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Quantitative versions of almost squareness and diameter 2 properties

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ABSTRACT. We introduce a quantitative version (using $s \in (0,1]$) of almost (local) squareness of Banach spaces. The latter concept (i.e., the $s = 1$ case) was introduced by Abrahamsen, Langemets, and Lima in 2016. Related diameter 2 properties (local, strong, and symmetric strong) are also relaxed correspondingly. Our note contains some (counter-)examples and results for the s-almost (local) squareness property.

1. Concepts

Almost square Banach spaces were introduced by Abrahamsen et al. [1] in 2016. These spaces have already got quite a lot of attention in the literature; see, e.g., [15] for results and references.

Let X be a Banach space over $\mathbb{R}$ and let S_X denote its unit sphere, B_X its closed unit ball and X^* its dual space. Following [1], we say that X is *almost square* (ASQ) if for every finite subset $\{x_1, \ldots, x_n\}$ of S_X and every $\varepsilon > 0$, there exists $y \in S_X$ such that $\|x_i + y\| \leqslant 1 + \varepsilon$ for all $i = 1, \ldots, n$.

Also, following [1], X is called *locally almost square* (LASQ) if for every $x \in S_X$ and every $\varepsilon > 0$, there exists $y \in S_X$ such that $\|x \pm y\| \leqslant 1 + \varepsilon$.

Definition 1.1. Let $s \in (0,1]$. A Banach space X is *s-locally almost square* (s-LASQ) if for every x in S_X and for every $\varepsilon > 0$ there exists $y \in S_X$ such that

$$\|x \pm sy\| \leqslant 1 + \varepsilon.$$

Note that 1-LASQ means precisely LASQ.

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Definition 1.2. Let $s \in (0,1]$. A Banach space X is *s-almost square* (s-ASQ) if for every finite family $x_1, \dots, x_n$ in S_X and for every $\varepsilon > 0$ there exists $y \in S_X$ such that

$$\|x_i + sy\| \leqslant 1 + \varepsilon, \quad i = 1, \dots, n.$$

Note that 1-ASQ means precisely ASQ. As in the case of the ASQ property, we can have for free the plus-minus sign in the definition of the s-ASQ property since we can take $-x_i$ together with x_i in the finite family of elements from S_X.

Note that the LASQ property occurs in [12], a paper by P. Harmand and Å. Lima from 1984, in the proof of the Harmand–Lima theorem: *if X is a non-reflexive M-ideal in its bidual X^{**}, then X contains almost isometric copies of c_0.* The Harmand–Lima theorem has been refined in [1] as follows: *every ASQ space X contains almost isometric copies of c_0, and every non-reflexive X which is M-ideal in X^{**}, is ASQ.* To complement [1], let us remark that the result [1, Corollary 2.3] that *every LASQ space contains almost isometric copies of ℓ_∞^2*, is present in the Harmand–Lima proof.

The LASQ property was first isolated and used in 2014 in [14] under the notation of *points of uniformly non-squareness.*

Recall that a set $S(x^*, \alpha) = \{x \in B_X : x^*(x) > 1 - \alpha\}$ (where $x^* \in S_{X^*}$ and $\alpha > 0$) is called a *slice* of B_X. According to [2], a Banach space X has the *local diameter 2 property* (LD2P) if every slice of B_X has diameter 2, and the *strong diameter 2 property* (SD2P) if every finite convex combination of slices has diameter 2.

If, for a finite family of slices $S_1, \dots, S_n$ of B_X, and for a number $\varepsilon > 0$, there exist elements $x_i \in S_i$ $(i = 1, \dots, n)$ and an element $y \in B_X$ with $\|y\| > 1 - \varepsilon$ such that $x_i \pm y \in S_i$ for all i, then X is said to have the *symmetric strong diameter 2 property* (SSD2P). This property was defined in [4] and was considered in [2, Lemma 4.1]. Very recently, SSD2P has been further characterized and investigated [11].

In 1988, Deville [9] investigated the following property that for the case $d = 2$ is equivalent to SD2P.

Definition 1.3. Let $d \in (0,2]$. A Banach space X has the *strong diameter d property* (SD(d)P) if the diameter of every convex combination of slices of B_X is greater than or equal to d.

The respective generalization of LD2P is the following.

Definition 1.4. Let $d \in (0,2]$. A Banach space X has the *local diameter d property* (LD(d)P) if the diameter of every slice of B_X is greater than or equal to d.

The symmetric version of SD(d)P is the following.

Definition 1.5. Let $d \in (0,2]$. A Banach space X has the *symmetric strong diameter d property* (SSD(d)P) if whenever $n \in \mathbb{N}$, $S_1, \dots, S_n$ are slices of B_X, and $\varepsilon > 0$, there exist elements $x_i \in S_i$ ($i = 1, \dots, n$) and an element $x \in B_X$, $\|x\| > 1 - \varepsilon$, such that $x_i \pm \frac{d}{2}x \in S_i$ for every $i = 1, \dots, n$.

The relations between these properties are as follows. (The case $d = 2$ was treated already in [2, Lemma 4.1].)

