Category theoretical view of I-cluster and I-limit points of subsequences

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In memory of Prof. Harry I. Miller, a true mathematician and inspiring teacher

Abstract. We study the concepts of I-limit and I-cluster points of a sequence, where I is an ideal with the Baire property. We obtain the relationship between I-limit and I-cluster points of a subsequence of a given sequence and the set of its classical limit points in the sense of category theory.

1. Introduction

The convergence of sequences lies at the heart of summability theory. It is well known that a sequence is convergent if and only if all of its subsequences are convergent. This has been extended to a result of great generality in [5, 6, 7] with the use of summability. Agnew [1], Miller and Orhan [16], Yurdakadim and Miller-Van Wieren [19], and Zeager [21] have studied the relationship between sequences and their subsequences.

There are many important types of convergence which generalize the ordinary convergence. One of such generalizations is the ideal convergence. It should be noted that the ideal convergence is a generalization of statistical convergence and has been studied by many authors (see, for example, [3, 12, 17, 18]).

In the earlier mentioned papers, relationships between a given sequence and its subsequences have mostly been studied from the perspective of Lebesgue measure. In the present paper, we will shift our attention to studying these notions from the perspective of Baire category.

Now let us recall the concept of ideal convergence which is the primary topic of this paper. A family $I \subseteq P(\mathbb{N})$ of subsets of $\mathbb{N}$ is said to be an *ideal*

on $\mathbb{N}$ if I is closed under subsets and finite unions, i.e., for each $A, B \in I$ we have $A \cup B \in I$ and for each $A \in I$ and $B \subset A$, we have $B \in I$. The ideal I is said to be *proper ideal* on $\mathbb{N}$ if $\mathbb{N} \notin I$. A proper ideal is said to be *admissible* if $\{n\} \in I$ for each $n \in \mathbb{N}$. It can be easily seen that an admissible ideal contains all finite subsets of $\mathbb{N}$.

In the remainder of the paper, we will assume that the ideal I is admissible. A sequence s is said to be *I-convergent* to l if for every ε the set $K_\varepsilon = \{n \in \mathbb{N} : |s_n - l| \geq \varepsilon\}$ belongs to I, and we write I-$\lim_{n \to \infty} s_n = l$. Notice that if $I = I_d = \{A \subseteq \mathbb{N} : d(A) = 0\}$, then I_d-convergence coincides with statistical convergence where $d(A)$ denotes the natural density of A [10], and if $I = I_u = \{A \subseteq \mathbb{N} : u(A) = 0\}$, then we obtain uniform statistical convergence where $u(A)$ denotes the uniform density of A [2]. Ideals on $\mathbb{N}$ can be regarded as subsets of the Polish space $\{0, 1\}^{\mathbb{N}}$. Hence they may have the Baire property or they may be Borel, analytic, coanalytic, and so on [9]. Throughout the paper the term meager will refer to sets of first Baire category, while the term comeager will refer to sets whose complement is of first category. The following lemma is well known.

Lemma 1 (see [3]). *Let I be an ideal on $\mathbb{N}$. The following conditions are equivalent:*

- *I has the Baire property;*
- *I is meager;*
- *There is a sequence $n_1 < n_2 < \ldots$ of integers in $\mathbb{N}$ such that no member of I contains infinitely many intervals $[n_k, n_{k+1})$ in $\mathbb{N}$.*

It is easy to check that I_d and I_u are both Borel, so they have the Baire property.

Since every $x \in (0, 1]$ has a unique binary expansion $x = \sum_{n=1}^{\infty} 2^{-n} d_n(x)$ such that $d_n(x) = 1$ for infinitely many positive integers n, for every $x \in (0, 1]$ and a sequence s we can generate a subsequence (sx) of s in such a way that if $d_n(x) = 1$, then $(sx)_n = s_n$. Therefore, there is a one to one correspondence between $x \in (0, 1]$ and subsequences of s.

As mentioned earlier, in many papers, relationships between sequences, their subsequences and different types of convergence have been studied using Lebesgue measure as a gauge of size. Balcerzak et al. [3] have studied similar relationships concerning ideal convergence using Baire category. Among other results, in the case that I has the Baire property, they have proved that if s is divergent, the set of $x \in (0, 1]$, for which (sx) is I-convergent, is meager. This generalizes the earlier result of Miller and Orhan [16] in the case of statistical convergence.

In a recent paper, Leonetti et al. [13] have obtained some results concerning subsequences, statistical cluster and limit points, also using Baire

category. Balcerzak and Leonetti [4] have studied some relationships between ideal cluster points and ideal limit points under suitable assumptions on the ideal.

In this paper we are able to generalize those results for ideal convergence, for ideals with the Baire property.

2. Main results

We start with the following definitions. These definitions have been studied by Fridy [11] in the case $I = I_d$ and have been formulated in the general case by Kostryko et al. [12]. They have also been studied by Demirci [8] in detail. Note that some results on I-convergence, I-limit points and I-cluster points can be found in [8, 12].

