ACTA ET COMMENTATIONES UNIVERSITATIS TARTUENSIS DE MATHEMATICA
Volume 24, Number 2, December 2020
Available online at http://acutm.math.ut.ee

When the annihilator graph of a commutative ring is planar or toroidal?

M. Bakhtyiari, R. Nikandish, and M. J. Nikmehr

Abstract. Let R be a commutative ring with identity, and let $Z(R)$ be the set of zero-divisors of R. The annihilator graph of R is defined as the undirected graph $AG(R)$ with the vertex set $Z(R)^* = Z(R) \setminus \{0\}$, and two distinct vertices x and y are adjacent if and only if $ann_R(xy) \neq ann_R(x) \cup ann_R(y)$. In this paper, all rings whose annihilator graphs can be embedded on the plane or torus are classified.

1. Introduction

Recently, a major part of research in algebraic combinatorics has been devoted to the application of graph theory and combinatorics in abstract algebra. There are a lot of papers which apply combinatorial methods to obtain algebraic results in ring theory (see [2], [3], [6] and [13]). Moreover, for most recent study in this field see [7] and [14].

Throughout this paper R is a commutative ring with identity which is not an integral domain. We denote by $\mathrm{Min}(R)$, $\mathrm{Nil}(R)$ and $\mathrm{U}(R)$, the set of all minimal prime ideals of R, the set of all nilpotent elements of R, and the set of all invertible elements of R, respectively. Also, the set of all zero-divisors of an R-module M, which is denoted by $Z(M)$, is the set

$$Z(M) = \{r \in R \,|\, rx = 0 \text{ for some nonzero element } x \in M\}.$$

A finite field of order n is denoted by $\mathbb{F}_n$. By $\dim(R)$ and $\mathrm{depth}(R)$, we mean the dimension and depth of R, see [16]. For every ideal I of R, we denote the annihilator of I by $\mathrm{Ann}(I)$. For a subset A of a ring R we let $A^* = A \setminus \{0\}$. The ring R is said to be *reduced* if it has no non-zero nilpotent elements. Let R be a Noetherian local ring. Then R is said to be a *Cohen–Macaulay*

Received November 30, 2019.
2010 *Mathematics Subject Classification.* Primary 13A99; Secondary 05C10.
Key words and phrases. Annihilator graph, planarity, toroidality.
https://doi.org/10.12697/ACUTM.2020.24.19
Corresponding author: R. Nikandish

ring if depht$(R) = \dim(R)$. In general, if R is a Noetherian ring, then R is a Cohen–Macaulay ring if $R_{\mathfrak{m}}$ is a Cohen–Macaulay ring, for all maximal ideals $\mathfrak{m}$, where $R_{\mathfrak{m}}$ is the localization of R at $\mathfrak{m}$. Also, a Noetherian local ring R is called *Gorenstein* if R is Cohen–Macaulay and $\dim_{R/\mathfrak{m}}(\mathrm{soc}(R)) = 1$, where $\mathfrak{m}$ is the unique maximal ideal of R. In general, if R is a Noetherian ring, then R is a Gorenstein ring if $R_{\mathfrak{m}}$ is a Gorenstein ring, for all maximal ideals $\mathfrak{m}$. For any undefined notation or terminology in ring theory, we refer the reader to [9, 16, 17].

Let $G = (V, E)$ be a graph, where $V = V(G)$ is the set of vertices and $E = E(G)$ is the set of edges. By K_n and $K_{m,n}$ we mean the complete graph of order n and the complete bipartite graph with part sizes m and n, respectively. Moreover, by $\overline{G}$ we denote the complement of G. The graph $H = (V_0, E_0)$ is a subgraph of G if $V_0 \subseteq V$ and $E_0 \subseteq E$. Moreover, H is called an *induced subgraph by* V_0, denoted by $G[V_0]$, if $V_0 \subseteq V$ and $E_0 = \{\{u, v\} \in E \mid u, v \in V_0\}$. Let G_1 and G_2 be two graphs. *The subdivision of a graph* G is a graph obtained from G by subdividing some of the edges, that is, by replacing the edges by paths having at most their endvertices in common. By $G_1 \vee G_2$ and $G_1 = G_2$, we mean the *join* of G_1, G_2 and G_1 is identical to G_2, respectively. Let S_k denote the sphere with k handles, where k is a non-negative integer, that is, S_k is an oriented surface of genus k. The *genus* of a graph G, denoted $\gamma(G)$, is the minimal integer n such that the graph can be embedded in S_n (see [17, Chapter 6]). Intuitively, G is embedded in a surface if it can be drawn in the surface so that its edges intersect only at their common vertices. A genus 0 graph is called a *planar graph* and a genus 1 graph is called a *toroidal graph*. It is well known that

$$\gamma(K_n) = \left\lceil \frac{(n-3)(n-4)}{12} \right\rceil \quad \text{if } n \geq 3$$

and

$$\gamma(K_{m,n}) = \left\lceil \frac{(m-2)(n-2)}{4} \right\rceil \quad \text{if } n,m \geq 2.$$

The *annihilator graph* of a ring R is defined as the graph $AG(R)$ with the vertex set $Z(R)^* = Z(R) \setminus \{0\}$, and two distinct vertices x and y are adjacent if and only if $ann_R(xy) \neq ann_R(x) \cup ann_R(y)$. This graph was first introduced and investigated in [6] and many of interesting properties of an annihilator graph were studied. This paper is devoted to classify all rings whose annihilator graphs are planar or toroidal.