Proposition 1.6. *The property SSD(d)P implies SD(d)P, which, in turn, implies LD(d)P.*

Proof. For the first implication, let a set $S = \sum_{i=1}^n \lambda_i S_i$ be a convex combination of slices $S_i = S(x_i^*, \alpha_i)$. By SSD(d)P, for every $i \in \{1, \dots, n\}$ and $\varepsilon > 0$, there exist elements $x_i \in S_i$ and $x \in B_X$ such that $\|x\| > 1 - \varepsilon$ and $x_i \pm \frac{d}{2}x \in S_i$. Therefore

$$\operatorname{diam} S \geqslant \left\| \sum_{i=1}^n \lambda_i \left(x_i + \frac{d}{2}x \right) - \sum_{i=1}^n \lambda_i \left(x_i - \frac{d}{2}x \right) \right\| > d(1 - \varepsilon),$$

implying that $\operatorname{diam} S \geqslant d$.

The second implication is clear from the definitions. □

The LD2P/SD2P case of the following result is known due to [14] (see also [1, Proposition 2.5]).

Proposition 1.7. *Let X be a Banach space. Let $s \in (0,1]$.*

(a) *If X has the s-LASQ property, then X has the LD($2s$)P.*
(b) *If X has the s-ASQ property, then X has the SSD($2s$)P.*

Proof. First, we prove (b).

Let $S_i = S(x_i^*, \alpha_i)$ (in this proof, we always have $i = 1, \dots, n$) be slices of B_X and let $\varepsilon > 0$. Denote $\delta = \min\left\{\frac{1}{4}\alpha_1, \dots, \frac{1}{4}\alpha_n, \varepsilon\right\}$. For every functional x_i^* there exists an element $y_i \in S_X$ such that $x_i^*(y_i) > 1 - \delta$. By the s-ASQ property of X, for the finite family of elements $\pm y_1, \dots, \pm y_n \in S_X$ there exists $y \in S_X$ such that $\|y_i \pm sy\| < 1 + \delta$.

Note that

$$\pm {x_i}^*(sy) = -x_i^*(y_i) + x_i^*(y_i \pm sy) < (\delta - 1) + (1 + \delta) = 2\delta.$$

Therefore, for the elements $x_i = \frac{y_i}{1+\delta}$ and $x = \frac{y}{1+\delta}$ we have

$$\|x\| = \|x_i\| = \frac{1}{1+\delta} < 1$$

and

$$\|x\| > \frac{1}{1+\varepsilon} > 1 - \varepsilon.$$

Now, the elements $x_i \pm sx$ belong to the respective slices S_i. Indeed,

$$x^*(x_i \pm sx) > \frac{(1-\delta) - 2\delta}{1+\delta} > 1 - 4\delta \geqslant 1 - \alpha_i.$$

The conditions of the SSD($2s$)P for X have been fulfilled.

For the assertion (a), we read the last proof with $n = 1$. After that, we read the proof of (first implication of) Proposition 1.6 with $n = 1$ and $\lambda_1 = 1$. □

2. Some results

In this section we rewrite some results on the ASQ property in the s-ASQ setting.

Let $r, s \in (0,1]$. Recall that a closed subspace Y of X is called an $M(r,s)$-*ideal* in X if there exists a norm one projection P on X^* with $\ker P = Y^\perp = \{x^* \in X^* \colon x^*|_Y = 0\}$ and $r\|Px^*\| + s\|x^* - Px^*\| \leqslant \|x^*\|$ for all $x^* \in X^*$.

$M(r,s)$-ideals were introduced by Cabello and Nieto [7] in 1998. M-ideals are precisely $M(1,1)$-ideals. A number of examples of $M(r,s)$-ideals can be found in [8].

It is said that Y is an *almost isometric ideal* (ai-ideal) in X [3] if for every finite dimensional subspace E of X and every $\delta > 0$ there exists a linear operator $U \colon E \to Y$ such that $Ue = e$ for every $e \in E \cap Y$ and $(1+\delta)^{-1}\|e\| \leqslant \|Ue\| \leqslant (1+\delta)\|e\|$ for all $e \in E$.

Note that a Banach space Y is always an ai-ideal in its bidual Y^{**}.

The following result is a quantitative version of [1, Theorem 4.2].

Theorem 2.1. *Let Y be a proper ai-ideal in an infinite-dimensional Banach space X, and let $s \in (0,1]$. If Y is an $M(1,s)$-ideal in X, then Y is s-ASQ.*

Proof. We follow the scheme of the proof for M-ideals due to Harmand and Lima [12, proof of Theorem 3.5], formalized in [1, Theorem 4.2]. However, we do it in a bit smoother way.

Assuming that Y is an $M(1,s)$-ideal in X, let P be a corresponding ideal projection on X^*. Then $\|P\| = 1$, $\ker P = Y^\perp$, and $X^* = \operatorname{ran} P \oplus \ker P$ with

$$\|Px^*\| + s\|x^* - Px^*\| \leqslant \|x^*\|, \quad x^* \in X^*.$$

Hence $X^{**} = \ker P^* \oplus \operatorname{ran} P^*$ with $\operatorname{ran} P^* = (\ker P)^\perp = (Y^\perp)^\perp = Y^{\perp\perp}$. Since $Y \neq X$, we have that $\ker P^* \neq \{0\}$. Choose any $x^{**} \in S_{\ker P^*}$.

