

## On the controllability of Hilfer–Katugampola fractional differential equations

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ABSTRACT. By employing Kuratowski’s measure of noncompactness together with Sadovskii’s fixed point theorem, sufficient conditions for controllability results of Hilfer–Katugampola fractional differential equations in Banach spaces are derived.

### 1. Introduction

In the last few years, Katugampola [5, 6] introduced the generalized fractional integral  ${}^{\rho}I_{0+}^{\mu}$  and derivative operator  ${}^{\rho}D_{0+}^{\mu}$ , which contain an extra parameter  $\rho > 0$ . Taking  $\rho \rightarrow 0$ , the operators  ${}^{\rho}I_{0+}^{\mu}$  and  ${}^{\rho}D_{0+}^{\mu}$  reduce to the Hadamard fractional operators, and for  $\rho = 1$  become the Riemann–Liouville fractional operators. These operators have applications in probability theory [2], Langevin equations [12], and theory of inequalities [13].

Oliveira et al. [10] introduced a new generalized fractional derivative, called the Hilfer–Katugampola fractional derivative, which interpolates the well-known fractional derivatives such as the Hilfer, Hilfer–Hadamard, Katugampola, and Riemann–Liouville derivatives. They studied the existence and uniqueness of solutions of fractional differential equations involving this generalized Katugampola derivative.

On the other hand, controllability is one of the fundamental notions of modern control theory, which enables one to steer the control system from an arbitrary initial state to an arbitrary final state, using the set of admissible controls, where the initial and final state may vary over the entire space. The problem of controllability of nonlinear systems represented by fractional differential equations has been extensively studied by several authors (see, for example, [1, 9, 14]).

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In this paper, we study the controllability of the Hilfer–Katugampola fractional differential equation

$$\begin{aligned} {}^\rho D_{0+}^{\mu,\nu} x(t) &= f(t, x(t)) + Bu(t), \quad t \in J = [0, b], \\ {}^\rho I_{0+}^{1-\gamma} x(t)|_{t=0} &= x_0, \quad \mu \leq \gamma = \mu + \nu(1 - \mu) < 1, \end{aligned} \quad (1.1)$$

in a Banach space  $(\mathbb{X}, \|\cdot\|)$ , where  ${}^\rho D_{0+}^{\mu,\nu}$  denotes the Hilfer–Katugampola fractional derivative of order  $\mu$  ( $0 < \mu < 1$ ) and type  $\nu$  ( $0 \leq \nu < 1$ ),  ${}^\rho I_{0+}^{1-\gamma}$  is the left-sided Katugampola fractional integral of order  $1 - \gamma$ , the function  $x$  takes values in  $\mathbb{X}$ ,  $B$  is a bounded linear operator from a Banach space  $U$  into  $\mathbb{X}$ , the control function  $u(\cdot)$  is given in  $L^2(J, U)$  (the Banach space of functions  $u : J \rightarrow U$  which are Bochner integrable endowed with the norm  $\|u\|_{L^2(J,U)} = \left(\int_0^b \|u(t)\|_U^2 dt\right)^{\frac{1}{2}} < \infty$ ), and  $f : J \times \mathbb{X} \rightarrow \mathbb{X}$  is a Carathéodory function. Our results are motivated by those in [12, 14].

## 2. Preliminaries

In this section, we collect some definitions and lemmas which will be useful in the sequel. Let  $X_c^p(a, b)$  be the normed space of complex-valued Lebesgue measurable functions  $h$  on  $[a, b]$  for which  $c \in \mathbb{R}$ ,  $1 \leq p < \infty$ , and

$$\|h\|_{X_c^p} = \left(\int_a^b |t^c h(t)|^p \frac{dt}{t}\right)^{\frac{1}{p}} < \infty.$$

If  $c = 1/p$ , then the space  $X_c^p(a, b)$  coincides with the classical  $L^p(a, b)$ -space, see Katugampola [5].