Definition 1. For a given sequence $s = (s_n)$, l is called an I-*cluster point* of s if $\{n : |s_n - l| < \varepsilon\} \notin I$ holds for every $\varepsilon > 0$.

Definition 2. For a given sequence $s = (s_n)$, l is called an I-*limit point* of s if there exists a sequence $\{n_i : i \in \mathbb{N}\} \notin I$ such that $\lim_{i \to \infty} s_{n_i} = l$.

Throughout the paper let us denote by Γ_s the set of all I-cluster points of s and by λ_s the set of all I-limit points of s. Clearly $\lambda_s \subseteq \Gamma_s$. We also introduce the following notation: for s and $x \in (0, 1]$, $\Gamma_s(x)$ is the set of I-cluster points of (sx), and $\lambda_s(x)$ is the set of I-limit points of (sx). It is obvious that $\lambda_s(x) \subseteq \Gamma_s(x)$.

Lemma 2. *Suppose s is a sequence with an ordinary limit point l, and I is an ideal with the Baire property. The sets $\{x \in (0, 1] : l \in \lambda_s(x)\}$ and $\{x \in (0, 1] : l \in \Gamma_s(x)\}$ are both comeager.*

Proof. Since I has the Baire property, we can fix a sequence $n_1 < n_2 < \cdots < n_k < \ldots$ of integers in $\mathbb{N}$ such that no member of I contains infinitely many intervals $[n_k, n_{k+1})$. Now, for $m, j \in \mathbb{N}$, let

$$A_{m,j}(l) = \{x \in (0, 1] : \text{ there exists } k \in \mathbb{N} \text{ such that } n_k > m$$
$$\text{and } |(sx)_i - l| < 1/j \text{ for } i \in [n_k, n_{k+1})\}.$$

Let m and j be fixed. We will show that $A_{m,j}(l)$ is comeager. Let $\overline{x} = (x_1, x_2, \ldots, x_d)$ be an arbitrary fixed finite sequence of $0's$ and $1's$. It is sufficient to show that there exists a finite extension x^* of $\overline{x}$ such that every $x \in (0, 1]$ starting with x^* is in $A_{m,j}(l)$. Without loss of generality we can assume $\overline{x}$ has t $1's$ where $t \geq m$. Let $k = \min\{i : n_i > t\}$. First we extend $\overline{x}$ to a sequence $(x_1, x_2, \ldots, x_g)$, $g \geq d$, that contains exactly $n_k - 1$ $1's$ and where $x_g = 1$. Since l is a limit point of s one can find indices $i_{n_k} < i_{n_k+1} < \cdots < i_{n_{k+1}-1}$ greater than g such that the terms of s

corresponding to those indices lie in $(l-1/j, l+1/j)$. Consider the following extension of $\overline{x}$

$$x^* = (x_1, x_2, \ldots, x_g, \ldots, x_{i_{n_k}}, \ldots, x_{i_{n_{k+1}-1}})$$

where for $i > g$, $x_i = 1$ for $i \in \{i_{n_k}, i_{n_k+1}, \ldots, i_{n_{k+1}-1}\}$ and $x_i = 0$ otherwise. Then every $x \in (0,1]$ that extends x^* is in $A_{m,j}(l)$. Hence $A_{m,j}(l)$ is comeager. Furthermore, $A(l) = \cap_m \cap_j A_{m,j}(l)$ will be comeager.

Suppose that $x \in A(l)$. By setting $m = j$ one can see that we have $x \in \cap_m \cap_m A_{m,m}(l)$. Therefore, for each m we can fix k_m so that $|(sx)_i - l| < 1/m$ for $i \in [n_{k_m}, n_{k_m+1})$. If we observe the subsequence of (sx) with indices $i \in \cup_m [n_{k_m}, n_{k_m+1})$, it is clear that it converges to l and $\cup_m [n_{k_m}, n_{k_m+1}) \notin I$. Hence l is I-limit point of (sx). We conclude that

$$A(l) \subseteq \{x \in (0,1] : l \in \lambda_s(x)\}.$$

Since $A(l)$ is comeager, so is $\{x \in (0,1] : l \in \lambda_s(x)\}$.

Additionally, since $\lambda_s(x) \subseteq \Gamma_s(x)$, we have

$$\{x \in (0,1] : l \in \lambda_s(x)\} \subseteq \{x \in (0,1] : l \in \Gamma_s(x)\},$$

so the second part of the lemma follows. $\qquad\square$

Theorem 1. *Suppose s is a sequence, L is the set of its limit points, and I is an ideal with the Baire property. Then the sets $\{x \in (0,1] : \Gamma_s(x) = L\}$ and $\{x \in (0,1] : \lambda_s(x) = L\}$ are both comeager.*

