \rho_1(q - p') = 0, \quad \rho_2(q - q') = 0.$

If $p = p'$ and $q = q'$, we have $\lambda_2(p - q') = 0$ and $\rho_2(q - q') = 0$. If $p \neq q$, then $\lambda_2 = \rho_1 = 0$. Hence the lemma with $\lambda = \lambda_1$ and $\rho = \rho_2$. □

Theorem 4.3. *Every 2-dimensional multiplicative Hom-associative algebra is isomorphic to one of the following pairwise non-isomorphic Hom-associative algebras $(A, *, \alpha)$, where $*$ is the multiplication, α is the structure map, and $\{e_1, e_2\}$ is a basis of $\mathbb{K}^2$:*

A_1^2 : $e_1 * e_1 = -e_1, \quad e_1 * e_2 = e_2, \quad e_2 * e_1 = e_2, \quad e_2 * e_2 = e_1,$
$\alpha(e_1) = e_1, \quad \alpha(e_2) = -e_2;$

A_2^2 : $e_1 * e_1 = e_1, \quad e_1 * e_2 = 0, \quad e_2 * e_1 = 0, \quad e_2 * e_2 = e_2,$
$\alpha(e_1) = e_1, \quad \alpha(e_2) = 0;$

A_3^2 : $e_1 * e_1 = e_1, \quad e_1 * e_2 = 0, \quad e_2 * e_1 = 0, \quad e_2 * e_2 = 0,$
$\alpha(e_1) = e_1, \quad \alpha(e_2) = 0;$

A_4^2 : $e_1 * e_1 = e_1, \quad e_1 * e_2 = e_2, \quad e_2 * e_1 = e_2, \quad e_2 * e_2 = 0,$
$\alpha(e_1) = e_1, \quad \alpha(e_2) = e_2;$

A_5^2 : $e_1 * e_1 = e_1, \quad e_1 * e_2 = 0, \quad e_2 * e_1 = 0, \quad e_2 * e_2 = 0,$
$\alpha(e_1) = 0, \quad \alpha(e_2) = ke_2;$

A_6^2 : $e_1 * e_1 = e_2, \quad e_1 * e_2 = 0, \quad e_2 * e_1 = 0, \quad e_2 * e_2 = 0,$
$\alpha(e_1) = e_1, \quad \alpha(e_2) = e_2;$
A_7^2 : $e_1 * e_1 = 0, \quad e_1 * e_2 = ae_1, \quad e_2 * e_1 = be_1, \quad e_2 * e_2 = ce_1,$
$\alpha(e_1) = 0, \quad \alpha(e_2) = e_1$, where $a, b, c, k \in \mathbb{C}$;
A_8^2 : $e_1 * e_1 = 0, \quad e_1 * e_2 = e_1, \quad e_2 * e_1 = 0, \quad e_2 * e_2 = e_1 + e_2,$
$\alpha(e_1) = e_1, \quad \alpha(e_2) = e_1 + e_2;$
A_9^2 : $e_1 * e_1 = 0, \quad e_1 * e_2 = 0, \quad e_2 * e_1 = e_1, \quad e_2 * e_2 = e_1 + e_2,$
$\alpha(e_1) = e_1, \quad \alpha(e_2) = e_1 + e_2.$

Proof. The proof follows from straightforward calculation using Definition 2.4 and Lemma 4.2. □

Proposition 4.4. *The Hom-associative algebras A_1, A_4, A_6, A_8, A_9 are of associative type.*

Proof. Indeed, we set in the following corresponding associative algebras:
$\tilde{A}_1^2$: $e_1 \cdot e_1 = -e_1, \quad e_1 \cdot e_2 = -e_2, \quad e_2 \cdot e_1 = -e_2, \quad e_2 \cdot e_2 = e_1;$
$\tilde{A}_4^2$: $e_1 \cdot e_1 = e_1, \quad e_1 \cdot e_2 = e_2, \quad e_2 \cdot e_1 = e_2, \quad e_2 \cdot e_2 = 0;$
$\tilde{A}_6^2$: $e_1 \cdot e_1 = e_2, \quad e_1 \cdot e_2 = 0, \quad e_2 \cdot e_1 = 0, \quad e_2 \cdot e_2 = 0;$
$\tilde{A}_8^2$: $e_1 \cdot e_1 = 0, \quad e_1 \cdot e_2 = e_1, \quad e_2 \cdot e_1 = 0, \quad e_2 \cdot e_2 = e_2;$
$\tilde{A}_9^2$: $e_1 \cdot e_1 = 0, \quad e_1 \cdot e_2 = e_1, \quad e_2 \cdot e_1 = 0, \quad e_2 \cdot e_2 = e_2.$