Let  $C(J, \mathbb{X})$  be the Banach space of all continuous functions  $g : J \rightarrow \mathbb{X}$  with the supremum norm

$$\|g\|_C = \sup_{t \in J} \|g(t)\|.$$

For  $0 \leq \gamma < 1$  we define the weighted space  $C_{1-\gamma,\rho}(J, \mathbb{X})$  of continuous functions  $g$  on the finite interval  $J$  by

$$C_{1-\gamma,\rho}(J, \mathbb{X}) = \left\{ g : (0, b] \rightarrow \mathbb{X} : \left(\frac{t^\rho}{\rho}\right)^{1-\gamma} g(t) \in C(J, \mathbb{X}) \right\},$$

and with the norm

$$\|g\|_{C_{1-\gamma,\rho}} = \left\| \left(\frac{t^\rho}{\rho}\right)^{1-\gamma} g(t) \right\|_C = \max_{t \in J} \left\| \left(\frac{t^\rho}{\rho}\right)^{1-\gamma} g(t) \right\|.$$

Obviously,  $C_{1-\gamma,\rho}(J, \mathbb{X})$  is a Banach space.

**Definition 2.1 (Katugampola fractional integral [5]).** Let  $\mu, c \in \mathbb{R}$  with  $\mu > 0$  and  $h \in X_c^p(a, b)$ . The generalized left- and right-sided fractional integrals  ${}^\rho I_{a^+}^\mu h(t)$ ,  ${}^\rho I_{b^-}^\mu h(t)$  are defined, respectively, by

$${}^\rho I_{a^+}^\mu h(t) = \frac{\rho^{1-\mu}}{\Gamma(\mu)} \int_a^t \frac{s^{\rho-1} h(s)}{(t^\rho - s^\rho)^{1-\mu}} ds, \quad t > a,$$

and

$${}^\rho I_{b^-}^\mu h(t) = \frac{\rho^{1-\mu}}{\Gamma(\mu)} \int_t^b \frac{s^{\rho-1} h(s)}{(t^\rho - s^\rho)^{1-\mu}} ds, \quad t < b,$$

where  $\rho > 0$  and  $\Gamma(\cdot)$  is the Gamma function.

**Definition 2.2 (Katugampola fractional derivative [6]).** Let  $\mu, \rho \in \mathbb{R}$  such that  $\mu, \rho > 0$ ,  $\mu \notin \mathbb{N}$ , and  $n = [\mu] + 1$ . The generalized left- and right-sided fractional derivatives  ${}^\rho D_{a^+}^\mu h(t)$ ,  ${}^\rho D_{b^-}^\mu h(t)$  are defined, respectively, by

$$\begin{aligned} {}^\rho D_{a^+}^\mu h(t) &= \left( t^{1-\rho} \frac{d}{dt} \right)^n {}^\rho I_{a^+}^{n-\mu} h(t) \\ &= \frac{\rho^{\mu-n+1}}{\Gamma(n-\mu)} \left( t^{1-\rho} \frac{d}{dt} \right)^n \int_a^t \frac{s^{\rho-1} h(s)}{(t^\rho - s^\rho)^{\mu-n+1}} ds \end{aligned}$$

and

$$\begin{aligned} {}^\rho D_{b^-}^\mu h(t) &= \left( -t^{1-\rho} \frac{d}{dt} \right)^n {}^\rho I_{b^-}^{n-\mu} h(t) \\ &= \frac{\rho^{\mu-n+1}}{\Gamma(n-\mu)} \left( -t^{1-\rho} \frac{d}{dt} \right)^n \int_t^b \frac{s^{\rho-1} h(s)}{(t^\rho - s^\rho)^{\mu-n+1}} ds, \end{aligned}$$

if the integrals exist.

**Definition 2.3 (The Hilfer–Katugampola fractional derivative).** Let the order  $\mu$  and the type  $\nu$  satisfy  $n-1 < \mu < n$  and  $0 \leq \nu < 1$ , where  $n \in \mathbb{N}$ . The Hilfer–Katugampola fractional derivative (left-sided/right-sided) of a function  $\phi \in C_{1-\gamma, \rho}(J, \mathbb{X})$ ,  $\rho > 0$ , is defined by

$${}^\rho D_{a^\pm}^{\mu, \nu} \phi(t) = \left( \pm^\rho I_{a^\pm}^{\nu(n-\mu)} \left( t^{1-\rho} \frac{d}{dt} \right)^n {}^\rho I_{a^\pm}^{(1-\nu)(n-\mu)} \phi \right) (t). \quad (2.1)$$

In this paper we consider only the case  $n = 1$ .