2. Planar annihilator graphs

In this section, we characterize all rings whose annihilator graphs are planar. Moreover, it is shown that the genus of the annihilator graph associated with an infinite ring is either zero or infinite. First, we recall a series of necessary results.

Lemma 1 (see [12], Lemma 2.1)**.** *Let R be a ring and let x, y be distinct elements of $Z(R)^*$. Then the following statements are equivalent.*
(1) *$x - y$ is an edge of $AG(R)$.*
(2) *$Rx \cap ann_R(y) \neq (0)$ and $Ry \cap ann_R(x) \neq (0)$.*
(3) *$x \in Z(Ry)$ and $y \in Z(Rx)$.*

Lemma 2 (see [12], Lemma 2.2)**.** *Let R be a ring.*
(1) *Let x, y be elements of $Z(R)^*$. If $ann_R(x) \nsubseteq ann_R(y)$ and $ann_R(y) \nsubseteq ann_R(x)$, then $x - y$ is an edge of $AG(R)$. Moreover, if R is a reduced ring, then the converse is also true.*
(2) *Let $R \cong R_1 \times \cdots \times R_n$, $x = (x_1, \ldots, x_n)$, and $y = (y_1, \ldots, y_n)$, where n is a positive integer, every R_i is a ring, and $x_i, y_i \in R_i$ for every $1 \leq i \leq n$. If $R_i x_i \cap ann_{R_i}(y_i) \neq (0)$ and $R_j y_j \cap ann_{R_j}(x_j) \neq (0)$, for some $1 \leq i, j \leq n$, then $x - y$ is an edge of $AG(R)$. In particular, if $x_i - y_i$ is an edge of $AG(R_i)$ or $x_i = y_i \in \mathrm{Nil}(R_i)^*$, for some $1 \leq i \leq n$, then $x - y$ is an edge of $AG(R)$.*

Lemma 3. *Let R be a reduced ring which contains a minimal ideal. Then R is decomposable.*

Proof. The proof is obtained by [19, 2.7]. □

To classify planar annihilator graphs, we need a celebrated theorem due to Kuratowski.

Theorem 1 (see [17], Theorem 6.2.2)**.** *A graph is planar if and only if it contains no subdivision of either $K_{3,3}$ or K_5.*

Theorem 2. *Let R be a ring such that $R \cong R_1 \times \cdots \times R_n$, where n is a positive integer and R_i is a ring for every $1 \leq i \leq n$. Then the following statements hold.*
(1) *If $n \geq 4$, then $AG(R)$ is not planar.*
(2) *If $n = 3$ and $AG(R)$ is planar, then $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.*

Proof. (1) We only need to show that $AG(R)$ is not planar for $n = 4$. Since the set $\{(1,1,0,0), (0,1,1,0), (0,0,1,1), (1,0,1,0), (0,1,0,1)\}$ is a complete subgraph of $AG(R)$, K_5 is a subgraph of $AG(R)$. The result now follows from Theorem 1.

(2) Let $R \cong R_1 \times R_2 \times R_3$. Assume to the contrary and without loss of generality, $R_1 \neq \mathbb{Z}_2$. Let $x \in R_1 \setminus \{0, 1\}$. Then it is not hard to check that the vertices of the set $\{(1,0,1), (x,0,0), (x,0,1)\}$ and the vertices of the set $\{(0,1,1), (1,1,0), (0,1,0)\}$ together with the path $(x,0,0) - (0,0,1) - (1,1,0)$ forms a subgraph that contains a subdivision of $K_{3,3}$, a contradiction. So $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. □

In the next theorem, we characterize reduced rings whose annihilator graphs are planar.

Theorem 3. *Let R be a reduced ring. Then $AG(R)$ is planar if and only if one of the following statements hold:*
(1) $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$*;*
(2) $|\mathrm{Min}(R)| = 2$ *and one of the minimal prime ideals of R has at most three distinct elements.*

Proof. Suppose that $AG(R)$ is planar and let $x \in Z(R)^*$. Since R is a reduced ring, we have $Rx \cap ann_R(x) = (0)$. If $|Rx| = |ann_R(x)| = \infty$, then obviously $AG(R)$ is not planar, a contradiction. If either $|Rx|$ or $|ann_R(x)|$ is finite, then R has a minimal ideal and so, by Lemma 3, R is decomposable. Assume that $R \cong R_1 \times R_2$, where R_1, R_2 are two rings. If $|\mathrm{Min}(R)| = 2$, then, by [6, Theorem 3.7], one of the minimal prime ideals of R has at most three distinct elements. If $|\mathrm{Min}(R)| \geq 3$, without loss of generality, we may assume that $|\mathrm{Min}(R_2)| \geq 2$. Thus $Z(R_2) \neq (0)$. By repeating the above argument we conclude that R_2 is decomposable. Therefore, one may assume that $R \cong R_1 \times R_2 \times R_3$, where R_1, R_2, R_3 are three rings. By part (2) of Theorem 2, $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.
Conversely, if $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, then one may easily see that $AG(R)$ is planar. Also, if $|\mathrm{Min}(R)| = 2$ and one of the minimal prime ideals of R has at most three distinct elements, then the result follows from [6, Theorem 3.7]. □

To characterize non-reduced rings whose annihilator graphs are planar we state the following lemmas.