□

Remark 4.5. It turns out that $A_2^2, A_3^2, A_5^2, A_7^2$ cannot be obtained by twisting of an associative algebra.

4.5. Classification of 3-dimensional Hom-associative algebras. We seek for all 3-dimensional Hom-associative algebras. We consider two classes of morphism which are given by Jordan form, namely they are represented by the matrices

$$\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}, \quad \begin{pmatrix} a & 1 & 0 \\ 0 & a & 0 \\ 0 & 0 & b \end{pmatrix}, \quad \begin{pmatrix} a & 1 & 0 \\ 0 & a & 1 \\ 0 & 0 & a \end{pmatrix}.$$

Using similar calculation as in the previous section, we obtain the following classification.

Theorem 4.6. *Every 3-dimensional multiplicative Hom-associative algebra is isomorphic to one of the following pairwise non-isomorphic Hom-associative algebras $(A, *, \alpha)$, where $*$ is the multiplication, α is the structure map (the non written products and images of α are equal to zero), and $\{e_1, e_2, e_3\}$ is a basis of $\mathbb{K}^3$:*

A_1^3 : $e_1 * e_1 = e_1, \quad e_2 * e_2 = e_2 + e_3 \quad e_2 * e_3 = e_2 + e_3, \quad e_3 * e_2 = e_2 + e_3,$
$e_3 * e_3 = e_2 + e_3, \quad \alpha(e_1) = e_1;$

A_2^3 : $e_1 * e_1 = p_1e_1, \quad e_2 * e_2 = p_2e_2, \quad e_3 * e_3 = p_3e_3,$
$\alpha(e_1) = e_1, \quad \alpha(e_2) = e_2;$
A_3^3 : $e_1 * e_1 = p_1e_1, \quad e_2 * e_2 = p_2e_2, \quad e_3 * e_3 = p_3e_3, \quad \alpha(e_1) = e_1,$
$\alpha(e_2) = e_2, \quad \alpha(e_3) = e_3;$
A_4^3 : $e_1 * e_2 = p_1e_1, e_1 * e_3 = p_2e_1, e_2 * e_2 = p_3e_1, \quad e_2 * e_3 = p_4e_1,$
$e_3 * e_1 = p_5e_1, \quad e_3 * e_2 = p_4e_1, \ e_3 * e_3 = p_6e_1, \quad \alpha(e_2) = e_1;$
A_5^3 : $e_2 * e_2 = p_1e_1, \quad e_3 * e_3 = p_2e_3, \quad \alpha(e_1) = e_1, \quad \alpha(e_2) = e_1 + e_2;$
A_6^3 : $e_1 * e_2 = e_1, \quad e_2 * e_2 = e_1, \quad e_2 * e_3 = e_1, \quad e_3 * e_2 = e_1,$
$\alpha(e_2) = e_1, \quad \alpha(e_3) = e_3;$
A_7^3 : $e_2 * e_2 = e_1, \quad e_2 * e_3 = e_1, e_3 * e_2 = e_1, \quad e_3 * e_3 = e_1, \quad \alpha(e_1) = e_1,$
$\alpha(e_2) = e_1 + e_2, \quad \alpha(e_3) = e_3;$
A_8^3 : $e_1 * e_2 = -e_3, \quad e_2 * e_1 = e_3, \quad e_2 * e_2 = e_3, \quad \alpha(e_1) = e_1,$
$\alpha(e_2) = e_1 + e_2, \quad \alpha(e_3) = e_3;$
A_9^3 : $e_2 * e_3 = p_1e_1, \quad e_3 * e_2 = p_2e_1, \quad \alpha(e_1) = ae_1,$
$\alpha(e_2) = e_1 + ae_2, \quad \alpha(e_3) = e_3;$
A_{10}^3 : $e_2 * e_2 = p_1e_1, \quad e_3 * e_3 = p_2e_1, \quad \alpha(e_1) = e_1,$
$\alpha(e_2) = e_1 + e_2, \quad \alpha(e_3) = -e_3;$
A_{11}^3 : $e_1 * e_3 = p_1e_1, \quad e_2 * e_3 = p_2e_1, \quad e_3 * e_3 = p_3e_1,$
$\alpha(e_2) = e_1, \quad \alpha(e_3) = e_2;$
A_{12}^3 : $e_2 * e_3 = -p_1e_1, \quad e_3 * e_2 = p_1e_1, \quad e_3 * e_3 = p_2e_1, \quad \alpha(e_1) = e_1,$
$\alpha(e_2) = e_1 + e_2, \quad \alpha(e_3) = e_2 + e_3.$