Let $y_1, \dots, y_n \in S_Y$ and let $\varepsilon > 0$. Choose $\delta > 0$ such that $(1+\delta)^2 \leqslant 1+\varepsilon$. Applying first the principle of local reflexivity to the subspace $E = \operatorname{span}\{y_1, \dots, y_n, x^{**}\}$ of X^{**} provides us a local reflexivity operator $S \colon E \to X$. Applying then the definition of an ai-ideal to the subspace $S(E)$ of X provides us an operator $T \colon S(E) \to Y$ such that $U = TS \colon E \to Y$ satisfies the conditions

$$Ue = e, \quad e \in E \cap Y$$

and

$$(1+\delta)^{-1}\|e\| \leqslant \|Ue\| \leqslant (1+\delta)\|e\|, \quad e \in E.$$

Put $y = \frac{Ux^{**}}{\|Ux^{**}\|}$. Then $y \in S_Y$. We shall verify that $\|y_i + sy\| \leqslant 1 + \varepsilon$ for all $i = 1, \dots, n$.

Firstly, let us show that $\|y_i + sx^{**}\| \leqslant 1$. Let $x^* \in X^*$ be arbitrary. Since $y_i \in S_Y \subset \operatorname{ran} P^*$ and $x^{**} \in S_{\ker P^*}$, we have

$$\begin{aligned} |(y_i + sx^{**})(x^*)| &= |(Px^*)(y_i) + sx^{**}(x^* - Px^*)| \\ &\leqslant \|Px^*\| + s\|x^* - Px^*\| \leqslant \|x^*\| \end{aligned}$$

as needed.

Secondly, using that $1 + \delta \geqslant \|Ux^{**}\|^{-1} \geqslant (1+\delta)^{-1} \geqslant 1 - \delta$, we have

$$\left\| \frac{x^{**}}{\|Ux^{**}\|} - x^{**} \right\| = \left| \frac{1}{\|Ux^{**}\|} - 1 \right| \leqslant \delta.$$

Putting these inequalities together implies that

$$\begin{aligned} \|y_i + sy\| &= \left\| U\left(y_i + s\frac{x^{**}}{\|Ux^{**}\|} \right) \right\| \\ &\leqslant (1+\delta)\left(\|y_i + sx^{**}\| + s\left\| \frac{x^{**}}{\|Ux^{**}\|} - x^{**} \right\| \right) \\ &\leqslant (1+\delta)(1+s\delta) \leqslant (1+\delta)^2 \leqslant 1 + \varepsilon. \end{aligned}$$

□

The s-ASQ analogues of [1, Lemma 2.2] and [1, Theorem 2.4] go as follows.

Lemma 2.2. *If $x, y \in S_X$ are such that $\|x \pm sy\| \leqslant 1 + \varepsilon$, then for all scalars α, β the following estimate holds:*

$$\left(\frac{1}{2-s} - \varepsilon \right) \max\{|\alpha|, |\beta|\} \leqslant \|\alpha x + \beta y\| \leqslant (2 - s + \varepsilon) \max\{|\alpha|, |\beta|\}.$$

Theorem 2.3. *If X has the s-ASQ property, then for every finite dimensional subspace $E \subseteq X$ and every $\varepsilon > 0$ there exists $y \in S_X$ such that for all scalars λ and all $x \in E$*

$$\left(\frac{1}{2-s} - \varepsilon \right) \max\{\|x\|, |\lambda|\} \leqslant \|x + \lambda y\| \leqslant (2 - s + \varepsilon) \max\{\|x\|, |\lambda|\}.$$

In Remark 3.4, we shall see that the bounds in Lemma 2.2 (hence also in Theorem 2.3) cannot, in general, be improved.

Proof of Lemma 2.2. We may assume $s < 1$ since $s = 1$ has already been treated in [1]. We can also assume that ε is small enough.

First note that

$$2 = \|(x + sy) + (x - sy)\| \leqslant \|x \pm sy\| + 1 + \varepsilon,$$

hence

$$\|x \pm sy\| \geqslant 1 - \varepsilon. \tag{1}$$

It is clear that if $\alpha = 0$ or $\beta = 0$ the lemma holds.

Case $|\beta| \geqslant |\alpha| > 0$**.** We need to show that

$$\frac{1}{2-s} - \varepsilon \leqslant \|\gamma x \pm y\| \leqslant 2 - s + \varepsilon,$$

where $\gamma = \left|\frac{\alpha}{\beta}\right| \in (0,1]$. By the triangle inequality, we get

$$\|\gamma x \pm y\| = \|\gamma(x \pm sy) \pm (1-\gamma s)y\| \leqslant \gamma(1+\varepsilon - s) + 1 \leqslant 2 - s + \varepsilon.$$

For $\gamma > \frac{1}{2-s}$ we have (due to (1))

$$\begin{aligned}\|\gamma x \pm y\| &= \frac{1}{s}\|\gamma s x \pm s y\| = \frac{1}{s}\|x \pm sy - (1-\gamma s)x\| \\ &\geqslant \frac{1}{s}(1 - \varepsilon - (1-\gamma s)) = \gamma - \frac{1}{s}\varepsilon \geqslant \frac{1}{2-s} - \varepsilon.\end{aligned}$$