Lemma 4 (see [1], Lemma 2.2)**.** *Let R be a ring and let $\mathfrak{m}$ be a maximal ideal in R. If $Ann(\mathfrak{m}) \neq 0$, then $\mathfrak{m} = Z(Ann(\mathfrak{m}))$.*

Lemma 5. *Let R be a ring and let $\mathfrak{m}_1, \mathfrak{m}_2$ be two maximal ideals of R such that $Ann(\mathfrak{m}_1) \neq (0), Ann(\mathfrak{m}_2) \neq (0)$. Then $K_{|\mathfrak{m}_1 \setminus \mathfrak{m}_2|, |\mathfrak{m}_2 \setminus \mathfrak{m}_1|}$ is a subgraph of $AG(R)$.*

Proof. Let $x \in \mathfrak{m}_1 \setminus \mathfrak{m}_2$ and $y \in \mathfrak{m}_2 \setminus \mathfrak{m}_1$. We claim that $Ann(\mathfrak{m}_2) \cap ann_R(x) = 0$. Assume to the contrary, there exists an element $z \in Ann(\mathfrak{m}_2) \cap ann_R(x)$. Hence $zx = 0$. Now, Lemma 4 implies that $x \in \mathfrak{m}_2$, a contradiction. Similarly, $Ann(\mathfrak{m}_1) \cap ann_R(y) = 0$. Since $\mathfrak{m}_1 + \mathfrak{m}_2 = R$, $Ann(\mathfrak{m}_1) \neq Ann(\mathfrak{m}_2)$. Hence $Ann(\mathfrak{m}_1) \subseteq ann_R(x) \nsubseteq ann_R(y)$, $Ann(\mathfrak{m}_2) \subseteq ann_R(y) \nsubseteq ann_R(x)$ and so by part (2) of Lemma 2, $x - y$ is an edge of $AG(R)$. □

Theorem 4. *Let R be a non-reduced ring. Then $AG(R)$ is planar if and only if one of the following statements hold:*
(1) *R is ring-isomorphic to either $\mathbb{Z}_2 \times \mathbb{Z}_4$ or $\mathbb{Z}_2 \times \mathbb{Z}_2[X]/(X^2)$;*
(2) *$Ann(Z(R))$ is a prime ideal of R and* $2 \leq |\mathrm{Nil}(R)| \leq 3$*;*
(3) $Z(R) = \mathrm{Nil}(R)$ *and* $4 \leq |\mathrm{Nil}(R)| \leq 5$.

Proof. Suppose that $AG(R)$ is planar. We consider following two cases.

Case 1. R is decomposable. Let $R \cong R_1 \times R_2$, where R_1, R_2 are two rings. One may assume that there exists a non-zero element $a \in \mathrm{Nil}(R_1)$. We show that $|Z(R_1)| = 2$. If $|Z(R_1)| \geq 3$, then by part (2) of Lemma 2, the vertices contained in the set $\{(1,0),(u,0),(a,0)\}$ and the vertices contained in the set $\{(0,1),(x,1),(a,1)\}$ form $K_{3,3}$, where $1 \neq u \in U(R_1)$ and x is a neighbor of a in $AG(R_1)$, a contradiction (note that $|\mathrm{Nil}(R_1)| \leq |U(R_1)|$). This implies that $|Z(R_1)| = 2$. Similarly, if $x \in R_2 \setminus \{0,1\}$, then the vertices of the set $\{(1,0),(u,0),(a,0)\}$ and the vertices of the set $\{(0,1),(a,x),(a,1)\}$ form $K_{3,3}$, a contradiction. So R is ring-isomorphic to either $\mathbb{Z}_2 \times \mathbb{Z}_4$ or $\mathbb{Z}_2 \times \mathbb{Z}_2[X]/(X^2)$.