Proposition 4.7. *The Hom-associative algebras* $A_3^3, A_7^3, A_8^3, A_9^3, A_{10}^3, A_{12}^3$ *are of associative type.*

Proof. Indeed, we set in the corresponding associative algebras

$\tilde{A}_3^3$: $e_1 \cdot e_1 = p_1e_1, \quad e_2 \cdot e_2 = p_2e_2, \quad e_3 \cdot e_3 = p_3e_3;$
$\tilde{A}_7^3$: $e_2 \cdot e_1 = e_1 \quad e_2 \cdot e_2 = e_1, \quad e_2 \cdot e_3 = e_1, \quad e_3 \cdot e_2 = e_1, \quad e_3 \cdot e_3 = e_1;$
$\tilde{A}_8^3$: $e_1 \cdot e_2 = -e_3, \quad e_2 \cdot e_1 = e_3 \quad e_2 \cdot e_2 = e_3;$
$\tilde{A}_9^3$: $e_2 \cdot e_3 = \frac{p_1}{a}e_1, \quad e_3 \cdot e_2 = \frac{p_2}{a}e_1;$
$\tilde{A}_{10}^3$: $e_2 \cdot e_2 = p_1e_1, \quad e_3 \cdot e_3 = p_2e_1;$
$\tilde{A}_{12}^3$: $e_2 \cdot e_1 = e_3 \quad e_2 \cdot e_3 = -p_1e_1, \quad e_3 \cdot e_2 = p_1e_1, \quad e_3 \cdot e_3 = p_2e_1,$

where p_i are parameters. ☐

Remark 4.8. It turns out that $A_1^3, A_2^3, A_4^3, A_5^3, A_6^3, A_{11}^3$ cannot be obtained by twisting of an associative algebra.

Theorem 4.9. *Every* 2*-dimensional unital multiplicative Hom-associative algebra is isomorphic to one of the following pairwise non-isomorphic Hom-associative algebras* $(A, *, \alpha)$*, where* $*$ *is the multiplication,* α *is the structure map, and* $\{e_1, e_2\}$ *is a basis of* $\mathbb{K}^2$ *with the unit vector* e_1*:*

$A_1'^2$: $e_1 * e_1 = e_1, \quad e_1 * e_2 = e_2, \quad e_2 * e_1 = e_2, \quad e_2 * e_2 = e_1 + e_2,$
$\alpha(e_1) = e_1, \quad \alpha(e_2) = e_2;$

A'^2_2 : $e_1 * e_1 = e_1, \quad e_1 * e_2 = -e_2, \quad e_2 * e_1 = -e_2, \quad e_2 * e_2 = e_1,$
$\alpha(e_1) = e_1, \quad \alpha(e_2) = -e_2;$
A'^2_3 : $e_1 * e_1 = e_1, \quad e_1 * e_2 = 0, \quad e_2 * e_1 = 0, \quad e_2 * e_2 = e_2,$
$\alpha(e_1) = e_1, \quad \alpha(e_2) = 0;$
A'^2_4 : $e_1 * e_1 = e_1, \quad e_1 * e_2 = e_2, \quad e_2 * e_1 = e_2, \quad e_2 * e_2 = 0,$
$\alpha(e_1) = e_1, \quad \alpha(e_2) = e_2.$