The last inequality holds as it is equivalent to

$$\varepsilon \leqslant \frac{s}{1-s}\left(\gamma - \frac{1}{2-s}\right).$$

For $\gamma \leqslant \frac{1}{2-s}$ we have

$$\|\gamma x \pm y\| = \|(1+\gamma s)y \pm \gamma(x \mp sy)\| \geqslant 1 + \gamma s - \gamma(1+\varepsilon) \geqslant \frac{1}{2-s} - \gamma\varepsilon.$$

Case $|\alpha| > |\beta| > 0$**.** Denote $\delta = \left|\frac{\beta}{\alpha}\right| \in (0,1)$. We shall show that

$$\frac{1}{2-s} - \varepsilon \leqslant \|x \pm \delta y\| \leqslant 2 - s + \varepsilon.$$

We have

$$\begin{aligned}\|x \pm \delta y\| &= \|\delta(x \pm sy) + (1-\delta)x \pm \delta(1-s)y\| \\ &\leqslant \delta(1 + \varepsilon - 1 + 1 - s) + 1 \leqslant 2 - s + \varepsilon\end{aligned}$$

and

$$\begin{aligned}\|x \pm \delta y\| &= \frac{\delta}{s}\left\|\left(1 + \frac{s}{\delta}\right)x - (x \mp sy)\right\| \geqslant \frac{\delta}{s}\left(1 + \frac{s}{\delta} - 1 - \varepsilon\right) \\ &\geqslant 1 - \frac{\varepsilon}{s} \geqslant \frac{1}{2-s} - \varepsilon.\end{aligned}$$

The last inequality holds as it is equivalent to $\varepsilon \leqslant \frac{s}{2-s}$. □

Proof of Theorem 2.3. The argumentation is an adapted version of that in [1]. We take an $\frac{\varepsilon}{2}$-net $\{x_1, \ldots, x_N\}$ of S_E. Due to the s-ASQ-ness of X, we can find $y \in S_X$ such that

$$1 - \frac{\varepsilon}{2} \leqslant \|x_i \pm sy\| \leqslant 1 + \frac{\varepsilon}{2}.$$

Now, for a $x \in S_E$, find i such that $\|x - x_i\| \leqslant \frac{\varepsilon}{2}$, hence

$$1 - \varepsilon \leqslant \|x_i \pm sy\| - \|x - x_i\| \leqslant \|x \pm sy\| \leqslant \|x_i \pm sy\| + \|x - x_i\| \leqslant 1 + \varepsilon.$$

By using Lemma 2.2 we obtain the result. □

Remark 2.4. Arguing analogously to [1, Theorem 2.4], one can prove more in Theorem 2.3: we can have that, for any finite dimensional subspace $F \subseteq X^*$ the element y can be taken so that $s|f(y)| \leqslant (1-s+\varepsilon)\|f\|$ for every $f \in F$.

(Unlike in the 1-ASQ case, in the general s-ASQ case such reasoning does not allow $\frac{|f(y)|}{\|f\|}$ to be arbitrarily small.)

A slight generalization of the argument in [1, Lemma 5.5] yields the following result.

Proposition 2.5. *Let X and Y be nontrivial Banach spaces. Then $X \oplus_1 Y$ fails the s-ASQ property for any $s \in (0,1]$.*

Proof. Let $Z = X \oplus_1 Y$, $x \in S_X$, $y \in S_Y$. Consider the elements $z_1 = (-tx, (1-t)y)$ and $z_2 = ((1-t)x, -ty)$ from S_Z where the exact value of $t \in (0,1)$ will be clarified later. Assume that there is a $w = (w_x, w_y) \in S_Z$ with $\|z_i \pm sw\| \leqslant 1+\varepsilon$ for a certain small ε. Then

$$\begin{aligned} s\|w_x\| + \|(1-t)y\| &\leqslant \frac{1}{2}\|-tx + sw_x\| + \frac{1}{2}\|tx + sw_x\| \\ &\quad + \frac{1}{2}\|(1-t)y - sw_y\| + \frac{1}{2}\|(1-t)y + sw_y\| \\ &\leqslant \max\{\|z_1 + sw\|, \|z_1 - sw\|\} \leqslant 1+\varepsilon. \end{aligned}$$

Hence $s\|w_x\| \leqslant 1+\varepsilon-(1-t) = t+\varepsilon$. Similarly $s\|w_y\| \leqslant t+\varepsilon$, giving

$$s\|w\| \leqslant 2(t+\varepsilon).$$

A contradiction has been reached if

$$2(t+\varepsilon) < s.$$

It suffices to take, e.g., $t = \varepsilon = \frac{s}{5}$. □

The following proposition is a s-LASQ version of [1, Proposition 5.7(i),(iii)].

Proposition 2.6. *Let X and Y be nontrivial Banach spaces. The direct sum $Z = X \oplus_\infty Y$ is s-ASQ (s-LASQ) if and only if either X or Y is s-ASQ (s-LASQ).*

Proof. We only prove the s-ASQ case – the s-LASQ case follows similarly.