Case 2. R is indecomposable. By [6, Theorem 3.10], $2 \leq |\mathrm{Nil}(R)| \leq 5$. Then either $2 \leq |\mathrm{Nil}(R)| \leq 3$ or $4 \leq |\mathrm{Nil}(R)| \leq 5$. First assume that $Z(R) = \mathrm{Nil}(R)$. If $4 \leq |\mathrm{Nil}(R)| \leq 5$, then (3) holds. If $2 \leq |\mathrm{Nil}(R)| \leq 3$, then $\mathrm{Nil}(R)^2 = (0)$ and since $Z(R) = \mathrm{Nil}(R)$, $Ann(Z(R))$ is a prime ideal of R and so (2) holds. Now, let $Z(R) \neq \mathrm{Nil}(R)$ and Ra be a minimal ideal, for some $a \in \mathrm{Nil}(R)^*$. Since R is indecomposable and $Z(R) \neq \mathrm{Nil}(R)$, we conclude that $|ann_R(a)|$ has infinitely many elements. If $xy = 0$, for some $x, y \in Z(R) \setminus \mathrm{Nil}(R)$, then the vertices of the set $\{x, x^2, x^3\}$ and the vertices of the set $\{y, y^2, y^3\}$ are adjacent, a contradiction (as R is indecomposable). So $ann_R(x) \subseteq \mathrm{Nil}(R)$, for every $x \in Z(R) \setminus \mathrm{Nil}(R)$. Now, let $a \neq b \in \mathrm{Nil}(R)^*$. We claim that b is adjacent to all vertices contained in $ann_R(a)$. To see this, we consider two subcases.

Subcase 1. $Ra \subseteq Rb$. Let x be an arbitrary element of $ann_R(a) \setminus \mathrm{Nil}(R)$. If $xb = 0$, then there is nothing to prove. So let $xb \neq 0$ and $xb^{n-1} \neq 0$, $xb^n = 0$, for a positive integer n. Thus $xb^{n-1} \in Rx \cap ann_R(b)$. Since $Ra \subseteq Rb$, we deduce that $Rb \cap ann_R(x) \neq (0)$. Now, by Lemma 1, $x - b$ is an edge of $AG(R)$.

Subcase 2. $Ra \nsubseteq Rb$. Since Ra is a minimal ideal, $Ra \cap Rb = (0)$. So Rb contains a minimal ideal, say Rc, for some $c \in \mathrm{Nil}(R)$. Thus $ann_R(c)$ is a maximal ideal of R. If $ann_R(a) \neq ann_R(c)$, then by Lemma 5, we get a contradiction (as $ann_R(a)$ is a maximal ideal, too). Thus $ann_R(a) = ann_R(c)$. The fact $Rc \subseteq Rb$ together with subcase 1 imply that b is adjacent to all vertices contained in $ann_R(a)$.

So the claim is proved. This together with the planarity of $AG(R)$ imply that $2 \leq |\mathrm{Nil}(R)| \leq 3$ and hence $\mathrm{Nil}(R)$ is a minimal ideal. Since $ann_R(x) \subseteq \mathrm{Nil}(R)$ for every $x \in Z(R) \setminus \mathrm{Nil}(R)$, we have $Ann(Z(R)) = \mathrm{Nil}(R)$ and $\mathrm{Nil}(R)$ is a prime ideal of R.

Conversely, if either (1) or (2) is hold, then obviously $AG(R)$ is planar. Moreover if $Ann(Z(R))$ is a prime ideal of R, then $Ann(Z(R)) = \mathrm{Nil}(R)$ and $ann_R(x) \subseteq \mathrm{Nil}(R)$, for every $x \in Z(R) \setminus \mathrm{Nil}(R)$. Since $\mathrm{Nil}(R)$ is a minimal ideal, $ann_R(x) = \mathrm{Nil}(R)$. Hence $AG(R) = K_{|\mathrm{Nil}(R)^*|} \vee \overline{K_n}$, where $n \in \{0, \infty\}$. Therefore, the condition $2 \leq |\mathrm{Nil}(R)| \leq 3$ implies that $AG(R)$ is planar. □

We are now in a position to classify all finite rings with planar annihilator graphs.

Corollary 1. *Let R be a finite ring. If $AG(R)$ is planar, then R is isomorphic to one of the following rings:*
$\mathbb{Z}_4,\ \mathbb{Z}_2[x]/(x^2),\ \mathbb{Z}_9,\ \mathbb{Z}_3[x]/(x^2),\ \mathbb{Z}_8,\ \mathbb{Z}_2[x]/(x^3),\ \mathbb{Z}_4[x]/(x^2-2, 2x),$
$\mathbb{Z}_2[x,y]/(x^2, xy, y^2),\ \mathbb{Z}_4[x]/(2x, x^2),\ \mathbb{F}_4[x]/(x^2), \mathbb{Z}_4[x]/(x^2+x+1),$
$\mathbb{Z}_{25},\ \mathbb{Z}_5[x]/(x^2),\ \mathbb{Z}_2 \times \mathbb{F}_{p^n},\ \mathbb{Z}_3 \times \mathbb{F}_{p^n},\ \mathbb{Z}_2 \times \mathbb{Z}_4,\ \mathbb{Z}_2 \times \mathbb{Z}_2[X]/(X^2),\ \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2.$

Proof. The proof follows from [15, Section 5], Theorems 3 and 4. □

The last result in this section states that the genus of the annihilator graph associated with an infinite ring is either zero or infinite.

Theorem 5. *Let R be an infinite ring. Then either $\gamma(AG(R)) = 0$ or $\gamma(AG(R)) = \infty$.*

Proof. Suppose to the contrary that $0 < \gamma(AG(R)) < \infty$. We consider the following two cases.