Proposition 4.10. *The unital Hom-associative algebras* $\tilde{A'}^2_1, \tilde{A'}^2_2, \tilde{A'}^2_4$ *are of associative type.*

Proof. Indeed, we set in the following the corresponding associative algebras:
$\tilde{A'}^2_1$: $e_1 \cdot e_1 = e_1, \quad e_1 \cdot e_2 = e_2, \quad e_2 \cdot e_1 = e_2, \quad e_2 \cdot e_2 = e_1 + e_2;$
$\tilde{A'}^2_2$: $e_1 \cdot e_1 = e_1, \quad e_1 \cdot e_2 = e_2, \quad e_2 \cdot e_1 = e_2, \quad e_2 \cdot e_2 = e_1;$
$\tilde{A'}^2_4$: $e_1 \cdot e_1 = e_1, \quad e_1 \cdot e_2 = e_2, \quad e_2 \cdot e_1 = e_2, \quad e_2 \cdot e_2 = 0.$

□

Remark 4.11. It turns out that $\tilde{A'}^2_3$ cannot be obtained by twisting of an associative algebra.

Theorem 4.12. *Every 3-dimensional unital multiplicative Hom-associative algebra is isomorphic to one of the following pairwise non-isomorphic Hom-associative algebras* $(A, *, \alpha)$*, where* $*$ *is the multiplication,* α *is the structure map, and* $\{e_1, e_2, e_3\}$ *is a basis of* $\mathbb{K}^3$ *with the unit* e_1 *(the non-written products and images of* α *are equal to zero):*

A'^3_1 : $e_1 * e_1 = e_1, \quad e_2 * e_2 = e_2 + e_3, \quad e_2 * e_3 = e_2 + e_3, \quad e_3 * e_2 = e_2 + e_3,$
$e_3 * e_3 = e_2 + e_3, \quad \alpha(e_1) = e_1;$
A'^3_2 : $e_1 * e_1 = e_1, \quad e_2 * e_2 = e_2, \quad e_3 * e_1 = e_3, \quad e_3 * e_3 = e_1 + e_3,$
$\alpha(e_1) = e_1, \quad \alpha(e_3) = e_3;$
A'^3_3 : $e_1 * e_1 = e_1, \quad e_2 * e_2 = e_2, \quad e_1 * e_3 = -e_3, \quad e_3 * e_1 = -e_3,$
$e_3 * e_3 = e_1, \quad \alpha(e_1) = e_1, \quad \alpha(e_3) = -e_3;$
A'^3_4 : $e_1 * e_1 = e_1, \quad e_1 * e_2 = e_2, \quad e_2 * e_1 = e_2, \quad e_2 * e_2 = e_1 + e_2,$
$e_3 * e_3 = e_3, \quad \alpha(e_1) = e_1, \quad \alpha(e_2) = e_2;$
A'^3_5 : $e_1 * e_1 = e_1, \quad e_1 * e_2 = -e_2, \quad e_2 * e_1 = -e_2, \quad e_2 * e_2 = e_1,$
$e_3 * e_3 = e_3, \quad \alpha(e_1) = e_1, \quad \alpha(e_2) = -e_2;$
A'^3_6 : $e_1 * e_1 = e_1, \quad e_1 * e_2 = e_2, \quad e_1 * e_3 = e_3, \quad e_2 * e_1 = e_2, \quad e_2 * e_2 = e_2,$
$e_2 * e_3 = e_3, e_3 * e_1 = e_3, e_3 * e_2 = e_3, e_3 * e_3 = e_2 + e_3, \alpha(e_1) = e_1,$
$\alpha(e_2) = e_2, \alpha(e_3) = e_3;$
A'^3_7 : $e_1 * e_1 = e_1, \quad e_1 * e_2 = e_2, e_1 * e_3 = -e_3, \quad e_2 * e_1 = e_2, e_2 * e_2 = -e_2,$
$e_2 * e_3 = e_3, e_3 * e_1 = -e_3, e_3 * e_2 = e_3, e_3 * e_3 = e_2,$
$\alpha(e_1) = e_1, \alpha(e_2) = e_2, \alpha(e_3) = -e_3;$
A'^3_8 : $e_1 * e_1 = e_1, e_1 * e_2 = -e_2, e_1 * e_3 = e_3, e_2 * e_1 = -e_2, e_2 * e_2 = e_3,$
$e_2 * e_3 = e_2, e_3 * e_1 = e_3, e_3 * e_2 = e_2, e_3 * e_3 = -e_3,$
$\alpha(e_1) = e_1, \alpha(e_2) = -e_2, \alpha(e_3) = e_3;$