Necessity. Assume that the sum $Z = X \oplus_\infty Y$ is s-ASQ. Suppose to the contrary that neither X nor Y is s-ASQ. Thus there are finite families $x_1, \ldots, x_n \in S_X$, $y_1, \ldots, y_m \in S_Y$, and $\varepsilon > 0$ such that for every $x \in S_X$ there exists an index $k \in \{1, \ldots, n\}$ and for every $y \in S_Y$ there exists an index $l \in \{1, \ldots, m\}$ such that

$$\|x_k + sx\| > 1+\varepsilon \tag{2}$$

and

$$\|y_l + sy\| > 1 + \varepsilon. \tag{3}$$

Suppose that $m \geqslant n$. Denote $x_i = 0$ for $i = n+1, \dots, m$. Consider a family $z_i = (x_i, y_i)$, $i = 1, \dots, m$. By our assumption, there is a $z = (u, v) \in S_Z$ such that

$$\|z_i + sz\| \leqslant 1 + \varepsilon \tag{4}$$

for every $i = 1, \dots, m$. The condition $z \in S_Z$ implies $u \in S_X$ or $v \in S_Y$. In the case $u \in S_X$ the inequality (4) is in contradiction with condition (2). In the case $v \in S_Y$ we get contradiction with (3).

Sufficiency. Suppose that X is s-ASQ. Let $z_i = (x_i, y_i) \in S_Z$ for $i = 1, \dots, N$ and let $\varepsilon > 0$. We may assume that $x_i \neq 0$ for $i = 1, \dots, N$. As X is s-ASQ, there exists $u \in S_X$ such that $\left\| \frac{x_i}{\|x_i\|} + su \right\| \leqslant 1 + \varepsilon$ for every $i = 1, \dots, N$. Then

$$\begin{aligned}\|x_i + su\| &= \left\| \|x_i\| \left(\frac{x_i}{\|x_i\|} + su \right) + su(1 - \|x_i\|) \right\| \leqslant \|x_i\|(1+\varepsilon) + s(1 - \|x_i\|) \\ &= (1 + \varepsilon - s)\|x_i\| + s \leqslant 1 + \varepsilon.\end{aligned}$$

Put $z = (u, 0) \in S_Z$. Now we have

$$\|z_i + sz\| \leqslant \max\{\|x_i + su\|, 1\} \leqslant 1 + \varepsilon$$

for every $i = 1, \dots, N$ and Z is s-ASQ. $\square$

Every (non-separable) s-ASQ space is saturated with separable s-ASQ subspaces, as is shown by the next result.

Proposition 2.7. *Let X have the s-ASQ property. For every separable subspace Y of X there exists a separable subspace Z having property s-ASQ such that $Y \subset Z \subset X$.*

We omit the proof as it is an almost verbatim copy of the proof of [1, Proposition 6.5] (only s must be added in front of every y).

3. Examples

Let $\lambda \in (0, 1)$; we denote $s = 1 - \lambda$. We consider an equivalent renorming of c_0 due to Johnson and Wolfe [13]: let

$$\|(a_k)\| = \sup \left\{ \frac{|a_1|}{\lambda}, |a_1 - a_2|, |a_1 - a_3|, \dots \right\}, \qquad (a_k) \in c_0, \tag{5}$$

and denote c_0 with respect to the norm (5) by $c_{0,\lambda}$.

Note that the information from [8, Example 4.3] together with [8, Corollary 2.4] and Theorem 2.1 shows that $c_{0,\lambda}$ has the $\frac{1-\lambda}{1+\lambda}$-ASQ property. However, we can say more.

Proposition 3.1. *The space $c_{0,\lambda}$ has the s-ASQ property.*

Proof. Take elements $x_i \in S_{c_{0,\lambda}}$, $i = 1, \ldots, n$, and a number $\varepsilon > 0$. Now there exists a natural number N such that $\|x_i - P_N x_i\| \leqslant \varepsilon$ for all $i = 1, \ldots, n$ where P_n, $n \in \mathbb{N}$, are the partial sum projections associated to the unit vector basis $(e_n)_{n=1}^{\infty}$ of $c_{0,\lambda}$.

Denote $y = e_{N+1}$, then $\|y\| = 1$ and $\|P_N x_i + sy\| \leqslant 1$ for all $i = 1, \ldots, n$. Indeed, let $x_i = (\xi_k^i)_{k=1}^{\infty}$, then $|\xi_1^i| \leqslant \lambda$ and $|\xi_1^i - \xi_k^i| \leqslant 1$ for all $k \in \mathbb{N}$ and $i = 1, \ldots, n$, and $P_N x_i + sy = (\xi_1^i, \ldots, \xi_N^i, s, 0, 0, \ldots)$. Now

$$\begin{aligned} \|P_N x_i + sy\| &= \max\left\{\frac{|\xi_1^i|}{\lambda}, |\xi_1^i - \xi_2^i|, \ldots, |\xi_1^i - \xi_N^i|, |\xi_1^i - s|, |\xi_1^i|\right\} \\ &\leqslant \max\{1, \lambda\} = 1 \end{aligned}$$

since $|\xi_1^i - s| \leqslant \lambda + s = 1$. □

Proposition 3.2. *The space $c_{0,\lambda}$ fails the $\tilde{s}$-LASQ property for any $\tilde{s} \in (s, 1]$.*