Case 1. R is indecomposable. The equality $|R| = \infty$ together with [6, Theorem 3.10] imply that $Z(R) \neq \mathrm{Nil}(R)$. Let $x \in Z(R) \setminus \mathrm{Nil}(R)$. Since R is indecomposable, $|Rx| = \infty$, and so $\gamma(AG(R)) < \infty$ shows that $|ann_R(x)| \leq 3$. So the indecomposability of R implies that $\mathrm{Nil}(R) \neq (0)$. We claim that for every $y \in Z(R) \setminus \mathrm{Nil}(R)$, $ann_R(x) = ann_R(y)$. If $ann_R(x) \neq ann_R(y)$ for some $y \in Z(R) \setminus \mathrm{Nil}(R)$, then since $ann_R(x)$ and $ann_R(y)$ are two minimal ideals, $ann_R(x) \cap ann_R(y) = (0)$. Now, let $0 \neq a \in ann_R(x)$ and $0 \neq b \in ann_R(y)$. Since Ra and Rb are two minimal ideals, both $ann_R(a)$ and $ann_R(b)$ are maximal ideals. So we put $ann_R(a) = \mathfrak{m}_1$ and $ann_R(b) = \mathfrak{m}_2$. We consider two subcases.

Subcase 1. $|\mathfrak{m}_1 \cap \mathfrak{m}_2| = \infty$. So the vertices contained in the set $\{a, b, a+b\}$ and the vertices contained in the set $\mathfrak{m}_1^* \cap \mathfrak{m}_2^* \setminus \{a, b, a+b\}$ form $K_{3,\infty}$, a contradiction.

Subcase 2. $|\mathfrak{m}_1 \cap \mathfrak{m}_2| < \infty$. The indecomposability of R implies that $\mathfrak{m}_1 \neq \mathfrak{m}_2$, $|\mathfrak{m}_1 \setminus \mathfrak{m}_2| = \infty$ and $|\mathfrak{m}_2 \setminus \mathfrak{m}_1| = \infty$. Thus Lemma 4 contradicts $\gamma(AG(R)) < \infty$. Hence for every $y \in Z(R) \setminus \mathrm{Nil}(R)$, $ann_R(x) = ann_R(y)$ and so the claim is proved. This implies that $AG(R) = K_{|ann_R(x)^*|} \vee \overline{K}_\infty$ and so $\gamma(AG(R)) = 0$, a contradiction.

Case 2. R is decomposable. Let $R \cong R_1 \times R_2$. Since $0 < \gamma(AG(R)) < \infty$, we may assume that $|R_1| \leq 3$, $|R_2| = \infty$. Therefore, $\gamma(AG(R)) = 0$, a contradiction. □

3. Toroidal annihilator graphs

In this section all rings with toroidal annihilator graphs are classified. We first study annihilator graphs associated with reduced rings.

Theorem 6. *Let R be a reduced ring. If $AG(R)$ is toroidal, then $R \cong R_1 \times \cdots \times R_n$, where $2 \leq n \leq 3$. Moreover, one of the following statements hold.*

(1) *If $n = 3$, then $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3$. Also, $AG(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3)$ is a toroidal graph.*

(2) *If $n = 2$, then R is one of the rings $\mathbb{F}_7 \times \mathbb{F}_4$, $\mathbb{F}_5 \times \mathbb{F}_5$, $\mathbb{F}_5 \times \mathbb{F}_4$, $\mathbb{F}_4 \times \mathbb{F}_4$.*

Proof. First we show that R is decomposable. By hypothesis, $AG(R)$ is a toroidal graph and so it follows from Theorem 5 that R is finite. Since R is a reduced ring, we deduce that $R \cong R_1 \times \cdots \times R_n$, where $2 \leq n$. If $n \geq 4$, then we prove that $AG(R)$ is not a toroidal graph. To see this, we only need to check the case $n = 4$. If $n = 4$, then it is not hard to check that the vertices of the set $\{(1,1,1,0),(1,1,0,0),(1,0,1,0),(0,1,1,0),(1,0,0,0)\}$ and the vertices contained in the set $\{(1,0,0,1),(0,1,0,1),(0,0,1,1),(0,0,0,1)\}$ together with the path $(1,0,0,0) - (0,1,0,0) - (1,0,0,1)$ form a subgraph which contains a subdivision of $K_{5,4}$, a contradiction. So $n \leq 3$.