A'^3_9 : $e_1 * e_1 = e_1, \quad e_1 * e_2 = ae_2, \quad e_1 * e_3 = e_3, \quad e_2 * e_1 = ae_2,$
$e_3 * e_1 = e_3, \quad e_3 * e_3 = e_3, \quad \alpha(e_1) = e_1, \alpha(e_2) = ae_2, \quad \alpha(e_3) = e_3;$
A'^3_{10} : $e_1*e_1 = e_1$, $e_1*e_2 = ae_2$, $e_1*e_3 = -e_3$, $e_2*e_1 = ae_2$, $e_3*e_1 = -e_3$,
$\alpha(e_1) = e_1$, $\alpha(e_2) = ae_2$, $\alpha(e_3) = -e_3$;
A'^3_{11} : $e_1 * e_1 = e_1, \quad e_1 * e_2 = ae_2, \quad e_1 * e_3 = a^2 e_3, \quad e_2 * e_1 = ae_2,$
$e_2 * e_2 = e_3$, $e_3 * e_1 = a^2 e_3$, $\alpha(e_1) = e_1, \alpha(e_2) = ae_2, \quad \alpha(e_3) = a^2 e_3;$
A'^3_{12} : $e_1 * e_1 = e_1$, $e_1 * e_2 = e_2$, $e_1 * e_3 = be_3$, $e_2 * e_1 = e_2$, $e_2 * e_2 = \frac{1}{b} e_2$,
$e_2 * e_3 = e_3$, $e_3 * e_1 = be_3$, $\alpha(e_1) = e_1$, $\alpha(e_2) = e_2$, $\alpha(e_3) = be_3$;
A'^3_{13} : $e_1*e_1 = e_1$, $e_1*e_2 = -e_2$, $e_1*e_3 = be_3$, $e_2*e_1 = -e_2$, $e_3*e_1 = be_3$,
$\alpha(e_1) = e_1$, $\alpha(e_2) = -e_2$, $\alpha(e_3) = be_3$;
A'^3_{14} : $e_1*e_1 = e_1$, $e_1*e_2 = b^2 e_2$, $e_1*e_3 = be_3$, $e_2*e_1 = b^2 e_2$, $e_3*e_1 = be_3$,
$e_3 * e_3 = e_2$, $\alpha(e_1) = e_1, \alpha(e_2) = b^2 e_2$, $\alpha(e_3) = be_3$;
A'^3_{15} : $e_1 * e_1 = e_1$, $e_1 * e_2 = ae_2$, $e_1 * e_3 = be_3$, $e_2 * e_1 = ae_2$, $e_3 * e_1 = be_3$,
$\alpha(e_1) = e_1$, $\alpha(e_2) = ae_2$, $\alpha(e_3) = be_3$.

Property 4.13. The unital Hom-associative algebras $\tilde{A'}^3_6, \tilde{A'}^3_7, \tilde{A'}^3_8, \tilde{A'}^3_9$, $\tilde{A'}^3_{10}, \tilde{A'}^3_{11}, \tilde{A'}^3_{12}$, $\tilde{A'}^3_{13}, \tilde{A'}^3_{14}, \tilde{A'}^3_{15}$ are of associative type.