Proof. Fix a number $\tilde{s} \in (s, 1]$. Consider an element $x = (\lambda, 0, 0, \ldots) \in c_{0,\lambda}$, then $\|x\| = 1$. Fix a number $\varepsilon > 0$ such that $\varepsilon < \tilde{s} - s$. Assume that there exists an element $y = (\eta_n) \in c_{0,\lambda}$, $\|y\| = 1$, such that $\|x \pm \tilde{s}y\| \leqslant 1 + \varepsilon$. Since $x \pm \tilde{s}y = (\lambda \pm \tilde{s}\eta_1, \pm\tilde{s}\eta_2, \pm\tilde{s}\eta_3, \ldots)$, we have

$$\left|1 \pm \frac{\tilde{s}}{\lambda}\eta_1\right| = \frac{|\lambda \pm \tilde{s}\eta_1|}{\lambda} \leqslant 1 + \varepsilon,$$

therefore $\frac{|\eta_1|}{\lambda} \leqslant \frac{\varepsilon}{\tilde{s}} < \frac{\tilde{s}-s}{\tilde{s}} < 1$. Now, as $\|y\| = 1$, $\frac{|\eta_1|}{\lambda} < 1$, and $|\eta_1 - \eta_n| \to_n |\eta_1| < \lambda$, there exists an index m such that $|\eta_1 - \eta_m| = 1$. If $\eta_1 - \eta_m = 1$, then

$$1 + \varepsilon \geqslant \|x + \tilde{s}y\| \geqslant |\lambda + \tilde{s}\eta_1 - \tilde{s}\eta_m| = \lambda + \tilde{s} = 1 - s + \tilde{s} > 1 + \varepsilon,$$

a contradiction. The case $\eta_m - \eta_1 = 1$ is treated similarly, using the element $x - \tilde{s}y$. □

Remark 3.3. Propositions 3.1 and 3.2 show that the Johnson–Wolfe spaces $c_{0,\lambda}$ offer exact examples in the full scale of the s-ASQ property (where $s \in (0, 1)$). (An example for 1-ASQ=ASQ is, of course, c_0.) Also note that if $\tilde{s} \in (0, 1)$ is such that $\tilde{s} > s$, then the space $c_{0,1-s}$ has the s-ASQ property, but fails even the $\tilde{s}$-LASQ property, hence fails the $\tilde{s}$-ASQ property, hence fails the (L)ASQ property.

Remark 3.4. Due to the spaces $c_{0,\lambda}$, the bounds $\frac{1}{2-s}$ and $2 - s$ in Lemma 2.2 and Theorem 2.3 cannot be improved. Indeed, take $x = (\lambda, 0, 0, \ldots)$ and $y = (0, 1, 0, 0, \ldots)$. Clearly $x, y \in S_{c_{0,\lambda}}$ and $\|x \pm sy\| = 1$. Now,

$$\|x - y\| = \|(\lambda, -1, 0, \ldots)\| = \max\{1, 1 + \lambda\} = 1 + \lambda = 2 - s,$$

$$\|x+(2-s)y\| = \|(\lambda, 1+\lambda, 0, \ldots)\| = 1 = \frac{1}{2-s} \cdot \max\{1, 2-s\}.$$

Proposition 1.7 yields that the space $c_{0,\lambda}$ also has the SSD$(2(1-\lambda))$P. The following result shows that it even has the SSD2P, hence also the SD2P.

Proposition 3.5. *The space $c_{0,\lambda}$ has the SSD2P.*

Proof. We are going to use [11, Theorem 2.1 (a)⇔(d)]: a Banach space X has the SSD2P iff, for every $n \in \mathbb{N}$ and every $x_1, \ldots, x_n \in X$, there exist nets $(y_\alpha^i) \subset S_X$ and $(z_\alpha) \subset S_X$ such that $y_\alpha^i \to x_i$ weakly, $z_\alpha \to 0$ weakly, and $\|y_\alpha^i \pm z_\alpha\| \to 1$ for all $i = 1, \ldots, n$.

So we have the elements $x_i = (\xi_k^i)_{k=1}^\infty \in c_{0,\lambda}$, $i = 1, \ldots, n$. Denote $y_N^i = x_i + (\xi_1^i - \xi_N^i)e_N$. Choose an index N' such that, for all i, we have $|\xi_N^i| < \frac{1-\lambda}{2}$ if $N > N'$. Hence, if $N > N'$, then $|\xi_1^i - \xi_N^i| < \lambda + \frac{1-\lambda}{2} < 1$, therefore the equality

$$\|x_i\| = \sup\left\{\frac{|\xi_1^i|}{\lambda}, |\xi_1^i - \xi_2^i|, \ldots, |\xi_1^i - \xi_N^i|, \ldots\right\} = 1$$

implies

$$\|y_N^i\| = \sup\left\{\frac{|\xi_1^i|}{\lambda}, |\xi_1^i - \xi_2^i|, \ldots, |\xi_1^i - \xi_{N-1}^i|, 0, |\xi_1^i - \xi_{N+1}^i|, \ldots\right\} = 1.$$

We also have that $e_N \to 0$ weakly, $y_N^i \to x_i$ weakly, and

$$\|y_N^i \pm e_N\| = 1$$

because $\left|\xi_1^i - \left(\xi_1^i \pm 1\right)\right| = 1$.