(1) Let $R \cong R_1 \times R_2 \times R_3$. The ring R_i is indecomposable and finite, for every $1 \leq i \leq 3$, so R_i is a field for every $1 \leq i \leq 3$. If $R_1 \cong R_2 \cong R_3 \cong \mathbb{Z}_2$, then by Theorem 2, $AG(R)$ is a planar graph, a contradiction. So, with no loss of generality, we can suppose that $|R_3| > 2$. We show that $R_1 \cong R_2 \cong \mathbb{Z}_2$. If $|R_2| > 2$, then the vertices of the set $\{(1,0,0),(1,0,1),(1,0,y),(1,1,0),(1,x,0)\}$ and the vertices of the set $\{(0,1,1),(0,1,y),(0,x,y),(0,x,1)\}$ form a subgraph which contains a subdivision of $K_{5,4}$, where $x \in R_2 \setminus \{0,1\}$ and $y \in R_3 \setminus \{0,1\}$, a contradiction. Thus $R_2 \cong \mathbb{Z}_2$. Similarly, $R_1 \cong \mathbb{Z}_2$. We only have to prove that $R_3 \cong \mathbb{Z}_3$ and $AG(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3)$ is a toroidal graph. If $x, y \in R_3 \setminus \{0,1\}$, then the vertices of the set $\{(0,1,x),(0,1,y),(0,1,1),(0,1,0)\}$ and the vertices contained in the set $\{(1,1,0),(1,0,1),(1,0,x),(1,0,y),(1,0,0)\}$ together with the path $(0,1,0) - (0,0,1) - (1,1,0)$ form a subgraph which contains a subdivision of $K_{5,4}$, a contradiction. Hence $R_3 \cong \mathbb{Z}_3$ and $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3$. The following Figure shows that $AG(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3)$ can, indeed, be drawn without crossing itself on a torus. Hence $AG(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3)$ is a toroidal graph.

(2) If $n = 2$, then the result follows from [6, Theorem 3.6], and part (3) of [18, Theorem 3.1]. □

To complete our classification, we state the following remark and lemma.

Remark 1. It is not hard to see that, if $(R, \mathfrak{m})$ is a finite local ring, then there exists a prime integer p and positive integers t, l, k such that $\mathrm{char}(R) = p^t$, $|\mathfrak{m}| = p^l$, $|R| = p^k$, and $\mathrm{char}(R/\mathfrak{m}) = p$.

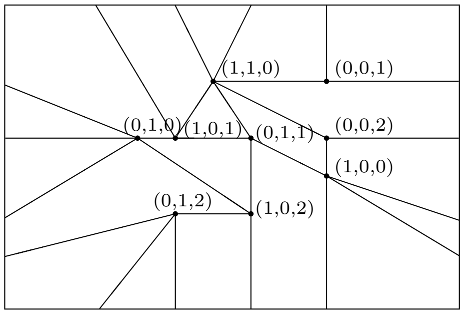

FIGURE 1. The annihilator graph of $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3$ on the torus.

Lemma 6. *Let $(R, \mathfrak{m})$ be a finite local ring. If $|\mathfrak{m}| \in \{7, 8\}$, then R is isomorphic to one of the following 22 rings:*
$\mathbb{Z}_{49}$, $\mathbb{Z}_7[x]/(x^2)$, $\mathbb{Z}_{16}$, $\mathbb{Z}_2[x]/(x^4)$, $\mathbb{Z}_4[x]/(x^2+2)$, $\mathbb{Z}_4[x]/(x^2+3x)$,
$\mathbb{Z}_4[x]/(x^3-2, 2x^2, 2x)$, $\mathbb{Z}_2[x,y]/(x^3, xy, y^2)$, $\mathbb{Z}_8[x]/(2x, x^2)$,
$\mathbb{Z}_4[x]/(x^3, 2x^2, 2x)$, $\mathbb{Z}_4[x]/(x^2+2x)$, $\mathbb{Z}_8[x]/(2x, x^2+4)$,
$\mathbb{Z}_2[x,y]/(x^2, y^2-xy)$, $\mathbb{Z}_4[x,y]/(x^2, y^2-xy, xy-2, 2x, 2y)$,
$\mathbb{Z}_4[x,y]/(x^3, y^2, xy-2, 2x, 2y)$, $\mathbb{Z}_2[x,y]/(x^2, y^2)$, $\mathbb{Z}_4[x]/(x^2)$,
$\mathbb{Z}_4[x]/(x^3-x^2-2, 2x^2, 2x)$, $\mathbb{Z}_2[x,y,z]/(x,y,z)^2$, $\mathbb{F}_8[x]/(x^2)$,
$\mathbb{Z}_4[x,y]/(x^2, y^2, xy, 2x, 2y)$, $\mathbb{Z}_4[x]/(x^3+x+1)$.

Proof. The proof follows from [15, Section 5]. □

We are now in a position to classify toroidal annihilator graphs associated with non-reduced ring.

Theorem 7. *Let R be a non-reduced ring. If $AG(R)$ is toroidal, then $R \cong R_1 \times \cdots \times R_n$, where $n \leq 2$. Moreover, one of the following statements hold.*