**Remark 2.4.** It is useful to mention that Definition 2.3 was introduced in [10] for complex valued functions from  $X_c^p(a, b)$ . In this paper, the function  $h$  in Definitions 2.1 and 2.2 has values in a Banach space, and the integrals there are taken in Bochner’s sense.

**Definition 2.5.** A solution  $x \in C_{1-\gamma, \rho}$  of the Hilfer–Katugampola fractional differential equation (1.1) is a measurable function satisfying the initial condition  ${}^\rho I_{0^+}^{1-\gamma} x(t)|_{t=0} = x_0$  and the equation  ${}^\rho D_{0^+}^{\mu, \nu} x(t) = f(t, x(t)) + Bu(t)$  on  $J$ .

According to Theorem 4.1 in [10], we conclude the following lemma.

**Lemma 2.6.** *Let  $\gamma = \mu + \nu(1 - \mu)$ , where  $0 < \mu < 1$ ,  $0 \leq \nu < 1$ , and  $\rho > 0$ . Let  $f : J \times \mathbb{X} \rightarrow \mathbb{X}$  be a function such that  $f(\cdot, x(\cdot)) \in C_{1-\gamma, \rho}(J, \mathbb{X})$  for any  $x \in C_{1-\gamma, \rho}(J, \mathbb{X})$ . Then  $x$  is a solution the Hilfer–Katugampola fractional differential equation (1.1) if and only if it satisfies the Volterra integral equation*

$$x(t) = \frac{x_0}{\Gamma(\gamma)} \left(\frac{t^\rho}{\rho}\right)^{\gamma-1} + \frac{1}{\Gamma(\mu)} \int_0^t s^{\rho-1} \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\mu-1} (f(s, x(s)) + Bu(s)) ds.$$

**Definition 2.7.** The Hilfer–Katugampola fractional differential equation (1.1) is said to be controllable on  $J$  if, for any initial and final one  $x_0, x_1 \in \mathbb{X}$ , there exists a control function  $u \in L^2(J, U)$  such that the solution of (1.1) satisfies  $x(b) = x_1$ .

### 3. Kuratowski's measure of noncompactness

The notion of the measure of noncompactness ( $\alpha$ -measure, or set measure) was first introduced by Kuratowski [7] in 1930.

**Definition 3.1.** Let  $Q$  be a bounded subset of a seminormed linear space  $E$ . The Kuratowski measure of noncompactness (the set-measure of noncompactness, or  $\alpha$ -measure) of  $Q$ , denoted by  $\alpha(Q)$ , is the infimum of the set of all numbers  $\epsilon > 0$  such that  $Q$  can be covered by a finite number of sets with diameters less than  $\epsilon$ , that is

$$\alpha(Q) = \inf \left\{ \epsilon > 0 : Q \subset \bigcup_{i=1}^n S_i, S_i \subset E, \text{diam}(S_i) < \epsilon \right\}.$$

**Lemma 3.2.** *Let  $Q_1$  and  $Q_2$  be two bounded sets in a Banach space  $\mathbb{X}$ . Then*

- (i)  $\alpha(Q_1) = 0$  if and only if  $\overline{Q_1}$  is compact ( $Q_1$  is relatively compact),
- (ii)  $\alpha(Q_1) = \alpha(\overline{Q_1})$ ,
- (iii)  $Q_1 \subset Q_2$  implies  $\alpha(Q_1) \leq \alpha(Q_2)$ ,
- (iv)  $\alpha(Q_1 + Q_2) \leq \alpha(Q_1) + \alpha(Q_2)$ .

For more properties of Kuratowski's measure of noncompactness, we refer to [3].

**Theorem 3.3 (Sadovskii's fixed point theorem [11]).** *Let  $\mathcal{K}$  be a condensing operator on a Banach space  $\mathbb{X}$ , i.e.,  $\mathcal{K}$  is continuous, maps bounded sets into bounded sets, and  $\alpha(\mathcal{K}(D)) < \alpha(D)$  for every bounded subset  $D$  of  $\mathbb{X}$  with  $\alpha(D) > 0$ . If  $\mathcal{K}(S) \subset S$  for a convex, closed, and bounded subset  $S$  of  $\mathbb{X}$ , then  $\mathcal{K}$  has a fixed point in  $S$ .*