We have verified that the nets $(y_N^i)_{N>N'}$ and $(e_N)_{N>N'}$ suit to the role of (y_α^i) and (z_α), respectively. □

The paper [8] offers yet another equivalent renorming of c_0. Fix a $\mu \in (0,1)$ such that $\mu = \sum_n \mu_n$ where $\mu_n > 0$ for every n. Denote $\check{c}_0 = (c_0, \|\cdot\|)$ where

$$\|(a_n)\| = \sup_n \left(|a_n| + \sum_{k=1}^n \mu_k |a_k|\right).$$

The "s-LASQ" analysis of $\check{c}_0$ remains inconclusive here, but some remarks will be made. We denote

$$s = 1 - \sum_k \frac{\mu_k}{1+\mu_k} \in (0,1).$$

Let the unit vector basis of $\check{c}_0$ be denoted by $(e_n)_{n=1}^\infty$ where

$$e_n = \left(0, \ldots, 0, \frac{1}{1+\mu_n}, 0, \ldots\right).$$

Note that the information from [8, Example 4.4] together with [8, Corollary 2.4] and Theorem 2.1 shows that $\check{c}_0$ has the $(1-\mu)$-ASQ property. However, we can say more.

Proposition 3.6. *The space $\check{c}_0$ has the s-ASQ property.*

Proof. We follow the scheme of the proof of Proposition 3.1. Take elements $x_1, \ldots, x_n \in S_{\check{c}_0}$ and a number $\varepsilon > 0$. There exists a natural number N such that $\|x_i - P_N x_i\| \leqslant \varepsilon$ for all $i = 1, \ldots, n$ where P_m, $m \in \mathbb{N}$, are the partial sum projections associated to the unit vector basis e_m, $m \in \mathbb{N}$, of $\check{c}_0$.

Denote $y = e_{N+1}$, then $\|y\| = 1$ and $\|P_N x_i + sy\| \leqslant 1$ for all $i = 1, \ldots, n$. Indeed, let $x_i = (\xi_k^i)_{k=1}^{\infty}$, then $\|P_N x_i + sy\|$ is the maximum of numbers (we let $j = 1, \ldots, N$)

$$(1+\mu_j)|\xi_j^i| + \sum_{k=1}^{j-1} \mu_k |\xi_k^i|, \quad \frac{1+\mu_{N+1}}{1+\mu_{N+1}} s + \sum_{k=1}^{N} \mu_k |\xi_k^i|, \quad \frac{\mu_{N+1}}{1+\mu_{N+1}} s + \sum_{k=1}^{N} \mu_k |\xi_k^i|.$$

Since

$$s + \sum_{k=1}^{N} \mu_k |\xi_k^i| \leqslant 1 - \mu + \sum_{k=1}^{N} \frac{\mu_k}{1+\mu_k} < 1,$$

we have $\|P_N x_i + sy\| \leqslant 1$. □

Proposition 3.7. *For any k, for $\tilde{s} \in \left(\frac{1}{1+\mu_k}, 1\right]$, the space $\check{c}_0$ fails the $\tilde{s}$-LASQ property.*

Proof. Let $\tilde{s} > \frac{1}{1+\mu_k}$ for some index k. Take $x = e_k \in S_{\check{c}_0}$. We denote $y = (a_n)$, $\|y\| = 1$, and prove that $\|x \pm \tilde{s}y\| \leqslant 1 + \varepsilon$ for a small $\varepsilon > 0$ is impossible.

Let m be an index, $m > k$, for which

$$\sum_{j=1}^{m-1} \mu_j |a_j| + (1+\mu_m)|a_m| > 1 - \varepsilon.$$

Let $a_k \geqslant 0$. Now

$$\begin{aligned} \|x + \tilde{s}y\| &\geqslant \frac{\mu_k}{1+\mu_k} + \tilde{s} a_k \mu_k + \sum_{\substack{j=1 \\ j \neq k}}^{m-1} \mu_j \tilde{s} |a_j| + (1+\mu_m)\tilde{s}|a_m| \\ &> \frac{\mu_k}{1+\mu_k} + \tilde{s}(1-\varepsilon) > 1 + \varepsilon, \end{aligned}$$

as the last inequality is equivalent to $\varepsilon < \frac{1}{1+\tilde{s}} \cdot \left(\tilde{s} - \frac{1}{1+\mu_k}\right)$.

For $a_k < 0$, we analogously show that $\|x - \tilde{s}y\| > 1 + \varepsilon$. □

We do not have the answer on the $\tilde{s}$-ASQ-ness of $\check{c}_0$ for any $\tilde{s} > s$. However, the next proposition pushes the lower bound towards s.

Proposition 3.8. *Let k and p be natural numbers such that $k > p$ and*

$$R(p,k) = 1 - \frac{1}{\prod_{j=1}^{k}(1+\mu_j)} - \sum_{i=1}^{p} \frac{\mu_i}{1+\mu_i} \geqslant 0.$$

Let

$$\tilde{s} > 1 - \sum_{j=1}^{p} \frac{\mu_j}{1+\mu_j}.$$

Then $\check{c}_0$ fails the $\tilde{s}$-ASQ property.