Proof. It is sufficient to show that

$$\{x \in (0,1] : \lambda_s(x) = L\}$$

is comeager. Assume L is infinite (the finite case is simpler). There exist $l_1, l_2, \ldots, l_n, \ldots$ such that $L = \overline{\{l_1, l_2, \ldots, l_n, \ldots\}}$. From the proof of the above lemma, $\cap_m \cap_j A_{m,j}(l_n)$ is comeager for each n, so $A = \cap_n \cap_m \cap_j A_{m,j}(l_n)$ is comeager. Suppose $x \in A$ and let $l \in L$ be arbitrary. Thus l is the limit of a subsequence of (l_n), for simpler notation, we will just write $l = \lim_{m \to \infty} l_m$. Then $x \in A$ implies $x \in \cap_m A_{m,m}(l_m)$ by setting $n = m = j$. For each m, fix $n_{k_m} > m$ so that $|(sx)_i - l_m| < \frac{1}{m}$ for $i \in [n_{k_m}, n_{k_m+1})$. If we observe the subsequence of (sx) with indices $i \in \cup_m [n_{k_m}, n_{k_m+1})$, it is clear that it converges to l and $\cup_m [n_{k_m}, n_{k_m+1}) \notin I$. Hence l is an I-limit point of (sx). So $A \subseteq \{x \in (0,1] : \lambda_x = L\}$ which implies that the second set is comeager. This completes the proof. $\qquad\square$

One can immediately obtain the following theorem as a consequence of the above result.

Theorem 2. *Suppose s is a sequence, L is the set of its limit points, and I is an ideal with the Baire property. Then the set $\{x \in (0,1] : \Gamma_s(x) = \Gamma_s\}$ is comeager if (and only if) $\Gamma_s = L$ and is meager if $\Gamma_s \subsetneq L$. Likewise*

$\{x \in (0,1] : \lambda_s(x) = \lambda_s\}$ *is comeager if (and only if)* $\lambda_s = L$ *and meager if* $\lambda_s \subsetneqq L$.

Proof. Suppose $\Gamma_s = L$. Then, from Theorem 1 we get that the set $\{x \in (0,1] : \Gamma_s(x) = \Gamma_s\}$ is comeager. For $\Gamma_s \subsetneqq L$, from Theorem 1 it follows that $\{x \in (0,1] : \Gamma_s(x) = \Gamma_s\}$ is meager. The second part is analogous. $\square$

Additionally we have the following result, reminiscent of some theorems concerning statistical and uniform statistical convergence (see [15, 16, 20]).

Theorem 3. *Let I be an ideal on $\mathbb{R}$ with the Baire property. Suppose $s = (s_n)$ is a bounded sequence and L is the set of its limit points. If $M \subseteq L$, M closed and nonempty, then there exists a subsequence of s, $y = (y_n)$ such that M is the set of I-limit points, as well as the set of I-cluster points of y.*

Proof. Since M is closed and separable, $M = \overline{\{l_m : m \in \mathbb{N}\}}$ for some $l_m \in M$. For $m \in \mathbb{N}$, fix a subsequence of (s_n), $(s_{n_{jm}})_{j=1}^{\infty}$ converging to l_m and contained in $(l_m - 1/m, l_m + 1/m)$. Since I has the Baire property we can fix $n_1 < n_2 < \cdots < n_k < \ldots$ such that no element of I contains infinitely many $[n_{k-1}, n_k)$, $k \geq 2$. Let us denote $I_1 = [1, n_1)$ and $I_k = [n_{k-1}, n_k)$ for $k \geq 2$. We construct $y = (y_n)$ as follows:

for $n \in \cup_{\nu=1}^{i} I_{2^{\mu-1}(2\nu+1)}$ and any fixed $\mu = 1, 2, \ldots, m$,

y_n will be chosen from $(s_{n_{j\mu}})$,

choosing each y_n inductively in such a way that its index in the sequence s is greater than the indices of $y_1, y_2, \ldots, y_{n-1}$ in terms of s. Now for each l_m, the subsequence of $\{y_n : n \in \cup_i I_{2^{m-1}(2i+1)}\}$ converges to l_m and $\cup_i I_{2^{m-1}(2i+1)} \notin I$ so l_m is an I-limit (and I-cluster) point of y. Similarly, for $l \in M$, $l = \lim_{j \to \infty} l_{m_j}$ and the subsequence $\{y_n : n \in \cup_j I_{2^{m_j-1}(2m_j+1)}\}$ converges to l and $\cup_j I_{2^{m_j-1}(2m_j+1)} \notin I$, so each $l \in M$ is an I-limit (and I-cluster) point of $y = (y_n)$.

Finally it is easy to see that since M is closed, if $l \in L \backslash M$, there exists $\varepsilon > 0$ such that $(l - \varepsilon, l + \varepsilon) \cap M = \emptyset$ and from the construction of y, l cannot be a limit point of y, and hence cannot be an I-limit or I-cluster point of y. So M is precisely the set of I-limit (I-cluster) points of y. This completes the proof. $\square$

Remark 1. Let $I = I_d$. Then the first two of our results coincide with the results obtained in [13].

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