(1) *If $n = 1$, then R is one of the following rings:*
$\mathbb{Z}_{49}$, $\mathbb{Z}_7[x]/(x^2)$, $\mathbb{Z}_{16}$, $\mathbb{Z}_2[x]/(x^4)$, $\mathbb{Z}_4[x]/(x^2+2)$, $\mathbb{Z}_4[x]/(x^2+3x)$,
$\mathbb{Z}_4[x]/(x^3-2, 2x^2, 2x)$, $\mathbb{Z}_2[x,y]/(x^3, xy, y^2)$, $\mathbb{Z}_8[x]/(2x, x^2)$,
$\mathbb{Z}_4[x]/(x^3, 2x^2, 2x)$, $\mathbb{Z}_4[x]/(x^2+2x)$, $\mathbb{Z}_8[x]/(2x, x^2+4)$,
$\mathbb{Z}_2[x,y]/(x^2, y^2-xy)$, $\mathbb{Z}_4[x,y]/(x^2, y^2-xy, xy-2, 2x, 2y)$,
$\mathbb{Z}_4[x,y]/(x^3, y^2, xy-2, 2x, 2y)$, $\mathbb{Z}_2[x,y]/(x^2, y^2)$, $\mathbb{Z}_4[x]/(x^2)$,
$\mathbb{Z}_4[x]/(x^3-x^2-2, 2x^2, 2x)$, $\mathbb{Z}_2[x,y,z]/(x,y,z)^2$, $\mathbb{F}_8[x]/(x^2)$,
$\mathbb{Z}_4[x,y]/(x^2, y^2, xy, 2x, 2y)$, $\mathbb{Z}_4[x]/(x^3+x+1)$

(2) *If $n = 2$, then either $R \cong \mathbb{Z}_4 \times \mathbb{Z}_3$ or $R \cong \mathbb{Z}_2[x]/(x^2) \times \mathbb{Z}_3$.*

Proof. By Theorem 5, R is finite and so is an Artinian ring. Thus $R \cong R_1 \times \cdots \times R_n$, where $n \geq 1$. Let $R \cong R_1 \times \cdots \times R_n$, where $n \geq 3$. Since R is a non-reduced ring, we can suppose that $\mathrm{Nil}(R_1) \neq (0)$. This implies that $|U(R_1)| \geq 2$. Let $a \in \mathrm{Nil}(R_1)^*$ and $1 \neq u \in U(R_1)$. We show that $AG(R)$ is not a toroidal graph. To see this, we only need to check the case $n = 3$. But if $n = 3$, then by Lemma 2, one may see that the vertices of the set $\{(1,0,0),(u,0,0),(1,1,0),(u,1,0),(a,1,0)\}$ and the vertices contained in the set $\{(0,1,1),(0,0,1),(a,0,1),(a,1,1)\}$ form a subgraph which contains a subdivision of $K_{5,4}$, a contradiction. So $n \leq 2$.

(1) Let $(R,\mathfrak{m})$ be a local ring. By Theorem 5, R is finite. So $|R| = p^k$ and $|\mathfrak{m}| = p^l$, for some prime number p and some integers k, l. If $|\mathfrak{m}| > 8$, then by [6, Theorem 3.10], $AG(R)$ is a not toroidal graph. Thus $|\mathfrak{m}| \leq 8$. Since $|\mathfrak{m}| \geq 6$ and $|\mathfrak{m}| = p^l$, for some prime number p and for some integer l, we deduce that either $|\mathfrak{m}| = 8$ or $|\mathfrak{m}| = 7$. Thus by Lemma 6, the result holds.

(2) Suppose that $R \cong R_1 \times R_2$, where $(R_i,\mathfrak{m}_i)$ is a finite local ring, for $1 \leq i \leq 2$. With no loss of generality, suppose that $\mathrm{Nil}(R_1) \neq (0)$. First, we show that $|\mathfrak{m}_1| = 2$. If $|\mathfrak{m}_1| > 2$, then $|R_1| \geq 9$. So the vertices of the set $\{(1,0),(a_1,0),(a_2,0),(a_3,0),(a_4,0),(a_5,0),(a_6,0)\}$ and vertices contained in the set $\{(0,1),(0,1),(a,1),(b,1)\}$ form a subgraph which contains a subdivision of $K_{7,4}$ where $a,b \in \mathrm{Nil}(R_1)^*$ and $a_i \in R_1 \setminus \{0,1\}$ for $1 \leq i \leq 6$, a contradiction. Hence $|\mathfrak{m}_1| = 2$. Thus either $R_1 = \mathbb{Z}_4$ or $R_1 = \mathbb{Z}_2[x]/(x^2)$. Next, we show that R_2 is a field. To see this, let $a \in \mathfrak{m}_2^*$ and $R_1 = \mathbb{Z}_4$. Then by Lemma 2, the vertices contained in two sets $\{(2,a),(2,1),(0,1),(0,a)\}$ and $\{(1,a),(3,a),(1,0),(3,0),(2,0)\}$ form a subgraph which contains a subdivision of $K_{5,4}$, a contradiction. Therefore, R_2 is a field. If $|R_2| \geq 5$ and $R_1 = \mathbb{Z}_4$, then the vertices contained in sets $\{(2,1),(2,a_1),(2,a_2),(2,a_3),(0,1),(0,a_1),(0,a_2),(0,a_3)\}$ and $\{(1,0),(2,0),(3,0)\}$ form a subgraph which contains a subdivision of $K_{8,3}$, a contradiction. This implies that $R_2 = \mathbb{Z}_2$, $R_2 = \mathbb{Z}_3$ or $R_2 = \mathbb{F}_4$. If $R_2 = \mathbb{Z}_2$, then by [6, Theorem 3.16], $AG(R) = K_{2,3}$ and so $AG(R)$ is not a toroidal graph. If $R_2 = \mathbb{Z}_3$, then we can easily check that $AG(R)$ contains $K_{3,3}$ as a subgraph and since in this case $|V(AG(R))| = 7$, we conclude that $AG(R)$ is a toroidal graph. Hence if $R_2 = \mathbb{Z}_3$, then there are two rings such that $AG(R)$ is a toroidal graph. They are: $\mathbb{Z}_4 \times \mathbb{Z}_3$ and $\mathbb{Z}_2[x]/(x^2) \times \mathbb{Z}_3$.