Note that the sufficient condition in Proposition 3.8 is non-void, i.e., there exist spaces $\check{c}_0$, where $R(p,k) \geqslant 0$. Indeed, for every $k \in \mathbb{N}$ we have

$$\prod_{j=1}^{k}(1+\mu_j) \geqslant 1 + \sum_{j=1}^{k} \mu_j,$$

therefore

$$R(p,k) > 1 - \frac{1}{1+\sum_{j=1}^{k}\mu_j} - \sum_{j=1}^{p} \mu_j \geqslant \frac{\sum_{j=1}^{k}\mu_j}{1+\mu} - \sum_{j=1}^{p} \mu_j.$$

For example, if we put $\mu_n = \mu q^{n-1}(1-q)$ $(0 < q < 1)$, then $\sum_{j=1}^{n}\mu_j = \mu(1-q^n)$. In this case

$$R(p,k) \geqslant \frac{\mu}{1+\mu}(q^p - q^k + \mu q^p - \mu).$$

Hence, the condition $R(p,k) \geqslant 0$ holds if μ, p, q and k satisfy the inequality

$$\mu < \frac{1-q^k}{1-q^p} - 1.$$

Proof of Prop. 3.8. Assume that under these conditions the space $\check{c}_0$ has the $\tilde{s}$-ASQ property. We fix a number of "bad" elements $x \in S_{\check{c}_0}$ and show that if there is an element $y = (a_n)$ such that $\|x + sy\| \leqslant 1 + \varepsilon$ holds for all of these "bad" elements x (and for a suitably small $\varepsilon > 0$) then one cannot have $\|y\| = 1$.

Denote $x_1 = \pm e_1, \ldots, x_k = \pm e_k$,

$$x_0 = \left(\frac{\pm 1}{1+\mu_1}, \frac{\pm 1}{(1+\mu_1)(1+\mu_2)}, \ldots, \pm \prod_{j=1}^{k} \frac{1}{1+\mu_j}, 0, 0, \ldots \right)$$

where all coordinates can take the $+$ or the $-$ sign independently.

Assume now that, for all choices of signs, $\|x_j + \tilde{s}y\| \leqslant 1+\varepsilon$, $j = 0, 1, \ldots, k$. The norm $\|y\|$ is the supremum of numbers $\sum_{i=1}^{m-1} \mu_i|a_i| + (1+\mu_m)|a_m|$, $m \in \mathbb{N}$.

We have, for a suitable choice of signs in x_m (where $m = 1, \dots, k$) that

$$\begin{aligned} 1 + \tilde{s} &\left(\sum_{j=1}^{m-1} \mu_j |a_j| + (1+\mu_m)|a_m| \right) \\ &= \sum_{j=1}^{m-1} \tilde{s}\mu_j |a_j| + \left| \frac{1}{1+\mu_m} \pm \tilde{s} a_m \right| (1+\mu_m) \\ &\leqslant \|x_m + \tilde{s}y\| \leqslant 1 + \varepsilon, \end{aligned}$$

therefore, for $\varepsilon < \tilde{s}/(1+\tilde{s})$,

$$\sum_{j=1}^{m-1} \mu_j |a_j| + (1+\mu_m)|a_m| < \frac{\varepsilon}{\tilde{s}} < 1 - \varepsilon.$$

Let $m > k$. For a suitable choice of signs in x_0,

$$\begin{aligned} \sum_{j=1}^{k} \frac{\mu_j}{\prod_{i=1}^{j}(1+\mu_i)} &+ \tilde{s} \left(\sum_{j=1}^{m-1} \mu_j |a_j| + (1+\mu_m)|a_m| \right) \\ &= \sum_{j=1}^{k} \left| \frac{1}{\prod_{i=1}^{j}(1+\mu_i)} \pm \tilde{s} a_j \right| \cdot \mu_j \\ &\quad + \sum_{j=k+1}^{m-1} \tilde{s}\mu_j |a_j| + \tilde{s}(1+\mu_m)|a_m| \\ &\leqslant \|x_0 + \tilde{s}y\| \leqslant 1 + \varepsilon. \end{aligned} \tag{6}$$

Since

$$\sum_{j=1}^{k} \frac{\mu_j}{\prod_{i=1}^{j}(1+\mu_i)} + \frac{1}{\prod_{j=1}^{k}(1+\mu_j)} = \|x_0\| = 1,$$

adding $\frac{1}{\prod_{j=1}^{k}(1+\mu_j)}$ to the inequalities (6) yields that

$$\tilde{s} \left(\sum_{j=1}^{m-1} \mu_j |a_j| + (1+\mu_m)|a_m| \right) \leqslant \frac{1}{\prod_{j=1}^{k}(1+\mu_j)} + \varepsilon.$$

If we had $\sum_{j=1}^{m-1} \mu_j |a_j| + (1+\mu_m)|a_m| > 1 - \varepsilon$, then

$$\tilde{s}(1-\varepsilon) < \frac{1}{\prod_{j=1}^{k}(1+\mu_j)} + \varepsilon$$

and, using that $R(p,k) \geqslant 0$, we would obtain

$$\tilde{s}(1-\varepsilon) < 1 - \sum_{i=1}^{p} \frac{\mu_i}{1+\mu_i} + \varepsilon$$

which would be equivalent to

$$\varepsilon > \frac{\sum_{i=1}^{p} \frac{\mu_i}{1+\mu_i} + \tilde{s} - 1}{\tilde{s}+1} > 0.$$

Therefore ε can not be made arbitrarily small, so $\check{c}_0$ is not $\tilde{s}$-ASQ. □

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