Proof. Indeed, we set in the following, the corresponding associative algebras:

$\tilde{A'}^3_6$: $e_1 \cdot e_1 = e_1$, $e_1 \cdot e_2 = e_2$, $e_1 \cdot e_3 = e_3$, $e_2 \cdot e_1 = e_2$ $e_2 \cdot e_2 = e_2$, $e_2 \cdot e_3 = e_3$,
$e_3 \cdot e_1 = e_3$, $e_3 \cdot e_2 = e_3$, $e_3 \cdot e_3 = e_2 + e_3$;
$\tilde{A'}^3_7$: $e_1 \cdot e_1 = e_1$, $e_1 \cdot e_2 = e_2$, $e_1 \cdot e_3 = e_3$, $\quad e_2 \cdot e_1 = e_2$ $e_2 \cdot e_2 = -e_2$,
$e_2 \cdot e_3 = -e_3, \quad e_3 \cdot e_1 = e_3$, $e_3 \cdot e_2 = -e_3, \quad e_3 \cdot e_3 = e_2$;
$\tilde{A'}^3_8$: $e_1 \cdot e_1 = e_1$, $e_1 \cdot e_2 = e_2$, $e_1 \cdot e_3 = e_3$, $e_2 \cdot e_1 = e_2$ $e_2 \cdot e_2 = e_3$, $e_2 \cdot e_3 = e_2$,
$e_3 \cdot e_1 = e_3$, $e_3 \cdot e_2 = -e_2$, $e_3 \cdot e_3 = -e_3$;
$\tilde{A'}^3_9$: $e_1 \cdot e_1 = e_1, \quad e_1 \cdot e_2 = e_2, \quad e_1 \cdot e_3 = e_3, \quad e_2 \cdot e_1 = e_2,$
$e_3 \cdot e_1 = e_3, e_3 \cdot e_3 = e_3;$
$\tilde{A'}^3_{10}$: $e_1 \cdot e_1 = e_1, \quad e_1 \cdot e_2 = e_2, \quad e_1 \cdot e_3 = e_3, \quad e_2 \cdot e_1 = e_2, \quad e_3 \cdot e_1 = e_3;$
$\tilde{A'}^3_{11}$: $e_1 \cdot e_1 = e_1, \quad e_1 \cdot e_2 = e_2, \quad e_1 \cdot e_3 = e_3, \quad e_2 \cdot e_1 = e_2$
$e_2 \cdot e_2 = e_3, \quad e_3 \cdot e_1 = e_3;$
$\tilde{A'}^3_{12}$: $e_1 \cdot e_1 = e_1, \quad e_1 \cdot e_2 = e_2, \quad e_1 \cdot e_3 = e_3, \quad e_2 \cdot e_1 = e_2 \quad e_2 \cdot e_2 = e_2,$
$e_2 \cdot e_3 = e_3, \quad e_3 \cdot e_1 = e_3;$
$\tilde{A'}^3_{14}$: $e_1 \cdot e_1 = e_1, \quad e_1 \cdot e_2 = e_2, \quad e_1 \cdot e_3 = e_3, \quad e_2 \cdot e_1 = e_2,$
$e_3 \cdot e_1 = e_3, \quad e_3 \cdot e_3 = e_2.$

□

Remark 4.14. It turns out that $\tilde{A'}^3_1, \tilde{A'}^3_2, \tilde{A'}^3_3, \tilde{A'}^3_4, \tilde{A'}^3_5$ cannot be obtained by twisting an associative algebra.

5. Deformations and irreducible components of Hom-associative algebras

In this section, we aim to discuss the geometric classification of $\mathcal{HA}ss_n$ and $\mathcal{UHA}ss_n$ for $n = 2, 3$. To this end we use one parameter formal deformation theory introduced first by Gerstenhaber for associative algebras and extended to Hom-associative algebras in [2, 8].

Definition 5.1. Let (A, μ, α) be a Hom-associative algebra. A formal deformation of the Hom-associative algebra $\mathcal{A}$ is given by a $\mathbb{K}[[t]]$-bilinear map $\mu_t : A[[t]] \times A[[t]] \longrightarrow A[[t]]$ of the form $\mu_t = \sum_{i\geq 0} t^i \mu_i$, where each μ_i is a $\mathbb{K}$-bilinear-map $\mu_i : A \times A \to A$ (extended to be $\mathbb{K}[[t]]$-bilinear) and $\mu_0 = \mu$ such that, for $x, y, z \in A$,