If $R_2 = \mathbb{F}_4$, then $R = \mathbb{Z}_4 \times \mathbb{F}_4$. Assume that $\mathbb{F}_4 = \{0,u_1,u_2,u_3\}$. Let $x = (1,0), y = (2,0), z = (3,0), a = (0,u_1), b = (0,u_2), c = (0,u_3), d = (2,u_1), e = (2,u_2), f = (2,u_3), V_1 = \{x,y,z\}$ and $V_2 = \{a,b,c,d,e,f\}$. It is not hard to check that $AG(R)$ is $K_{|V_1|,|V_2|}$ together with a triangle in V_2. Therefore, $AG(R)$ is not a toroidal graph and so the proof is complete. □

Acknowledgements. The research of M. Bakhtyiari was partly funded by the Iranian National Science Foundation (INSF) under the contract No. 95003759. Also, the authors thank the referee for careful reading and excellent suggestions.

References

[1] M. Afkhami, M. Karimi, and K. Khashyarmanesh, *On the regular digraph of ideals of commutative rings*, Bull. Aust. Math. Soc. **88** (2013), 177–189.

[2] S. Akbari and R. Nikandish, *Some results on the intersection graphs of ideals of matrix algebras*, Linear Multilinear Algebra **62**(2) (2014), 195–206.

[3] A. Alibemani, M. Bakhtyiari, R. Nikandish, and M. J. Nikmehr, *The annihilator ideal graph of a commutative ring*, J. Korean Math. Soc. **52**(2) (2015), 417–429.

[4] D. F. Anderson and P. S. Livingston, *The zero-divisor graph of a commutative ring*, J. Algebra **217** (1999), 434–447.

[5] D. D. Anderson and M. Naseer, *Beck's coloring of a commutative ring*, J. Algebra **159** (1993), 500–514.

[6] A. Badawi, *On the annihilator graph of a commutative ring*, Comm. Algebra **42** (2014), 108–121.

[7] A. Badawi, *On the dot product graph of a commutative ring*, Comm. Algebra **43** (2015), 43–50.

[8] I. Beck, *Coloring of commutative rings*, J. Algebra **116** (1988), 208–226.

[9] W. Bruns and J. Herzog, *Cohen-Macaulay Rings*, Cambridge University Press, Cambridge, 1993.

[10] B. Corbas, *Rings with few zero divisors*, Math. Ann, **181** (1969), 1–7.

[11] B. Corbas and G. D. Williams, *Rings of order p^5. I. Nonlocal rings*, J. Algebra **231** (2000), 677–690.

[12] R. Nikandish, M. J. Nikmehr, and M. Bakhtyiari, *Coloring of the annihilator graph of a commutative ring*, J. Algebra Appl. **15**(7) (2016), 1650124, 13 pp.

[13] M. J. Nikmehr and F. Heydari, *The M-principal graph of a commutative ring*, Period. Math. Hung. **68** (2014), 185–192.

[14] A. Patil and B. N. Waphare, *On the zero-divisor graph of Rickart $*$-rings*, Asian-Eur. J. Math. **10**(1) (2017), 1750015, 17 pp.

[15] S. P. Redmond, *On zero-divisor graphs of small finite commutative rings*, Discrete Math. **307** (2007), 1155–1166.

[16] R. Y. Sharp, *Steps in Commutative Algebra*, London Mathematical Society Student Texts 51, Cambridge University Press, Cambridge, 2000.

[17] D. B. West, *Introduction to Graph Theory*, Prentice Hall, Upper Saddle River, 2001.

[18] C. Wickham, *Classification of rings with genus one zero-divisor graphs*, Comm. Algebra **36** (2008), 325–345.

[19] R. Wisbauer, *Foundations of Module and Ring Theory*, Gordon and Breach Science Publishers, Philadelphia, PA, 1991.

Faculty of Mathematics, K.N. Toosi University of Technology, P.O. BOX 16315-1618, Tehran, Iran

E-mail address: m.bakhtyiari55@gmail.com

Department of Mathematics, Jundi-Shapur University of Technology, P.O. BOX 64615-334, Dezful, Iran

E-mail address: r.nikandish@ipm.ir

Faculty of Mathematics, K.N. Toosi University of Technology, P.O. BOX 16315-1618, Tehran, Iran

E-mail address: nikmehr@kntu.ac.ir