### 4. Controllability results

In this section we investigate sufficient conditions for the controllability of the Hilfer–Katugampola fractional differential equation (1.1). Thereat we use the notation

$$\Delta(t, s) = \left( \frac{t^\rho - s^\rho}{\rho} \right)^{\mu-1}.$$

We also need to impose the following definitons and conditions:

- (A1) Let  $f : J \times \mathbb{X} \rightarrow \mathbb{X}$  be a Carathéodory function, i.e., for each  $t \in J$  the function  $f(t, \cdot) : \mathbb{X} \rightarrow \mathbb{X}$  is continuous and for each  $x \in \mathbb{X}$  the function  $f(\cdot, x) : J \rightarrow \mathbb{X}$  is measurable.
- (A2) There exists a continuous function  $\Phi : J \rightarrow [0, \infty[$  such that

$$\|f(t, x)\| \leq \Phi(t)\|x\| \text{ for all } t \in J \text{ and } x \in \mathbb{X}$$

with

$$\Phi^* = \sup_{t \in J} \Phi(t) < \infty.$$

- (A3) The linear operator  $\mathcal{W} : L^2(J, U) \rightarrow \mathbb{X}$ , defined by

$$\mathcal{W}u = \frac{1}{\Gamma(\mu)} \int_0^b s^{\rho-1} \Delta(b, s) B u(s) ds,$$

has an induced inverse operator  $\mathcal{W}^{-1}$  which takes values in the space  $L^2(J, U)/\ker \mathcal{W}$ , where the kernel space of  $\mathcal{W}$  is defined by  $\ker \mathcal{W} = \{x \in L^2(J, U) : \mathcal{W}x = 0\}$ , and there exist constants  $M_1, M_2 > 0$  such that  $\|B\| \leq M_1$  and  $\|\mathcal{W}^{-1}\| \leq M_2$  (see [4]).

- (A4) Let  $\mathcal{B}_k = \{y \in C_{1-\gamma, \rho}(J, \mathbb{X}) : \|y\|_{C_{1-\gamma, \rho}} \leq k\}$  with

$$k \geq \frac{\frac{M_1 M_2}{\Gamma(\mu+1)} \left(\frac{b^\rho}{\rho}\right)^{1-\gamma+\mu} \left(\|x_1\| + \frac{\|x_0\|}{\Gamma(\gamma)} \left(\frac{b^\rho}{\rho}\right)^{\gamma-1}\right)}{1 - \left(\frac{M_1 M_2 \Phi^*}{(\Gamma(\mu+1))^2} \left(\frac{b^\rho}{\rho}\right)^{1-\gamma+2\mu}\right)}.$$

Define the control function  $u \in L^2(J, U)$  by

$$u(t) = \mathcal{W}^{-1} \left[ x_1 - \frac{x_0}{\Gamma(\gamma)} \left(\frac{b^\rho}{\rho}\right)^{\gamma-1} - \frac{1}{\Gamma(\mu)} \int_0^b s^{\rho-1} \Delta(b, s) f(s, x(s)) ds \right] (t).$$

We need the following lemma.

**Lemma 4.1.** *If  $x \in \mathcal{B}_k$  and  $t \in J$ , then*

$$\|u(t)\| \leq M_2 \left( \|x_1\| + \frac{\|x_0\|}{\Gamma(\gamma)} \left(\frac{b^\rho}{\rho}\right)^{\gamma-1} + \frac{\Phi^*}{\Gamma(\mu+1)} \left(\frac{b^\rho}{\rho}\right)^\mu k \right).$$

*Proof.* One has

$$\begin{aligned}
\|u(t)\| &\leq \|W^{-1}\| \left( \|x_1\| + \frac{\|x_0\|}{\Gamma(\gamma)} \left(\frac{b^\rho}{\rho}\right)^{\gamma-1} \right. \\
&\quad \left. + \frac{1}{\Gamma(\mu)} \int_0^b s^{\rho-1} \Delta(b, s) \|f(s, x(s))\| ds \right) \\
&\leq M_2 \left( \|x_1\| + \frac{\|x_0\|}{\Gamma(\gamma)} \left(\frac{b^\rho}{\rho}\right)^{\gamma-1} + \frac{1}{\Gamma(\mu)} \int_0^b s^{\rho-1} \Delta(b, s) \Phi(s) \|x(s)\| ds \right) \\
&\leq M_2 \left( \|x_1\| + \frac{\|x_0\|}{\Gamma(\gamma)} \left(\frac{b^\rho}{\rho}\right)^{\gamma-1} + \frac{k\Phi^*}{\Gamma(\mu)} \int_0^b s^{\rho-1} \Delta(b, s) ds \right) \\
&\leq M_2 \left( \|x_1\| + \frac{\|x_0\|}{\Gamma(\gamma)} \left(\frac{b^\rho}{\rho}\right)^{\gamma-1} + \frac{\Phi^*}{\Gamma(\mu+1)} \left(\frac{b^\rho}{\rho}\right)^\mu k \right).
\end{aligned}$$