$$\mu_t(\mu_t(x, y), \alpha(z)) = \mu_t(\alpha(x), \mu_t(y, z)).$$

Suppose that $(A\,[[t]]\,, \mu_{1,t}, \alpha_{1,t})$ and $(A\,[[t]]\,, \mu'_{1,t}, \alpha'_{1,t})$ are Hom-associative deformations of the Hom-associative algebras (A, μ, α). They are said to be equivalent if there exists a formal isomorphism between them, i.e., a $\mathbb{K}\,[[t]]$-linear map φ_t, compatible with both the deformed multiplications and the deformed twisting maps, of the form $\varphi_t = \sum_{i\geq 0} t^i \varphi_i$, where the $\varphi_i \colon A \to A$ are linear maps and $\varphi_0 = id_A$. Compatibility with the deformed multiplications means that $\varphi_t \circ \mu_t = \mu' \circ (\varphi_t \otimes \varphi_t)$, compatibility with the twisting maps means $\varphi_t \circ \alpha_t = \alpha' \circ \varphi_t$.

Proposition 5.2. *Let $\mu_{1,t} = \phi^{-1} \circ \mu_2 \circ (\phi \otimes \phi)$ and $\alpha_{1,t} = \phi^{-1} \circ \alpha_2 \circ \phi$. If (A, μ_2, α_2) is Hom-associative, then $(A\,[[t]]\,, \mu_{1,t}, \alpha_{1,t})$ is Hom-associative.*

Proof. By straightforward computation, we have

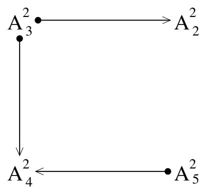
$$\begin{aligned}
\mu_{1,t}(\alpha_{1,t}(x), \mu_{1,t}(y, z)) &= \phi^{-1}\mu_2(\phi(\phi^{-1} \circ \alpha_2 \circ \phi(x)), \phi\phi^{-1} \circ \mu_2(\phi(y), \phi(z))) \\
&= \phi^{-1}\mu_2(\alpha_2 \circ \phi(x), \mu_2(\phi(y), \phi(z))) \\
&= \phi^{-1}\mu_2(\mu_2(\phi(x), \phi(y)), \alpha_2 \circ \phi(z)) \\
&= \phi^{-1}\mu_2(\phi \circ \phi^{-1}(\mu_2(\phi(x), \phi(y))), \phi \circ \phi^{-1}\alpha_2\phi(z))) \\
&= \mu_{1,t}(\mu_{1,t}(x, y), \alpha_{1,t}(z)).
\end{aligned}$$

□

Definition 5.3. A Hom-associative algebra A is called formally rigid, if every formal deformation of A is trivial. It is called geometrically rigid, if its orbit $\vartheta(\mu)$ is open in $\mathcal{HA}ss_n$. Then $\overline{\vartheta(\mu)}$ is an irreducible component of $\mathcal{HA}ss_n$.

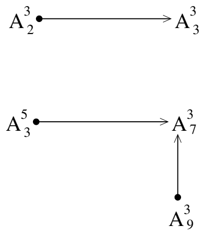
Remark 5.4. Any irreducible component $\mathcal{C}$ of $\mathcal{HA}ss_n$ containing A also contains all degenerations of A. Indeed, we have $\vartheta(\mu) \subset \mathcal{C}$ so that $\overline{\vartheta(\mu)}$ is contained in $\mathcal{C}$, since $\mathcal{C}$ is closed.

Proposition 5.5. *The irreducible components of $\mathcal{HA}ss_2$ are the Zariski closure of orbits of Hom-associative algebras $\Omega = \left\{A_3^2, A_5^2\right\}$.*



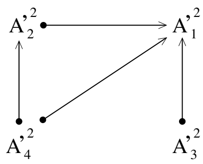
Irreducible components of $\mathcal{HA}ss_2$.

Proposition 5.6. *The irreducible components of $\mathcal{HA}ss_3$ are the Zariski closure of orbits of Hom-associative algebras $\Omega = \left\{A_2^3, A_5^3, A_9^3\right\}$.*



Irreducible components of $\mathcal{A}ss_3$.