□

Now we are ready to prove our main theorem.

**Theorem 4.2.** *Under the assumptions (A1) – (A4), let*

$$\frac{\Phi^*}{\Gamma(\mu+1)} \left(\frac{b^\rho}{\rho}\right)^{1-\gamma+\mu} \leq 1. \tag{4.1}$$

*Then the Hilfer–Katugampola fractional differential equation (1.1) is controllable on  $J$ .*

*Proof.* Using the control function  $u$ , we define the operator  $\mathcal{K} : C_{1-\gamma, \rho}(J, \mathbb{X}) \rightarrow C_{1-\gamma, \rho}(J, \mathbb{X})$  by

$$\begin{aligned}
(\mathcal{K}x)(t) &= \frac{x_0}{\Gamma(\gamma)} \left(\frac{t^\rho}{\rho}\right)^{\gamma-1} + \frac{1}{\Gamma(\mu)} \int_0^t s^{\rho-1} \Delta(t, s) f(s, x(s)) ds \\
&\quad + \frac{1}{\Gamma(\mu)} \int_0^t s^{\rho-1} \Delta(t, s) Bu(s) ds.
\end{aligned}$$

The operator  $\mathcal{K}$  is well defined and the fixed points of  $\mathcal{K}$  are solutions to (1.1). Indeed,  $x \in \mathcal{B}_k$  is a solution of (1.1) if and only if  $x$  is a solution of the operator equation  $x = \mathcal{K}x$ . Therefore, the existence of a solution of (1.1) is equivalent to determining a positive constant  $k$  such that  $\mathcal{K}$  has a fixed point on  $\mathcal{B}_k$ .

We decompose the operator  $\mathcal{K}$  into two operators  $\mathcal{K}_1$  and  $\mathcal{K}_2$  ( $\mathcal{K} = \mathcal{K}_1 + \mathcal{K}_2$ ) on  $\mathcal{B}_k$ , where

$$(\mathcal{K}_1x)(t) = \frac{1}{\Gamma(\mu)} \int_0^t s^{\rho-1} \Delta(t, s) Bu(s) ds, \quad t \in J,$$

and

$$(\mathcal{K}_2x)(t) = \frac{x_0}{\Gamma(\gamma)} \left(\frac{t^\rho}{\rho}\right)^{\gamma-1} + \frac{1}{\Gamma(\mu)} \int_0^t s^{\rho-1} \Delta(t,s) f(s,x(s)) ds, \quad t \in J.$$

In order to apply Theorem 3.3, the proof is divided into four steps.

**Step 1.** The operator  $\mathcal{K}_1$  maps  $\mathcal{B}_k$  into itself.

For each  $t \in J$  and  $x \in \mathcal{B}_k$ , by Lemma 4.1 we have

$$\begin{aligned} \left\| \left(\frac{t^\rho}{\rho}\right)^{1-\gamma} (\mathcal{K}_1x)(t) \right\|_C &= \max_{t \in J} \left\| \frac{1}{\Gamma(\mu)} \left(\frac{t^\rho}{\rho}\right)^{1-\gamma} \int_0^t s^{\rho-1} \Delta(t,s) Bu(s) ds \right\| \\ &\leq \frac{1}{\Gamma(\mu)} \left(\frac{b^\rho}{\rho}\right)^{1-\gamma} \int_0^t s^{\rho-1} \Delta(t,s) \|B\| \|u(s)\| ds \\ &\leq \frac{M_1M_2}{\Gamma(\mu+1)} \left(\frac{b^\rho}{\rho}\right)^{1-\gamma+\mu} \left( \|x_1\| + \frac{\|x_0\|}{\Gamma(\gamma)} \left(\frac{b^\rho}{\rho}\right)^{\gamma-1} \right. \\ &\quad \left. + \frac{\Phi^*}{\Gamma(\mu+1)} \left(\frac{b^\rho}{\rho}\right)^\mu k \right) \leq k, \end{aligned}$$

which implies  $\|(\mathcal{K}_1x)\|_{C_{1-\gamma,\rho}} \leq k$ . Thus  $\mathcal{K}_1$  maps  $\mathcal{B}_k$  into itself.

**Step 2.** The operator  $\mathcal{K}_2$  is continuous.

Let  $\{x_n\}$  be a sequence in  $\mathcal{B}_k$  satisfying  $x_n \rightarrow x$  as  $n \rightarrow \infty$ . Then, for each  $t \in J$ , we have

$$\begin{aligned} &\left\| \left(\frac{t^\rho}{\rho}\right)^{1-\gamma} (\mathcal{K}_2x_n)(t) - \left(\frac{t^\rho}{\rho}\right)^{1-\gamma} (\mathcal{K}_2x)(t) \right\|_C \\ &= \max_{t \in J} \left\| \frac{1}{\Gamma(\mu)} \left(\frac{t^\rho}{\rho}\right)^{1-\gamma} \int_0^t s^{\rho-1} \Delta(t,s) (f(s,x_n(s)) - f(s,x(s))) ds \right\| \\ &\leq \frac{1}{\Gamma(\mu+1)} \left(\frac{b^\rho}{\rho}\right)^{1-\gamma+\mu} \|f(\cdot, x_n(\cdot)) - f(\cdot, x(\cdot))\|. \end{aligned}$$

By the Lebesgue dominated convergence theorem, we get

$$\|(\mathcal{K}_2x_n) - (\mathcal{K}_2x)\|_{C_{1-\gamma,\rho}} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

This means that  $\mathcal{K}_2$  is continuous.

**Step 3.** We show that  $\mathcal{K}_2(\mathcal{B}_k) \subset \mathcal{B}_k$ .

We prove this by contradiction, supposing that there exists a function  $\eta(\cdot) \in \mathcal{B}_k$  such that  $\|(\mathcal{K}_2\eta)\|_{C_{1-\gamma,\rho}} > k$ . Thus, for each  $t \in J$ , we have

$$\begin{aligned} k &< \left\| \left(\frac{t^\rho}{\rho}\right)^{1-\gamma} (\mathcal{K}_2\eta)(t) \right\|_C \\ &\leq \frac{\|x_0\|}{\Gamma(\mu)} + \frac{1}{\Gamma(\mu)} \left(\frac{t^\rho}{\rho}\right)^{1-\gamma} \int_0^t s^{\rho-1} \Delta(t,s) \|f(s,\eta(s))\| ds \end{aligned}$$

$$\begin{aligned}
&\leq \frac{\|x_0\|}{\Gamma(\mu)} + \frac{1}{\Gamma(\mu)} \left(\frac{t^\rho}{\rho}\right)^{1-\gamma} \int_0^t s^{\rho-1} \Delta(t,s) \Phi(s) \|\eta(s)\| ds \\
&\leq \frac{\|x_0\|}{\Gamma(\mu)} + \frac{k\Phi^*}{\Gamma(\mu)} \left(\frac{t^\rho}{\rho}\right)^{1-\gamma} \int_0^t s^{\rho-1} \Delta(t,s) ds \\
&\leq \frac{\|x_0\|}{\Gamma(\mu)} + \frac{\Phi^*}{\Gamma(\mu+1)} \left(\frac{b^\rho}{\rho}\right)^{1-\gamma+\mu} k.
\end{aligned}$$

Dividing both sides by  $k$ , and taking the limit as  $k \rightarrow \infty$ , we get

$$\frac{\Phi^*}{\Gamma(\mu+1)} \left(\frac{b^\rho}{\rho}\right)^{1-\gamma+\mu} \geq 1,$$

which contradicts (4.1). This shows that  $\mathcal{K}_2(\mathcal{B}_k) \subset \mathcal{B}_k$ .

**Step 4.**  $\mathcal{K}_2(\mathcal{B}_k)$  is bounded and equicontinuous.

From Step 3, it is clear that  $\mathcal{K}_2(\mathcal{B}_k)$  is bounded. It remains to show that  $\mathcal{K}_2(\mathcal{B}_k)$  is equicontinuous.