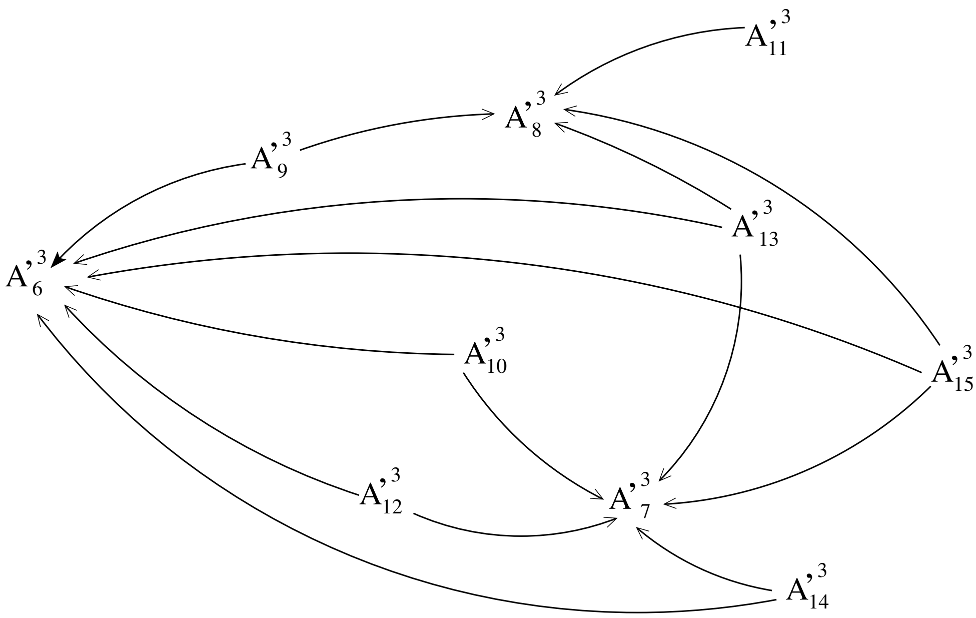
Proposition 5.7. *The irreducible components of $\mathcal{UHA}ss_2$ are the Zariski closure of orbits of Hom-associative algebras $\Omega = \left\{A_3'^2, A_4'^2\right\}$.*



Irreducible components of $\mathcal{UHA}ss_2$.

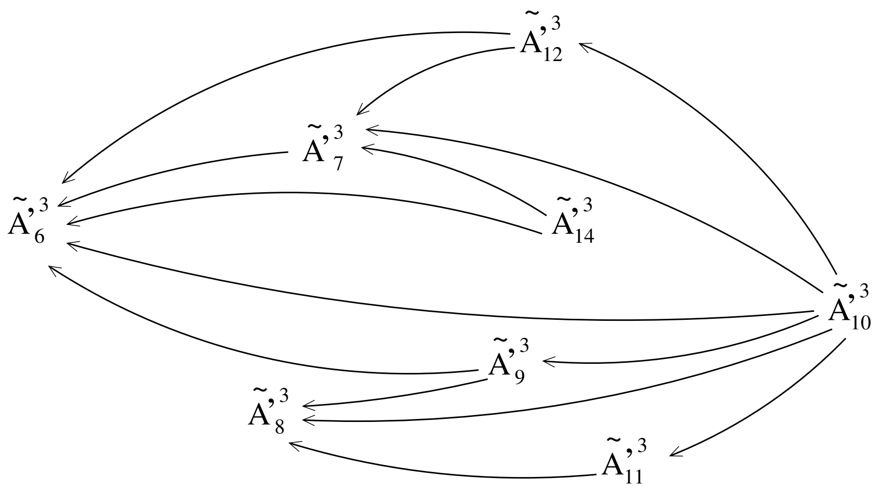
Proposition 5.8. *The irreducible components of $\mathcal{UHA}ss_3$ are the Zariski closure of orbits of Hom-associative algebras*

$$\Omega = \left\{A_9'^3, A_{10}'^3, A_{11}'^3, A_{12}'^3, A_{13}'^3, A_{14}'^3, A_{15}'^3\right\}.$$



Irreducible components of $\mathcal{UHA}ss_3$.

Proposition 5.9. *The irreducible components of* $\mathcal{UA}ss_3$ *are the Zariski closure of orbits of Hom-associative algebras* $\Omega = \left\{\tilde{A}^3_{10}, \tilde{A}^3_{14}\right\}$.



Irreducible components of $\mathcal{A}ss_3$.

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