Let  $t_1, t_2 \in J$ ,  $t_1 < t_2$ . For each  $x \in \mathcal{B}_k$  we have

$$\begin{aligned}
&\left\| \left(\frac{t_2^\rho}{\rho}\right)^{1-\gamma} (\mathcal{K}_2 x)(t_2) - \left(\frac{t_1^\rho}{\rho}\right)^{1-\gamma} (\mathcal{K}_2 x)(t_1) \right\|_C \\
&= \max_{t \in J} \left\| \frac{1}{\Gamma(\mu)} \left(\frac{t_2^\rho}{\rho}\right)^{1-\gamma} \int_0^{t_2} s^{\rho-1} \Delta(t_2, s) f(s, x(s)) ds \right. \\
&\quad \left. - \frac{1}{\Gamma(\mu)} \left(\frac{t_1^\rho}{\rho}\right)^{1-\gamma} \int_0^{t_1} s^{\rho-1} \Delta(t_1, s) f(s, x(s)) ds \right\| \\
&\leq \left\| \frac{1}{\Gamma(\mu)} \left( \left(\frac{t_2^\rho}{\rho}\right)^{1-\gamma} - \left(\frac{t_1^\rho}{\rho}\right)^{1-\gamma} \right) \int_0^{t_2} s^{\rho-1} \Delta(t_2, s) f(s, x(s)) ds \right\| \\
&\quad + \left\| \frac{1}{\Gamma(\mu)} \left(\frac{t_1^\rho}{\rho}\right)^{1-\gamma} \int_{t_1}^{t_2} s^{\rho-1} \Delta(t_2, s) f(s, x(s)) ds \right\| \\
&\quad + \left\| \frac{1}{\Gamma(\mu)} \left(\frac{t_1^\rho}{\rho}\right)^{1-\gamma} \int_0^{t_1} s^{\rho-1} (\Delta(t_2, s) - \Delta(t_1, s)) f(s, x(s)) ds \right\| \\
&= I_1 + I_2 + I_3,
\end{aligned}$$

where

$$\begin{aligned}
I_1 &= \left\| \frac{1}{\Gamma(\mu)} \left( \left(\frac{t_2^\rho}{\rho}\right)^{1-\gamma} - \left(\frac{t_1^\rho}{\rho}\right)^{1-\gamma} \right) \int_0^{t_2} s^{\rho-1} \Delta(t_2, s) f(s, x(s)) ds \right\|, \\
I_2 &= \left\| \frac{1}{\Gamma(\mu)} \left(\frac{t_1^\rho}{\rho}\right)^{1-\gamma} \int_{t_1}^{t_2} s^{\rho-1} \Delta(t_2, s) f(s, x(s)) ds \right\|,
\end{aligned}$$



$$I_3 = \left\| \frac{1}{\Gamma(\mu)} \left(\frac{t_1^\rho}{\rho}\right)^{1-\gamma} \int_0^{t_1} s^{\rho-1} (\Delta(t_2, s) - \Delta(t_1, s)) f(s, x(s)) ds \right\|.$$

By the assumptions (A2) and (A4) we have

$$\begin{aligned} I_1 &\leq \frac{1}{\Gamma(\mu)} \left( \left(\frac{t_2^\rho}{\rho}\right)^{1-\gamma} - \left(\frac{t_1^\rho}{\rho}\right)^{1-\gamma} \right) \int_0^{t_2} s^{\rho-1} \Delta(t_2, s) \|f(s, x(s))\| ds \\ &\leq \frac{k\Phi^*}{\Gamma(\mu)} \left( \left(\frac{t_2^\rho}{\rho}\right)^{1-\gamma} - \left(\frac{t_1^\rho}{\rho}\right)^{1-\gamma} \right) \int_0^{t_2} s^{\rho-1} \Delta(t_2, s) ds \\ &= \frac{k\Phi^*}{\Gamma(\mu+1)} \left(\frac{t_2^\rho}{\rho}\right)^\mu \left( \left(\frac{t_2^\rho}{\rho}\right)^{1-\gamma} - \left(\frac{t_1^\rho}{\rho}\right)^{1-\gamma} \right) \rightarrow 0 \text{ as } t_2 \rightarrow t_1, \end{aligned}$$

$$\begin{aligned} I_2 &\leq \frac{1}{\Gamma(\mu)} \left(\frac{t_1^\rho}{\rho}\right)^{1-\gamma} \int_{t_1}^{t_2} s^{\rho-1} \Delta(t_2, s) \|f(s, x(s))\| ds \\ &\leq \frac{k\Phi^*}{\Gamma(\mu)} \left(\frac{t_1^\rho}{\rho}\right)^{1-\gamma} \int_{t_1}^{t_2} s^{\rho-1} \Delta(t_2, s) ds \\ &= \frac{k\Phi^*}{\Gamma(\mu+1)} \left(\frac{1}{\rho}\right)^\mu (t_2^\rho - t_1^\rho)^\mu \rightarrow 0 \text{ as } t_2 \rightarrow t_1, \end{aligned}$$

$$\begin{aligned} I_3 &\leq \frac{1}{\Gamma(\mu)} \left(\frac{t_1^\rho}{\rho}\right)^{1-\gamma} \int_0^{t_1} s^{\rho-1} (\Delta(t_2, s) - \Delta(t_1, s)) \|f(s, x(s))\| ds \\ &\leq \frac{k\Phi^*}{\Gamma(\mu)} \left(\frac{t_1^\rho}{\rho}\right)^{1-\gamma} \int_0^{t_1} s^{\rho-1} (\Delta(t_2, s) - \Delta(t_1, s)) ds \\ &= \frac{k\Phi^*}{\Gamma(\mu+1)} \left(\frac{1}{\rho}\right)^\mu ((t_2^{\rho\mu} - t_1^{\rho\mu}) - (t_2^\rho - t_1^\rho)^\mu) \\ &\leq \frac{k\Phi^*}{\Gamma(\mu+1)} \left(\frac{1}{\rho}\right)^\mu (t_2^{\rho\mu} - t_1^{\rho\mu}) \rightarrow 0 \text{ as } t_2 \rightarrow t_1. \end{aligned}$$

Combining these estimations of  $I_1$ ,  $I_2$ , and  $I_3$ , we deduce that

$$\|(\mathcal{K}_2x)(t_2) - (\mathcal{K}_2x)(t_1)\|_{C_{1-\gamma,\rho}} \rightarrow 0 \text{ as } t_2 \rightarrow t_1.$$

Hence  $\mathcal{K}_2(\mathcal{B}_k)$  is equicontinuous.

As a consequence of Steps 2–4, together with the Arzelà–Ascoli theorem, we deduce that  $\mathcal{K}_2$  is compact. Hence, from Steps 1–4 and Lemma 3.2, we conclude that  $\mathcal{K} = \mathcal{K}_1 + \mathcal{K}_2$  is continuous and takes bounded sets into bounded sets. Also, one can verify the validity of  $\alpha(\mathcal{K}_2(\mathcal{B}_k)) = 0$  since  $\mathcal{K}_2(\mathcal{B}_k)$  is relatively compact.

It follows from the inclusion  $\mathcal{K}_1(\mathcal{B}_k) \subset \mathcal{B}_k$  and the equality  $\alpha(\mathcal{K}_2(\mathcal{B}_k)) = 0$  that

$$\alpha(\mathcal{K}(\mathcal{B}_k)) \leq \alpha(\mathcal{K}_1(\mathcal{B}_k)) + \alpha(\mathcal{K}_2(\mathcal{B}_k)) \leq \alpha(\mathcal{B}_k)$$

for every bounded set  $\mathcal{B}_k$  of  $C_{1-\gamma,\rho}(J, \mathbb{X})$  with  $\alpha(\mathcal{B}_k) > 0$ .

Since  $\mathcal{K}(\mathcal{B}_k) \subset \mathcal{B}_k$  for a convex, closed, and bounded set  $\mathcal{B}_k$  of  $C_{1-\gamma,\rho}(J, \mathbb{X})$ , all conditions of Theorem 3.3 are satisfied, and we conclude that the operator  $\mathcal{K}$  has a fixed point  $x \in \mathcal{B}_k$  which, in the same time, is a solution of the Hilfer–Katugampola fractional differential equation (1.1) such that  $x(b) = x_1$ . Therefore, the Hilfer–Katugampola fractional differential equation (1.1) is controllable on  $J$ .  $\square$

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